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Klaus-Jochen Engel
Rainer Nagel

# One-Parameter Semigroups for Linear Evolution Equations

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*(continued after index)*

Klaus-Jochen Engel
Rainer Nagel

# One-Parameter Semigroups for Linear Evolution Equations

With Contributions by
S. Brendle, M. Campiti, T. Hahn, G. Metafune,
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*To*
*Carla and Ursula*

# Preface

The theory of one-parameter semigroups of linear operators on Banach spaces started in the first half of this century, acquired its core in 1948 with the Hille–Yosida generation theorem, and attained its first apex with the 1957 edition of *Semigroups and Functional Analysis* by E. Hille and R.S. Phillips. In the 1970s and 80s, thanks to the efforts of many different schools, the theory reached a certain state of perfection, which is well represented in the monographs by E.B. Davies [Dav80], J.A. Goldstein [Gol85], A. Pazy [Paz83], and others.

Today, the situation is characterized by manifold applications of this theory not only to the traditional areas such as partial differential equations or stochastic processes. Semigroups have become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. Semigroup methods are also applied with great success to concrete equations arising, e.g., in population dynamics or transport theory. It is quite natural, however, that semigroup theory is in competition with alternative approaches in all of these fields, and that as a whole, the relevant functional-analytic toolbox now presents a highly diversified picture.

At this point we decided to write a new book, reflecting this situation but based on our personal mathematical taste. Thus, it is a book on semigroups or, more precisely, on one-parameter semigroups of bounded linear operators. In our view, this reflects the basic philosophy, first and strongly emphasized by A. Hadamard (see p. 152), that an autonomous deterministic system is described by a one-parameter semigroup of transformations.

Among the many continuity properties of these semigroups that were

already studied by E. Hille and R.S. Phillips in [HP57], we deliberately concentrate on strong continuity and show that this is the key to a deep and beautiful theory. Referring to many concrete equations, one might object that semigroups, and especially strongly continuous semigroups, are of limited value, and that other concepts such as integrated semigroups, regularized semigroups, cosine families, or resolvent families are needed. While we do not question the good reasons leading to these concepts, we take a very resolute stand in this book insofar as we put strongly continuous semigroups of bounded linear operators into the undisputed center of our attention. Around this concept we develop techniques that allow us to obtain

- a semigroup on an appropriate Banach space even if at first glance the semigroup property does not hold, and
- strong continuity in an appropriate topology where originally only weaker regularity properties are at hand.

In Chapter VI we then show how these constructions allow the treatment of many different evolution equations that initially do not have the form of a homogeneous abstract Cauchy problem and/or are not "well-posed" in a strict sense.

*Structure of the Book*

This is not a research monograph but an introduction to the theory of semigroups. After developing the fundamental results of this theory we emphasize spectral theory, qualitative properties, and the broad range of applications. Moreover, our book is written in the spirit of functional analysis. This means that we prefer abstract constructions and general arguments in order to underline basic principles and to minimize computations. Some of the required tools from functional analysis, operator theory, and vector-valued integration are collected in the appendices.

In Chapter I, we intentionally take a slow start and lead the reader from the finite-dimensional and uniformly continuous case through multiplication and translation semigroups to the notion of a strongly continuous semigroup.

To these semigroups we associate a generator in Chapter II and characterize these generators in the Hille–Yosida generation theorem and its variants. Semigroups having stronger regularity properties such as analyticity, eventual norm continuity, or compactness are then characterized, whenever possible, in a similar way. A special feature of our approach is the use of a rich scale of interpolation and extrapolation spaces associated to a strongly continuous semigroup. A comprehensive treatment of these "Sobolev towers" is presented by Simon Brendle in Section II.5.

In Chapter III we start with the classical Bounded Perturbation Theorem III.1.3, but then present a new simultaneous treatment of unbounded Desch–Schappacher and Miyadera–Voigt perturbations in Section III.3. In the remaining Sections III.4 and 5 it was our goal to discuss a broad range

of applications of the Trotter–Kato Approximation Theorem III.4.8.

Spectral theory is the core of our approach, and in Chapter IV we discuss in great detail under what conditions the so-called spectral mapping theorem is valid. A first payoff is the complete description of the structure of periodic groups in Theorem IV.2.27.

On the basis of this spectral theory we then discuss in Chapter V qualitative properties of the semigroup such as stability, hyperbolicity, and mean ergodicity. Inspired by the classical Liapunov stability theorem we try to describe these properties by the spectrum of the generator. It is rewarding to see how a combination of spectral theory with geometric properties of the underlying Banach space can help to overcome many of the typical difficulties encountered in the infinite-dimensional situation.

Only at the end of Chapter II do differential equations and initial value problems appear explicitly in our text. This does not mean that we neglect this aspect. On the contrary, the many applications of semigroup theory to all kinds of evolution equations elaborated in Chapter VI are the ultimate goal of our efforts. However, we postpone this discussion until a powerful and systematic theory is at hand.

In the final chapter, Chapter VII, Tanja Hahn and Carla Perazzoli try to embed today's theory into a historical perspective in order to give the reader a feeling for the roots and the raison d'être of semigroup theory.

Furthermore, we add to our exposition of the mathematical theory an epilogue by Gregor Nickel, in which he discusses the philosophical question concerning the relationship between semigroups and evolution equations and the philosophical concept of "determinism." This is certainly a matter worth considering, but regrettably not much discussed in the mathematical community. For this reason, we encourage the reader to grapple and come to terms with this genuine philosophical question. It is enlightening to see how such questions were formulated and resolved in different epochs of the history of thought. Perhaps a deeper understanding will emerge of how one's own contemporary mathematical concepts and theories are woven into the broad tapestry of metaphysics.

*Guide to the Reader*

The text is not meant to be read in a linear manner. Thus, the reader already familiar with, or not interested in, the finite-dimensional situation and the detailed discussion of examples may start immediately with Section I.5 and then proceed quickly to the Hille–Yosida Generation Theorems II.3.5 and II.3.8 via Section II.1. To indicate other shortcuts, several sections, subsections, and paragraphs are given in small print.

Such an individual reading style is particularly appropriate with regard to Chapter VI, since our applications of semigroup theory to the various evolution equations are more or less independent of each other. The reader should select a section according to his/her interest and then continue with the more specialized literature indicated in the notes. Or, he/she may

even start with a suitable section of Chapter VI and then follow the back references in the text in order to understand our arguments.

The exercises at the end of each section should lead to a better understanding of the theory. Occasionally, we state interesting recent results as an exercise marked by *.

The notes are intended to identify our sources, to integrate the text into a broader picture, and to suggest further reading. Inevitably, they also reflect our personal perspective, and we apologize for omissions and inaccuracies. Nevertheless, we hope that the interested reader will be put on the track to uncover additional information.

*Acknowledgments*
Our research would not have been possible without the invaluable help from many colleagues and friends. We are particularly grateful to

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*Simon Brendle (Tübingen), Michele Campiti (Bari), Tanja Hahn (Frankfurt), Giorgio Metafune (Lecce), Gregor Nickel (Tübingen), Diego Pallara (Lecce), Carla Perazzoli (Rome), Abdelaziz Rhandi (Marrakesh), Silvia Romanelli (Bari), and Roland Schnaubelt (Tübingen)*

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*Klaus-Jochen Engel*
*Rainer Nagel*
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# Prelude

## An Excerpt from *Der Mann ohne Eigenschaften* (*The Man Without Qualities*) by Robert Musil*

in German, followed by the English Translation

Es läßt sich verstehen, daß ein Ingenieur in seiner Besonderheit aufgeht, statt in die Freiheit und Weite der Gedankenwelt zu münden, obgleich seine Maschinen bis an die Enden der Erde geliefert werden; denn er braucht ebensowenig fähig zu sein, das Kühne und Neue der Seele seiner Technik auf seine Privatseele zu übertragen, wie eine Maschine imstande ist, die ihr zugrunde liegenden Infinitesimalgleichungen auf sich selbst anzuwenden. Von der Mathematik aber läßt sich das nicht sagen; da ist die neue Denklehre selbst, der Geist selbst, liegen die Quellen der Zeit und der Ursprung einer ungeheuerlichen Umgestaltung.

Wenn es die Verwirklichung von Urträumen ist, fliegen zu können und mit den Fischen zu reisen, sich unter den Leibern von Bergriesen durchzubohren, mit göttlichen Geschwindigkeiten Botschaften zu senden, das Unsichtbare und Ferne zu sehen und sprechen zu hören, Tote sprechen zu hören, sich in wundertätigen Genesungsschlaf versenken zu lassen, mit lebenden Augen erblicken zu können, wie man zwanzig Jahre nach seinem Tode aussehen wird, in flimmernden Nächten tausend Dinge über und unter dieser Welt zu wissen, die früher niemand gewußt hat, wenn Licht, Wärme, Kraft, Genuß, Bequemlichkeit Urträume der Menschheit sind,—dann ist die

* Rowohlt Verlag, Hamburg 1978, by permission.

heutige Forschung nicht nur Wissenschaft, sondern ein Zauber, eine Zeremonie von höchster Herzens- und Hirnkraft, vor der Gott eine Falte seines Mantels nach der anderen öffnet, eine Religion, deren Dogmatik von der harten, mutigen, beweglichen, messerkühlen und -scharfen Denklehre der Mathematik durchdrungen und getragen wird.

Allerdings, es ist nicht zu leugnen, daß alle diese Urträume nach Meinung der Nichtmathematiker mit einemmal in einer ganz anderen Weise verwirklicht waren, als man sich das ursprünglich vorgestellt hatte. Münchhausens Posthorn war schöner als die fabriksmäßige Stimmkonserve, der Siebenmeilenstiefel schöner als ein Kraftwagen, Laurins Reich schöner als ein Eisenbahntunnel, die Zauberwurzel schöner als ein Bildtelegramm, vom Herz seiner Mutter zu essen und die Vögel zu verstehen schöner als eine tierpsychologische Studie über die Ausdrucksbewegung der Vogelstimme. Man hat Wirklichkeit gewonnen und Traum verloren. Man liegt nicht mehr unter einem Baum und guckt zwischen der großen und der zweiten Zehe hindurch in den Himmel, sondern man schafft; man darf auch nicht hungrig und verträumt sein, wenn man tüchtig sein will, sondern muß Beefsteak essen und sich rühren. (...). Man braucht wirklich nicht viel darüber zu reden, es ist den meisten Menschen heute ohnehin klar, daß die Mathematik wie ein Dämon in alle Anwendungen unseres Lebens gefahren ist. Vielleicht glauben nicht alle diese Menschen an die Geschichte vom Teufel, dem man seine Seele verkaufen kann; aber alle Leute, die von der Seele etwas verstehen müssen, weil sie als Geistliche, Historiker, Künstler gute Einkünfte daraus beziehen, bezeugen es, daß sie von der Mathematik ruiniert worden sei und daß die Mathematik die Quelle eines bösen Verstandes bilde, der den Menschen zwar zum Herrn der Erde, aber zum Sklaven der Maschine macht. Die innere Dürre, die ungeheuerliche Mischung von Schärfe im Einzelnen und Gleichgültigkeit im Ganzen, das ungeheure Verlassensein des Menschen in einer Wüste von Einzelheiten, seine Unruhe, Bosheit, Herzensgleichgültigkeit ohnegleichen, Geldsucht, Kälte und Gewalttätigkeit, wie sie unsre Zeit kennzeichnen, sollen nach diesen Berichten einzig und allein die Folge der Verluste sein, die ein logisch scharfes Denken der Seele zufügt! Und so hat es auch schon damals, als Ulrich Mathematiker wurde, Leute gegeben, die den Zusammenbruch der europäischen Kultur voraussagten, weil kein Glaube, keine Liebe, keine Einfalt, keine Güte mehr im Menschen wohne, und bezeichnenderweise sind sie alle in ihrer Jugend- und Schulzeit schlechte Mathematiker gewesen. Damit war später für sie bewiesen, daß die Mathematik, Mutter der exakten Naturwissenschaft, Großmutter der Technik, auch Erzmutter jenes Geistes ist, aus dem schließlich auch Giftgase und Kampfflieger aufgestiegen sind.

In Unkenntnis dieser Gefahren lebten eigentlich nur die Mathematiker selbst und ihre Schüler, die Naturforscher, die von alledem so wenig in ihrer Seele verspüren wie Rennfahrer, die fleißig darauf los treten und nichts in der Welt bemerken als das Hinterrad ihres Vordermanns. Von Ulrich dagegen konnte man mit Sicherheit sagen, daß er die Mathematik liebte,

wegen der Menschen, die sie nicht ausstehen mochten. Er war weniger wissenschaftlich als menschlich verliebt in die Wissenschaft. Er sah, daß sie in allen Fragen, wo sie sich für zuständig hält, anders denkt als gewöhnliche Menschen. Wenn man statt wissenschaftlicher Anschauungen Lebensanschauung setzen würde, statt Hypothese Versuch und statt Wahrheit Tat, so gäbe es kein Lebenswerk eines ansehnlichen Naturforschers oder Mathematikers, das an Mut und Umsturzkraft nicht die größten Taten der Geschichte weit übertreffen würde. Der Mann war noch nicht auf der Welt, der zu seinen Gläubigen hätte sagen können: Stehlt, mordet, treibt Unzucht—unserer Lehre ist so stark, daß sie aus der Jauche eurer Sünden schäumend helle Bergwässer macht; aber in der Wissenschaft kommt es alle paar Jahre vor, daß etwas, das bis dahin als Fehler galt, plötzlich alle Anschauungen umkehrt oder daß ein unscheinbarer und verachteter Gedanke zum Herrscher über ein neues Gedankenreich wird, und solche Vorkommnisse sind dort nicht bloß Umstürze, sondern führen wie eine Himmelsleiter in die Höhe. Es geht in der Wissenschaft so stark und unbekümmert und herrlich zu wie in einem Märchen. Und Ulrich fühlte: die Menschen wissen das bloß nicht; sie haben keine Ahnung, wenn man sie neu denken lehren könnte, würden sie auch anders leben.

Nun wird man sich freilich fragen, ob es denn auf der Welt so verkehrt zugehe, daß sie immerdar umgedreht werden müsse? Aber darauf hat die Welt längst selbst zwei Antworten gegeben. Denn seit sie besteht, sind die meisten Menschen in ihrer Jugend für das Umdrehen gewesen. Sie haben es lächerlich empfunden, daß die Älteren am Bestehenden hingen und mit ihrem Herzen dachten, einem Stück Fleisch, statt mit dem Gehirn. (...). Dennoch haben sie, sobald sie in die Jahre der Verwirklichung gekommen sind, nichts mehr davon gewußt und noch weniger wissen wollen. Darum werden auch viele, denen Mathematik oder Naturwissenschaft einen Beruf bedeuten, es als einen Mißbrauch empfinden, sich aus solchen Gründen wie Ulrich für eine Wissenschaft zu entscheiden.

## *The Man Without Qualities**

It is understandable that an engineer should be completely absorbed in his speciality, instead of pouring himself out into the freedom and vastness of the world of thought, even though his machines are being sent off to the ends of the earth; for he no more needs to be capable of applying to his own personal soul what is daring and new in the soul of his subject than a machine is in fact capable of applying to itself the differential calculus on which it is based. The same thing cannot, however, be said about mathematics; for here we have the new method of thought, pure intellect, the

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* From *The Man Without Qualities* by Robert Musil, trans. Sophie Wilkins.

very wellspring of the times, the *fons et origo* of an unfathomable transformation.

If the realization of primordial dreams is flying, traveling with the fishes, boring one's way under the bodies of mountain-giants, sending messages with godlike swiftness, seeing what is invisible and what is in the distance and hearing its voice, hearing the dead speak, having oneself put into a wonder-working healing sleep, being able to behold with living eyes what one will look like twenty years after one's death, in glimmering nights to know a thousand things that are above and below this world, things that no one ever knew before, if light, warmth, power, enjoyment, and comfort are mankind's primordial dreams, then modern research is not only science but magic, a ritual involving the highest powers of heart and brain, before which God opens one fold of His mantle after another, a religion whose dogma is permeated and sustained by the hard, courageous, mobile, knife-cold, knife-sharp mode of thought that is mathematics.

Admittedly, it cannot be denied that in the nonmathematician's opinion all these primordial dreams were suddenly realized in quite a different way from what people had once imagined. Baron Münchhausen's post-horn was more beautiful than mass-produced canned music, the Seven-League Boots were more beautiful than a motor-car, Dwarf-King Laurin's realm more beautiful than a railway-tunnel, the magic mandrake-root more beautiful than a telegraphed picture, to have eaten of one's mother's heart and so to understand the language of birds more beautiful than an animal psychologist's study of the expressive values in bird-song. We have gained in terms of reality and lost in terms of the dream. We no longer lie under a tree, gazing up at the sky between our big toe and second toe; we are too busy getting on with our jobs. And it is no good being lost in dreams and going hungry, if one wants to be efficient; one must eat steak and get a move on. (...). There is really no need to say much about it. It is in any case quite obvious to most people nowadays that mathematics has entered like a daemon into all aspects of our life. Perhaps not all of these people believe in that stuff about the Devil to whom one can sell one's soul; but all those who have to know something about the soul, because they draw a good income out of it as clergy, historians, or artists, bear witness to the fact that it has been ruined by mathematics and that in mathematics is the source of a wicked intellect that, while making man the lord of the earth, also makes him the slave of the machine. The inner drought, the monstrous mixture of acuity in matters of detail and indifference as regards the whole, man's immense loneliness in a desert of detail, his restlessness, malice, incomparable callousness, his greed for money, his coldness and violence, which are characteristic of our time, are, according to such surveys, simply and solely the result of the losses that logical and accurate thinking has inflicted on the soul! And so it was that even at that time, when Ulrich became a mathematician, there were people who were prophesying the collapse of European civilization on the grounds that there was no longer

any faith, any love, any simplicity or any goodness left in mankind; and it is significant that these people were all bad at mathematics at school. This only went to convince them, later on, that mathematics, the mother of the exact natural sciences, the grandmother of engineering, was also the arch-mother of that spirit from which, in the end, poison-gases and fighter aircraft have been born.

Actually, the only people living in ignorance of these dangers were the mathematicians themselves and their disciples, the natural scientists, who felt no more of all this in their souls than racing-cyclists who are pedaling away hard with no eyes for anything in the world but the back wheel of the man in front. As far as Ulrich was concerned, however, it could at least definitely be said that he loved mathematics because of the people who could not endure it. He was not so much scientifically as humanly in love with science. He could see that in all the problems that came into its orbit science thought differently from the way ordinary people thought. If for "scientific attitude" one were to read "attitude to life," for "hypothesis" "attempt" and for "truth" "action," then there would be no considerable natural scientist or mathematician whose life's work did not in courage and revolutionary power far outmatch the greatest deeds in history. The man was not yet born who could have said to his disciples: "Rob, murder, fornicate—our teaching is so strong that it will transform the cesspool of your sins into clear, sparkling mountain-rills." But in science it happens every few years that something that up to then was held to be error suddenly revolutionizes all views or that an unobtrusive, despised idea becomes the ruler over a new realm of ideas; and such occurrences are not mere upheavals but lead up into the heights like Jacob's ladder. In science the way things happen is as vigorous and matter-of-fact and glorious as in a fairy-tale. "People simply don't know this," Ulrich felt. "They have no glimmer of what can be done with thinking. If one could teach them to think in a new way, they would also live differently."

Now someone is sure to ask, of course, whether the world is so topsy-turvy that it is always having to be turned up the other way again. But the world itself long ago gave two answers to this question. For ever since it has existed most people have in their youth been in favor of turning things upside-down. They have always felt that their elders were ridiculous in being so attached to the established order of things and in thinking with their heart—a mere lump of flesh—instead of with their brains. (...). Nevertheless, by the time they reach years of fulfillment they have forgotten all about it and are far from wishing to be reminded of it. That is why many people for whom mathematics or natural science is a job feel it is almost an outrage if someone goes in for science for reasons like Ulrich's.

# Chapter I

# Linear Dynamical Systems

There are many good reasons—the reader may consult Section 1 of the Epilogue for details—why an "autonomous deterministic system" should be described by maps $T(t)$, $t \geq 0$, satisfying the functional equation

$$\text{(FE)} \qquad T(t+s) = T(t)T(s).$$

Here, $t$ is the time parameter, and each $T(t)$ maps the "state space" of the system into itself. These maps completely determine the time evolution of the system in the following way: If the system is in state $x_0$ at time $t_0 = 0$, then at time $t$ it is in state $T(t)x_0$.

However, in most cases a complete knowledge of the maps $T(t)$ is hard, if not impossible, to obtain. It was one of the great discoveries of mathematical physics, based on the invention of calculus, that, as a rule, it is much easier to understand the "infinitesimal changes" occurring at any given time. In this case, the system can be described by a differential equation replacing the functional equation (FE).

In this chapter we analyze this phenomenon in the mathematical context of linear operators on Banach spaces.

For this purpose, we take two opposite views.

**$V_1$.** *We start with a solution $t \mapsto T(t)$ of the above functional equation* (FE) *and ask which assumptions imply that it is differentiable and satisfies a differential equation.*

**$V_2$.** *We start with a differential equation and ask how its solution can be related to a family of mappings satisfying* (FE).

In the following we treat the finite-dimensional and the uniformly continuous situation in some detail, then discuss further examples in Section 4. On the basis of this information we try to explain why strongly continuous semigroups as introduced in Section 5 correspond to both views.

However, the impatient reader who does not need this kind of motivation should start immediately with Section 5.

# 1. Cauchy's Functional Equation

As a warm-up, this program will be performed in the scalar-valued case first. In fact, it was A. Cauchy who in 1821 asked in his *Cours d'Analyse*, without any further motivation, the following question:

> *Déterminer la fonction $\varphi(x)$ de manière qu'elle reste continue entre deux limites réelles quelconques de la variable $x$, et que l'on ait pour toutes les valeurs réelles des variables $x$ et $y$*
>
> $$\varphi(x+y) = \varphi(x)\varphi(y).$$ [1]
>
> (A. Cauchy, [Cau21, p. 100])

Using modern notation, we restate his question as follows dropping the continuity requirement for the moment.

**1.1 Problem.** *Find all maps $T(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ satisfying the functional equation*

$$\text{(FE)} \qquad \begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = 1. \end{cases}$$

Evidently, the exponential functions

$$\text{(EXP)} \qquad t \mapsto \mathrm{e}^{ta}$$

satisfy (FE) for any $a \in \mathbb{C}$. With his question, Cauchy suggested that these *canonical* solutions should be *all* solutions of (FE).

Before giving an answer to Problem 1.1, we take a closer look at the exponential functions (EXP) and observe that they, besides solving the algebraic identity (FE), also enjoy some important analytic properties.

[1] Determine the function $\varphi(x)$ in such a way that it remains continuous between two arbitrary real limits of the variable $x$, and that, for all real values of the variables $x$ and $y$, one has

$$\varphi(x+y) = \varphi(x)\varphi(y).$$

**1.2 Proposition.** *Let $T(t) := \mathrm{e}^{ta}$ for some $a \in \mathbb{C}$ and all $t \geq 0$. Then the function $T(\cdot)$ is differentiable and satisfies the differential equation (or, more precisely, the initial value problem)*

$$\text{(DE)} \qquad \begin{cases} \frac{d}{dt}T(t) = aT(t) & \text{for } t \geq 0, \\ T(0) = 1. \end{cases}$$

*Conversely, the function $T(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ defined by $T(t) = \mathrm{e}^{ta}$ for some $a \in \mathbb{C}$ is the only differentiable function satisfying* (DE). *Finally, we observe that $a = {}^{d}\!/\!_{dt}T(t)\big|_{t=0}$.*

Proof. We show only the assertion concerning uniqueness. Let $S(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ be another differentiable function satisfying (DE). Then the new function $Q(\cdot) : [0,t] \to \mathbb{C}$ defined by

$$Q(s) := T(s)S(t-s) \quad \text{for } 0 \leq s \leq t$$

for some fixed $t > 0$ is differentiable with derivative ${}^{d}\!/\!_{ds}Q(s) \equiv 0$. This shows that

$$T(t) = Q(t) = Q(0) = S(t)$$

for arbitrary $t > 0$. □

This proposition shows that, in our scalar-valued case, $V_2$ can be answered easily using the exponential function. It is now our main point that *continuity* is already sufficient to obtain differentiability in $V_1$.

**1.3 Proposition.** *Let $T(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ be a continuous function satisfying* (FE). *Then $T(\cdot)$ is differentiable, and there exists a unique $a \in \mathbb{C}$ such that* (DE) *holds.*

Proof. Since $T(\cdot)$ is continuous on $\mathbb{R}_+$, the function $V(\cdot)$ defined by

$$V(t) := \int_0^t T(s)\,ds, \qquad t \geq 0,$$

is differentiable with $\dot{V}(t) = T(t)$. This implies[2]

$$\lim_{t \downarrow 0} \frac{1}{t} V(t) = \dot{V}(0) = T(0) = 1.$$

Therefore, $V(t_0)$ is different from zero, hence invertible, for some small $t_0 > 0$.

[2] In the sequel, we often denote a derivative with respect to the real variable $t$ by "$\cdot$", i.e., $\dot{V}(0) = {}^{d}\!/\!_{dt}V(t)\big|_{t=0}$.

The functional equation (FE) now yields

$$\begin{aligned} T(t) &= V(t_0)^{-1}V(t_0)T(t) = V(t_0)^{-1}\int_0^{t_0} T(t+s)\,ds \\ &= V(t_0)^{-1}\int_t^{t+t_0} T(s)\,ds = V(t_0)^{-1}\big(V(t+t_0) - V(t)\big) \end{aligned}$$

for all $t \geq 0$. Hence, $T(\cdot)$ is differentiable with derivative

$$\begin{aligned} \tfrac{d}{dt}T(t) &= \lim_{h\downarrow 0} \frac{T(t+h) - T(t)}{h} \\ &= \lim_{h\downarrow 0} \frac{T(h) - T(0)}{h} T(t) = \dot{T}(0)T(t) \quad \text{for all } t \geq 0. \end{aligned}$$

This shows that $T(\cdot)$ satisfies (DE) with $a := \dot{T}(0)$. □

The combination of both results leads to a satisfactory answer to Cauchy's Problem 1.1.

**1.4 Theorem.** *Let $T(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ be a continuous function satisfying* (FE). *Then there exists a unique $a \in \mathbb{C}$ such that*

$$T(t) = \mathrm{e}^{ta} \quad \textit{for all } t \geq 0.$$

With this answer we stop our discussion of this elementary situation and close this section with some further comments on Cauchy's Problem 1.1.

**1.5 Comments.** (i) Once shown, as in Theorem 1.4, that a certain function $T(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ is of the form $T(t) = \mathrm{e}^{ta}$, it is clear that it can be extended to all $t \in \mathbb{R}$ and even all $t \in \mathbb{C}$ still satisfying the functional equation (FE) for all $t, s \in \mathbb{C}$. In other words, this extension becomes a homomorphism from the additive group $(\mathbb{C}, +)$ into the multiplicative group $(\mathbb{C} \setminus \{0\}, \cdot)$.

(ii) Much weaker conditions than continuity, e.g., local integrability, are sufficient to obtain the conclusion of Theorem 1.4. For a detailed account on this subject we refer to [Acz66] and Exercise 1.7.

(iii) Even *noncanonical* solutions of (FE) can be found using a result of Hamel. In [Ham05] he considered $\mathbb{R}$ as a vector space over $\mathbb{Q}$. By linearly extending an arbitrary function on the elements of a $\mathbb{Q}$-vector basis of $\mathbb{R}$ he obtained all additive functions. Composition of the exponential function with the additive functions then yields the solutions of (FE). Again see Exercise 1.7 and [Acz66] for further details.

(iv) It is important to keep in mind that (FE) is not just any formal identity but gains its significance from the description of dynamical systems. If we identify $\mathbb{C}$ with the space $\mathcal{L}(\mathbb{C})$ of all linear operators on $\mathbb{C}$, we see that a

map $T(\cdot)$ satisfying (FE) describes the time evolution (for time $t \geq 0$) of a linear dynamical system on $\mathbb{C}$. More precisely, let $x_0 \in \mathbb{C}$ be the state of our system at time $t = 0$. Then

$$x(t) := T(t)x_0$$

is the state at $t \geq 0$. Then (FE) means that

$$x(t+s) = T(t+s)x_0 = T(t)T(s)x_0 = T(t)x(s);$$

hence the state $x(t+s)$ at time $t+s$ is the same as the state at time $t$ starting from $x(s)$. In the Epilogue we try to explain how (FE) appears in any mathematical description of deterministic dynamical systems.

**1.6 Perspective.** The basis for our solution of Problem 1.1 was the fact that a solution of the algebraic equation (FE) that is continuous must already be differentiable (even analytic) and therefore solves (DE). The phenomenon

$$\text{continuity} + \text{(FE)} \;\Rightarrow \text{differentiability}$$

will be a fundamental and recurrent theme for our further investigations. We already refer to Theorem 3.7, Lemma II.1.3.(ii), or Theorem II.4.6 for particularly important manifestations of this phenomenon. It thus seems justified to consider the subsequent theory of *one-parameter semigroups* as a contribution to what Hilbert suggested at the 1900 International Congress of Mathematicians at Paris in the second part of his fifth problem:

> *Überhaupt werden wir auf das weite und nicht uninteressante Feld der Funktionalgleichungen geführt, die bisher meist nur unter Voraussetzung der Differenzierbarkeit der auftretenden Funktionen untersucht worden ist. Insbesondere die von Abel*[3] *mit so vielem Scharfsinn behandelten Funktionalgleichungen, die Differenzengleichungen und andere in der Literatur vorkommende Gleichungen weisen an sich nichts auf, was zur Forderung der Differenzierbarkeit der auftretenden Funktionen zwingt. . . . In allen Fällen erhebt sich daher die Frage, inwieweit etwa die Aussagen, die wir im Falle der Annahme differenzierbarer Funktionen machen können, unter geeigneten Modifikationen ohne diese Voraussetzung gültig sind.*[4]
>
> (David Hilbert [Hil70, p. 20])

[3] Werke Vol. 1, pages 1, 61, and 389.

[4] Moreover, we are thus led to the wide and interesting field of functional equations, which have been heretofore investigated usually only under the assumption of the differentiability of the functions involved. In particular, the functional equations treated by Abel with so much ingenuity, the difference equations . . . and other equations occurring in the literature of mathematics, do not directly involve anything that necessitates the requirement of the differentiability of the accompanying functions. . . . In all these cases, then, the problem arises: *To what extent are the assertions that we can make in the case of differentiable functions true under proper modifications without this assumption?*

**1.7 Exercises.** (1) A function $f : \mathbb{R} \to \mathbb{R}$ is called *additive* if it satisfies the functional equation

$$f(s+t) = f(s) + f(t) \qquad \text{for all } s,\, t \in \mathbb{R}.$$

Show that the following assertions are true.

(i) The function $f : \mathbb{R}_+ \to \mathbb{R}$ is additive if and only if $T(\cdot) := \exp \circ f$ solves (FE).

(ii) There exist discontinuous additive functions on $\mathbb{R}$. (Hint: Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space and choose an arbitrary basis $\mathcal{B}$ of $\mathbb{R}$. Now take an arbitrary real-valued function defined on $\mathcal{B}$ and extend it linearly.)

(iii) There exist discontinuous solutions of (FE) that are not identically zero for $t > 0$.

(2) Show that any measurable solution $T(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ of (FE) either is given by

$$T(t) := \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t > 0, \end{cases} \tag{1.1}$$

or there exists $a \in \mathbb{R}$ such that $T(t) = \exp(ta)$ for all $t \in \mathbb{R}$. In particular, a solution of (FE) which is discontinuous in some $t > 0$ cannot be measurable. (Hint: First show that every solution $T(\cdot)$ different from (1.1) has no zeros. Hence for those $T(\cdot)$ the functions $g : t \mapsto \exp(\mathrm{i}\cdot \log T(t))$ are well-defined and are locally integrable solutions of (FE). Now a modification of the proof of Theorem 1.4 shows that $g$ is given by $g(t) = \exp(\mathrm{i}ta)$ for some $a \in \mathbb{R}$. Finally, use the fact that the maps $t \mapsto \log T(t)$ and $t \mapsto at$ are additive in order to derive the assertion.)

# 2. Finite-Dimensional Systems: Matrix Semigroups

In this section we pass to a more general setting and consider finite-dimensional vector spaces $X := \mathbb{C}^n$. The space $\mathcal{L}(X)$ of all linear operators on $X$ will then be identified with the space $\mathrm{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices, and a *linear dynamical system* on $X$ will be given by a *matrix*-valued function

$$T(\cdot) : \mathbb{R}_+ \to \mathrm{M}_n(\mathbb{C})$$

satisfying the functional equation

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = I. \end{cases} \tag{FE}$$

As before, the variable $t$ will be interpreted as "time." The "time evolution" of a state $x_0 \in X$ is then given by the function $\xi_{x_0} : \mathbb{R}_+ \to X$ defined as

$$\xi_{x_0}(t) := T(t)x_0.$$

We also call $\{T(t)x_0 : t \geq 0\}$ the *orbit* of $x_0$ under $T(\cdot)$. From the functional equation (FE) it follows that an initial state $x_0$ arrives after an elapsed time $t + s$ at the same state as the initial state $y_0 := T(s)x_0$ after time $t$. See also the considerations in the Epilogue, Section 1.

In this new context we study $V_1$ and $V_2$ from Section 1 and restate Cauchy's Problem 1.1.

**2.1 Problem.** *Find all maps* $T(\cdot) : \mathbb{R}_+ \to \mathrm{M}_n(\mathbb{C})$ *satisfying the functional equation* (FE).

Imitating the arguments from Section 1 we first look for "canonical" solutions of (FE) and then hope that these exhaust all (natural) linear dynamical systems.

As in Section 1 the candidates for solutions of (FE) are the "exponential functions," and G. Peano seems to have been the first who in 1887 gave a precise definition of *matrix-valued* exponential functions.

*Se ... le equazioni differenziali proposte sono a coefficienti costanti ... si ricava*

$$x = \left(1 + \alpha t + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \cdots\right) a;$$

*e se si conviene di rappresentare con* $\mathrm{e}^{\alpha t}$, *anche quando* $\alpha$ *è un complesso qualunque, la somma della serie* $1 + \alpha t + \frac{(\alpha t)^2}{2!} + \cdots$, *l'integrale dell'equazione differenziale proposta diventa*

$$x = \mathrm{e}^{\alpha t} a.$$ [5]

(G. Peano [Pea87], see also [Pea88])

In modern notation, Peano's definition takes the following form.

**2.2 Definition.** *For any* $A \in \mathrm{M}_n(\mathbb{C})$ *and* $t \in \mathbb{R}$ *the matrix exponential* $\mathrm{e}^{tA}$ *is defined by*

$$\mathrm{e}^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}. \tag{2.1}$$

Taking any norm on $\mathbb{C}^n$ and the corresponding matrix-norm on $\mathrm{M}_n(\mathbb{C})$ one shows that the partial sums of the series above form a Cauchy sequence; hence the series converges and satisfies

$$\left\|\mathrm{e}^{tA}\right\| \le \mathrm{e}^{t\|A\|} \tag{2.2}$$

for all $t \ge 0$. Moreover, the map $t \mapsto \mathrm{e}^{tA}$ has the following properties.

[5] If ... the equations considered have constant coefficients ... one obtains

$$x = \left(1 + \alpha t + \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^3}{3!} + \cdots\right) a;$$

and, if we agree to write, even for $\alpha$ an arbitrary matrix, $\mathrm{e}^{\alpha t}$ for the sum of the series $1 + \alpha t + \frac{(\alpha t)^2}{2!} + \cdots$, the integral of the differential equation considered becomes

$$x = \mathrm{e}^{\alpha t} a.$$

**2.3 Proposition.** *For any $A \in \mathrm{M}_n(\mathbb{C})$ the map*

$$\mathbb{R}_+ \ni t \mapsto \mathrm{e}^{tA} \in \mathrm{M}_n(\mathbb{C})$$

*is continuous and satisfies*

$$\text{(FE)} \qquad \begin{cases} \mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA} & \text{for } t, s \geq 0, \\ \mathrm{e}^{0A} = I. \end{cases}$$

PROOF. Since the series $\sum_{k=0}^{\infty} t^k \|A\|^k / k!$ converges, one can show, as for the Cauchy product of scalar series, that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{s^k A^k}{k!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^{n-k} A^{n-k}}{(n-k)!} \cdot \frac{s^k A^k}{k!} \\ &= \sum_{n=0}^{\infty} \frac{(t+s)^n A^n}{n!}. \end{aligned}$$

This proves (FE). In order to show that $t \mapsto \mathrm{e}^{tA}$ is continuous, we first observe that by (FE) one has

$$\mathrm{e}^{(t+h)A} - \mathrm{e}^{tA} = \mathrm{e}^{tA}\big(\mathrm{e}^{hA} - I\big)$$

for all $t, h \in \mathbb{R}$. Therefore, it suffices to show that $\lim_{h\to 0} \mathrm{e}^{hA} = I$. This follows from the estimate

$$\begin{aligned} \big\|\mathrm{e}^{hA} - I\big\| &= \left\| \sum_{k=1}^{\infty} \frac{h^k A^k}{k!} \right\| \\ &\leq \sum_{k=1}^{\infty} \frac{|h|^k \cdot \|A\|^k}{k!} = \mathrm{e}^{|h|\cdot\|A\|} - 1. \end{aligned}$$

□

At this point, it is good to pause for a moment and try to understand the meaning of the functional equation (FE) in terms of linear operators (or matrices) on $\mathbb{C}^n$. Obviously, the range of the function $t \mapsto T(t) := \mathrm{e}^{tA}$ in $\mathrm{M}_n(\mathbb{C})$ is a commutative semigroup of matrices depending continuously on the parameter $t \in \mathbb{R}_+$. In fact, this is a straightforward consequence of the following decisive property:

*The mapping $t \mapsto T(t)$ is a homomorphism from the additive semigroup $(\mathbb{R}_+, +)$ into the multiplicative semigroup $\big(\mathrm{M}_n(\mathbb{C}), \cdot\big)$.*

Keeping this in mind, we start to use the following terminology.

**2.4 Definition.** *We call $\big(\mathrm{e}^{tA}\big)_{t\geq 0}$ the (one-parameter) semigroup generated by the matrix $A \in \mathrm{M}_n(\mathbb{C})$.*

As the reader may have already realized, there is no need in Definition 2.2 (and 2.4) to restrict the (time) parameter $t$ to $\mathbb{R}_+$. The definition, the continuity, and the functional equation (FE) hold for any real and even complex $t$. Then the map

$$T(\cdot) : t \mapsto \mathrm{e}^{tA}$$

extends to a continuous (even analytic) homomorphism from the additive group $(\mathbb{R}, +)$ (or, $(\mathbb{C}, +)$) into the multiplicative group $\mathrm{GL}(n, \mathbb{C})$ of all invertible, complex $n \times n$ matrices. We call $\big(\mathrm{e}^{tA}\big)_{t\in\mathbb{R}}$ the *(one-parameter) group* generated by $A$.

Before proceeding with the abstract theory, the reader might appreciate some examples of matrix semigroups.

**2.5 Examples.** (i) The (semi) group generated by a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ is given by

$$\mathrm{e}^{tA} = \mathrm{diag}\big(\mathrm{e}^{ta_1}, \ldots, \mathrm{e}^{ta_n}\big).$$

(ii) Less trivial is the case of a $k \times k$ Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}_{k\times k}$$

with eigenvalue $\lambda \in \mathbb{C}$. Decompose $A$ into a sum $A = D + N$ where $D = \lambda I$. Then the $k$th power of $N$ is zero, and the power series (2.1) (with $A$ replaced by $N$) becomes

$$\mathrm{e}^{tN} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{k-2}}{(k-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}_{k\times k}. \tag{2.3}$$

Since $D$ and $N$ commute, we obtain

$$\mathrm{e}^{tA} = \mathrm{e}^{t\lambda}\mathrm{e}^{tN} \tag{2.4}$$

(see Exercise 2.12.(1)).

For arbitrary matrices $A$, the direct computation of $\mathrm{e}^{tA}$ (using the above definition) is very difficult if not impossible. Fortunately, thanks to the existence of the Jordan normal form, the following lemma shows that in a certain sense the Examples 2.5.(i) and (ii) suffice.

**2.6 Lemma.** *Let $B \in \mathrm{M}_n(\mathbb{C})$ and take an invertible matrix $S \in \mathrm{M}_n(\mathbb{C})$. Then the (semi) group generated by the matrix $A := S^{-1}BS$ is given by*

$$\mathrm{e}^{tA} = S^{-1}\mathrm{e}^{tB}S.$$

PROOF. Since $A^k = S^{-1}B^kS$ for all $k \in \mathbb{N}$ and since $S$, $S^{-1}$ are continuous operators, we obtain

$$\begin{aligned} \mathrm{e}^{tA} &= \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k S^{-1} B^k S}{k!} \\ &= S^{-1} \left( \sum_{k=0}^{\infty} \frac{t^k B^k}{k!} \right) S = S^{-1}\mathrm{e}^{tB}S. \end{aligned}$$

□

The content of this lemma is that *similar* matrices (for the definition of *similarity* see 5.10) generate similar (semi) groups. Since we know that any complex $n \times n$ matrix is similar to a direct sum of Jordan blocks, we conclude that any matrix (semi) group is similar to a direct sum of (semi) groups as in Example 2.5.(ii). Already in the case of $2 \times 2$ matrices, the necessary computations are lengthy; however, they yield explicit formulas for the matrix exponential function.

**2.7 More Examples.** (iii) Take an arbitrary $2 \times 2$ matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define $\delta := ad - bc$, $\tau := a + d$, and take $\gamma \in \mathbb{C}$ such that $\gamma^2 = {}^1\!/\!_4(\tau^2 - 4\delta)$. Then the (semi) group generated by $A$ is given by the matrices

(2.5)
$$\mathrm{e}^{tA} = \begin{cases} \mathrm{e}^{t\tau/2}\Big( {}^1\!/\!_\gamma \sinh(t\gamma)A + \big(\cosh(t\gamma) - {}^{2\tau}\!/\!_\gamma \sinh(t\gamma)\big)I \Big) & \text{if } \gamma \neq 0, \\ \mathrm{e}^{t\tau/2}\,(tA + (1 - t\tau/2)I) & \text{if } \gamma = 0. \end{cases}$$

We list some special cases yielding simpler formulas:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \mathrm{e}^{tA} &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}, \\ A &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \mathrm{e}^{tA} &= \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}, \\ A &= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, & \mathrm{e}^{tA} &= \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}. \end{aligned}$$

We now return to the theory of the matrix exponential functions $t \mapsto \mathrm{e}^{tA}$. We know from Proposition 2.3 that they are continuous and satisfy the functional equation (FE). In the next proposition we see that they are even differentiable and satisfy the differential equation (DE) (compare to Proposition 1.2).

**2.8 Proposition.** *Let $T(t) := \mathrm{e}^{tA}$ for some $A \in \mathrm{M}_n(\mathbb{C})$. Then the function $T(\cdot) : \mathbb{R}_+ \to \mathrm{M}_n(\mathbb{C})$ is differentiable and satisfies the differential equation*

$$\text{(DE)} \qquad \begin{cases} \frac{d}{dt}T(t) = AT(t) & \text{for } t \geq 0, \\ T(0) = I. \end{cases}$$

*Conversely, every differentiable function $T(\cdot) : \mathbb{R}_+ \to \mathrm{M}_n(\mathbb{C})$ satisfying* (DE) *is already of the form $T(t) = \mathrm{e}^{tA}$ for some $A \in \mathrm{M}_n(\mathbb{C})$. Finally, we observe that $A = \dot{T}(0)$.*

Proof. We start by showing that $T(\cdot)$ satisfies (DE). Since the functional equation (FE) in Proposition 2.3 implies

$$\frac{T(t+h) - T(t)}{h} = \frac{T(h) - I}{h} \cdot T(t)$$

for all $t, h \in \mathbb{R}$, (DE) is proved if $\lim_{h \to 0} \frac{T(h)-I}{h} = A$. This, however, follows, since

$$\begin{aligned} \left\| \frac{T(h) - I}{h} - A \right\| &\leq \sum_{k=2}^{\infty} \frac{|h|^{k-1} \cdot \|A\|^k}{k!} \\ &= \frac{\mathrm{e}^{|h| \cdot \|A\|} - 1}{|h|} - \|A\| \to 0 \quad \text{as } h \to 0. \end{aligned}$$

The remaining assertions are proved as in Proposition 1.2 by replacing the complex number $a$ by the matrix $A$. ☐

After these preparations, we are now ready to give an answer to Problem 2.1 that is in complete analogy to the result in Section 1.

**2.9 Theorem.** *Let $T(\cdot) : \mathbb{R}_+ \to \mathrm{M}_n(\mathbb{C})$ be a continuous function satisfying* (FE)*. Then there exists $A \in \mathrm{M}_n(\mathbb{C})$ such that*

$$T(t) = \mathrm{e}^{tA} \quad \textit{for all } t \geq 0.$$

Proof. Since $T(\cdot)$ is continuous and $T(0) = I$ is invertible, the matrices

$$V(t_0) := \int_0^{t_0} T(s)\, ds$$

are invertible for sufficiently small $t_0 > 0$ (use that $\lim_{t \downarrow 0} {}^1\!/_t V(t) = T(0) = I$). Now repeat the computations from the proof of Theorem 1.4. ☐

With this theorem we have characterized all continuous one-parameter (semi) groups on $\mathbb{C}^n$ as matrix-valued exponential functions $\left(\mathrm{e}^{tA}\right)_{t \geq 0}$. As in the scalar case, one can weaken the continuity assumption (see Comment 1.5.(ii)), but not drop it entirely (see Comment 1.5.(iii)). However, since continuity seems to be the natural assumption for our interpretation of $\left(\mathrm{e}^{tA}\right)_{t \geq 0}$ as a dynamical system, we will not enter this discussion.

Instead, we pursue another direction and are interested in the qualitative behavior, in particular as $t \to \infty$, of $\mathrm{e}^{tA}$. Convergence, boundedness, and unboundedness as $t \to \infty$ are properties of $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$ with a natural interpretation in terms of dynamical systems.

In case we have an explicit formula for $\mathrm{e}^{tA}$, it may be a good idea to try to check these properties directly. However, these cases are rather rare, and therefore it is important to understand the influence of properties of the matrix $A$ on $\mathrm{e}^{tA}$ without explicitly calculating $\mathrm{e}^{tA}$. We give an example of this procedure here.

Let us call a continuous one-parameter semigroup $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$ *stable* if

$$\lim_{t\to\infty} \left\|\mathrm{e}^{tA}\right\| = 0,$$

where $\|\cdot\|$ stands for any matrix norm on $\mathrm{M}_n(\mathbb{C})$. Since uniform and pointwise convergence on $\mathrm{M}_n(\mathbb{C})$ coincide, stability can also be defined by the fact that

$$\lim_{t\to\infty} \left\|\mathrm{e}^{tA}x\right\| = 0$$

for each $x \in \mathbb{C}^n$. The classical *Liapunov stability theorem* [Lia92] now characterizes the stability of $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$ in terms of the location of the eigenvalues of $A$.

**2.10 Theorem.** (Liapunov 1892). *Let $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$ be the one-parameter semigroup generated by $A \in \mathrm{M}_n(\mathbb{C})$. Then the following assertions are equivalent.*

(a) *The semigroup is stable, i.e.,* $\lim_{t\to\infty} \left\|\mathrm{e}^{tA}\right\| = 0$.

(b) *All eigenvalues of $A$ have negative real part, i.e.,* $\operatorname{Re}\lambda < 0$ *for all* $\lambda \in \sigma(A)$.

Proof. The key to the following simple proof is the observation that stability remains invariant under similarity. Thus we can assume that $A$ has Jordan normal form. Then the semigroup $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$ is stable if and only if all semigroups $\left(\mathrm{e}^{tA_k}\right)_{t\geq 0}$ generated by the Jordan blocks $A_k$ of $A$ are stable. Due to the explicit calculation of $\mathrm{e}^{tA_k}$ in Example 2.5.(ii) we see immediately that this is the case if and only if the diagonal elements in the Jordan blocks have negative real part. However, these diagonal elements are exactly the eigenvalues of $A$. □

This theorem is of great theoretical and practical importance. Its main purpose here is to serve as a first sample for results relating properties of $A$ to properties of $\left(\mathrm{e}^{tA}\right)_{t\geq 0}$. Later (see Section V.1) we will devote great effort to find the appropriate generalizations of this theorem in the infinite-dimensional case.

Before closing this section we state another result that can be proved exactly as Theorem 2.10 was.

**2.11 Corollary.** *For the semigroup* $\big(e^{tA}\big)_{t\geq 0}$ *generated by the matrix* $A \in \mathrm{M}_n(\mathbb{C})$, *the following assertions are equivalent.*

(a) *The semigroup is bounded, i.e.,* $\|e^{tA}\| \leq M$ *for all* $t \geq 0$ *and some* $M \geq 1$.

(b) *All eigenvalues* $\lambda$ *of* $A$ *satisfy* $\operatorname{Re}\lambda \leq 0$, *and whenever* $\operatorname{Re}\lambda = 0$, *then* $\lambda$ *is a simple eigenvalue (i.e., the Jordan blocks corresponding to* $\lambda$ *have size* $1$*).*

**2.12 Exercises.** (1) If $A, B \in \mathrm{M}_n(\mathbb{C})$ commute, then $e^{A+B} = e^A e^B$.

(2) Let $A \in \mathrm{M}_n(\mathbb{C})$ be an $n \times n$ matrix and denote by $m_A$ its minimal polynomial. If $p$ is a polynomial such that $p \equiv \exp\,(\operatorname{mod} m_A)$, i.e., if the function ${}^{(p-\exp)}\!/_{m_A}$ can be analytically extended to $\mathbb{C}$, then $p(A) = \exp(A)$. Use this fact in order to verify formula (2.5).

(3) Use Corollary 2.11 to show that $A \in \mathrm{M}_n(\mathbb{C})$ generates a bounded group, i.e., $\|e^{tA}\| \leq M$ for all $t \in \mathbb{R}$ and some $M \geq 1$, if and only if $A$ is similar to a diagonal matrix with purely imaginary entries.

(4) Characterize semigroups $(e^{tA})_{t\geq 0}$ satisfying $e^A = I$ in terms of the eigenvalues of the matrix $A \in \mathrm{M}_n(\mathbb{C})$.

(5) A semigroup $(e^{tA})_{t\geq 0}$ for $A \in \mathrm{M}_n(\mathbb{C})$ is called *hyperbolic* if there exists a direct decomposition $\mathbb{C}^n = X_s \oplus X_u$ into $A$-invariant subspaces $X_s$ and $X_u$ and constants $M \geq 1$, $\varepsilon > 0$ such that

$$\big\|e^{tA}x\big\| \leq M e^{-\varepsilon t}\|x\| \qquad \text{for all } x \in X_s,\ t \geq 0,$$

and

$$\big\|e^{tA}y\big\| \geq \frac{1}{M} e^{\varepsilon t}\|y\| \qquad \text{for all } y \in X_u,\ t \geq 0.$$

Use the idea of the proof of Theorem 2.10 to show that the following properties are equivalent.

(a) The semigroup $(e^{tA})_{t\geq 0}$ is hyperbolic.

(b) The matrix $e^{tA}$ has no eigenvalue of modulus 1, i.e., $\sigma(e^{tA}) \cap \Gamma = \emptyset$ for some/all $t > 0$, where $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle in $\mathbb{C}$.

(c) The matrix $A$ has no purely imaginary eigenvalue, i.e., $\sigma(A) \cap i\mathbb{R} = \emptyset$.

(6) For $A \in \mathrm{M}_n(\mathbb{C})$, we call $\lambda \in \sigma(A) \cap \mathbb{R}$ a *dominant eigenvalue* if

$$\operatorname{Re}\mu < \lambda \qquad \text{for all } \mu \in \sigma(A) \setminus \{\lambda\}$$

and if the Jordan blocks corresponding to $\lambda$ are all $1 \times 1$. Show that the following properties are equivalent.

(a) The eigenvalue $0 \in \sigma(A)$ is dominant.

(b) There exist $P = P^2 \in \mathrm{M}_n(\mathbb{C})$ and $M \geq 1$, $\varepsilon > 0$ such that

$$\big\|e^{tA} - P\big\| \leq M e^{-\varepsilon t} \qquad \text{for all } t \geq 0.$$

## 3. Uniformly Continuous Operator Semigroups

With this section the level of technical prerequisites increases considerably. In fact, we now turn our attention to dynamical systems (or semigroups) on infinite-dimensional spaces. As a consequence, the reader has to be familiar with the basic theory of Banach spaces and bounded linear operators thereon. In particular, we will use various topologies on these spaces, like the norm and the weak topology or the uniform, strong, and weak operator topology (see Appendix A).

From now on, we take $X$ to be a complex Banach space with norm $\|\cdot\|$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators[6] on $X$ endowed with the operator norm, which again is denoted by $\|\cdot\|$. In analogy to Sections 1 and 2, we can restate Cauchy's question in this new context.

**3.1 Problem.** *Find all maps* $T(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X)$ *satisfying the functional equation*

$$\text{(FE)} \qquad \begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = I. \end{cases}$$

The search for answers to this question will be the main theme of this book, and due to the infinite-dimensional framework, the answers will be much more complex than what we encountered up to now. Fortunately, there are again simple "typical" examples of functions $T(\cdot)$ satisfying (FE). Before discussing these we introduce the terminology that we will adopt throughout the following.

As observed before Definition 2.4, for every function $T(\cdot) : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfying (FE) the set $\{T(t) : t \geq 0\}$ is a *commutative subsemigroup* of $(\mathcal{L}(X), \cdot)$. In addition, the map $t \mapsto T(t)$ is a homomorphism from $(\mathbb{R}_+, +)$ into $(\mathcal{L}(X), \cdot)$. This justifies calling the functional equation (FE) the *semigroup law* and using the following terminology.

**3.2 Definition.** *A family* $\big(T(t)\big)_{t\geq 0}$ *of bounded linear operators on a Banach space* $X$ *is called a (one-parameter) semigroup (or linear dynamical system) on* $X$ *if it satisfies the functional equation* (FE). *If* (FE) *holds even for all* $t, s \in \mathbb{R}$, *we call* $\big(T(t)\big)_{t\in\mathbb{R}}$ *a (one-parameter) group on* $X$.

**3.3 Remark.** The aspect that a semigroup $\big(T(t)\big)_{t\geq 0}$ can be viewed as a linear dynamical system should always be kept in mind and will dominate much of the further discussion. In particular, the interpretation (see Epilogue) of

- $t$ as "time,"
- the functional equation (FE) as the "law of determinism,"
- $\{T(t)x : t \in \mathbb{R}_+\}$ as the "orbit of the initial value $x$,"

[6] In the sequel "operator" always means "linear operator."

is fundamental and serves as a guiding principle for the development of the theory.

We now introduce the "typical" examples of one-parameter semigroups of operators on a Banach space $X$. Take any operator $A \in \mathcal{L}(X)$. As in the matrix case (see Definition 2.2), we can define an operator-valued exponential function by

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!},$$

where the convergence of this series takes place in the Banach algebra $\mathcal{L}(X)$. Using the same arguments as in Propositions 2.3 and 2.8, one shows that $\bigl(e^{tA}\bigr)_{t\geq 0}$ satisfies the functional equation (FE) and the differential equation (DE), and hence Theorem 3.7 below follows as in Section 2.

However, for readers already possessing a solid knowledge of spectral theory, we choose a different approach based on the functional calculus for bounded linear operators on Banach spaces (see [DS58, Sec. VII.3] or [TL80, Sec. V.8]). We briefly state the necessary notions.

For an operator $A \in \mathcal{L}(X)$, we denote by $\sigma(A)$ its *spectrum*, while $\rho(A) := \mathbb{C} \setminus \sigma(A)$ is the *resolvent set* of $A$. Since $\sigma(A)$ is a nonempty, compact subset of $\mathbb{C}$, $\rho(A)$ is open, and one can show that the *resolvent*

$$R(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{L}(X)$$

yields an analytic map from $\rho(A)$ into $\mathcal{L}(X)$ (see Section IV.1).

Consider now for each $t \geq 0$ the function $\lambda \mapsto e^{t\lambda}$, which is analytic on all of $\mathbb{C}$. Therefore, one can define (see [DS58, Def. VII.3.9] or [TL80, Sec. V.8, (8-3)]) the exponential of $A$ through the operator-valued version of Cauchy's integral formula.

**3.4 Definition.** *Let $A \in \mathcal{L}(X)$ and choose an open neighborhood $U$ of $\sigma(A)$ with smooth, positively oriented boundary $+\partial U$. Then we define*

$$e^{tA} := \frac{1}{2\pi i} \int_{+\partial U} e^{t\lambda} R(\lambda, A)\, d\lambda \tag{3.1}$$

*for each $t \geq 0$.*

It follows from the general theory that $e^{tA}$ is a bounded operator on $X$ and that its definition does not depend on the particular choice of $U$. Moreover, since the functional calculus yields a homomorphism from an algebra of (analytic) functions into the algebra $\mathcal{L}(X)$ (cf. [DS58, Thm. VII.3.10] or [TL80, Chap. V, Thm. 8.1]), we obtain from $e^{(t+s)\lambda} = e^{t\lambda}e^{s\lambda}$ for $t, s \geq 0$ the functional equation (FE) for $(e^{tA})_{t\geq 0}$. Similarly, the continuity of $t \mapsto e^{tA} \in \mathcal{L}(X)$ follows from the continuity of $t \mapsto e^{t\,\cdot}$ for the topology of uniform convergence on compact subsets of $\mathbb{C}$ (see [DS58, Lem. VII.3.13]). These arguments immediately yield (most of) the following assertions.

**3.5 Proposition.** *For $A \in \mathcal{L}(X)$ define $(\mathrm{e}^{tA})_{t\geq 0}$ by (3.1). Then the following properties hold.*

(i) $(\mathrm{e}^{tA})_{t\geq 0}$ *is a semigroup on $X$ such that*

$$\mathbb{R}_+ \ni t \mapsto \mathrm{e}^{tA} \in (\mathcal{L}(X), \|\cdot\|)$$

*is continuous.*

(ii) *The map $\mathbb{R}_+ \ni t \mapsto T(t) := \mathrm{e}^{tA} \in (\mathcal{L}(X), \|\cdot\|)$ is differentiable and satisfies the differential equation*

$$\text{(DE)} \qquad \begin{aligned} \tfrac{d}{dt} T(t) &= AT(t) \quad \text{for } t \geq 0, \\ T(0) &= I. \end{aligned}$$

*Conversely, every differentiable function $T(\cdot) : \mathbb{R}_+ \to (\mathcal{L}(X), \|\cdot\|)$ satisfying (DE) is already of the form $T(t) = \mathrm{e}^{tA}$ for some $A \in \mathcal{L}(X)$.*

*Finally, we observe that $A = \dot{T}(0)$.*

Proof. We only sketch the proof of (ii). The resolvent of $A$ satisfies

$$\lambda R(\lambda, A) = AR(\lambda, A) + I \quad \text{for all } \lambda \in \rho(A).$$

Therefore, we obtain by using Cauchy's integral theorem that

$$\begin{aligned} \frac{d}{dt}\mathrm{e}^{tA} &= \frac{d}{dt}\frac{1}{2\pi \mathrm{i}} \int_{+\partial U} \mathrm{e}^{t\lambda} R(\lambda, A)\, d\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{+\partial U} \lambda \mathrm{e}^{t\lambda} R(\lambda, A)\, d\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{+\partial U} \mathrm{e}^{t\lambda} AR(\lambda, A)\, d\lambda + \frac{1}{2\pi \mathrm{i}} \int_{+\partial U} \mathrm{e}^{t\lambda}\, d\lambda \\ &= A\mathrm{e}^{tA} \end{aligned}$$

for all $t \geq 0$. The uniqueness is again proved as for Proposition 1.2. □

The above properties of $\big(\mathrm{e}^{tA}\big)_{t\geq 0}$ for $A \in \mathcal{L}(X)$, proved using power series as in Section 2 or via the functional calculus, will permit a simple and satisfying answer to Problem 3.1. We will give it in terms of semigroups using the following terminology.

**3.6 Definition.** *A one-parameter semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called uniformly continuous (or norm continuous) if*

$$\mathbb{R}_+ \ni t \mapsto T(t) \in \mathcal{L}(X)$$

*is continuous with respect to the uniform operator topology on $\mathcal{L}(X)$.*

With this terminology, we can restate Proposition 3.5.(i) by saying that $\big(\mathrm{e}^{tA}\big)_{t\geq 0}$ is a uniformly continuous semigroup for any $A \in \mathcal{L}(X)$. The converse is also true.

**3.7 Theorem.** *Every uniformly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is of the form*

$$T(t) = \mathrm{e}^{tA}, \quad t \geq 0,$$

*for some bounded operator $A \in \mathcal{L}(X)$.*

PROOF. Since the following arguments were already used in the scalar and matrix-valued cases (see Sections 1 and 2), we think that a brief outline of the proof is sufficient.

For a uniformly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ the operators

$$V(t) := \int_0^t T(s)\, ds, \quad t \geq 0,$$

are well-defined, and ${}^1\!/_t V(t)$ converges (in norm!) to $T(0) = I$ as $t \downarrow 0$. Hence, for $t > 0$ sufficiently small, the operator $V(t)$ becomes invertible. Repeat now the computations from the proof of Theorem 1.4 in order to obtain that $t \mapsto T(t)$ is differentiable and satisfies (DE). Then Proposition 3.5 yields the assertion. □

Before adding some comments and further properties of uniformly continuous semigroups we recall that in finite dimensions the only "noncontinuous" semigroups were quite pathological (see Exercises 1.7.(1) and (2)). Therefore, the following question is natural and leads directly to the objects forming the main objects of this book.

**3.8 Problem.** *Do there exist "natural" one-parameter semigroups of linear operators on Banach spaces that are not uniformly continuous?*

**3.9 Comments.** (i) The operator $A$ in Theorem 3.7 is determined uniquely as the derivative of $T(\cdot)$ at zero, i.e., $A = \dot{T}(0)$. We call it the *generator* of $\big(T(t)\big)_{t\geq 0}$.

(ii) Since Definition 3.4 for $\mathrm{e}^{tA}$ works also for $t \in \mathbb{R}$ and even for $t \in \mathbb{C}$, it follows that each uniformly continuous semigroup can be extended to a uniformly continuous group $\big(\mathrm{e}^{tA}\big)_{t\in\mathbb{R}}$, or to $\big(\mathrm{e}^{tA}\big)_{t\in\mathbb{C}}$, respectively.

(iii) From the differentiability of $t \mapsto T(t)$ it follows that for each $x \in X$ the orbit map $\mathbb{R}_+ \ni t \mapsto T(t)x \in X$ is differentiable as well. Therefore, the map $x(t) := T(t)x$ is the unique solution of the $X$-valued initial value problem (or *abstract Cauchy problem*)

$$\text{(ACP)} \qquad \begin{cases} \dot{x}(t) = Ax(t) & \text{for } t \geq 0, \\ x(0) = x. \end{cases}$$

**3.10 Example.** Only in few cases it is possible to find the explicit form of $\mathrm{e}^{tA}$ for a given operator $A$. As one source of examples we refer to *multiplication operators* (see Sections 4.a and 4.b below). Here we study an operator given by an infinite matrix. On $X := \ell^p$, $1 \le p \le \infty$, take the (shift) operator given by the infinite matrix

$$A = (a_{ij}) \quad \text{with } a_{ij} = \begin{cases} 1 & \text{if } j = i+1, \quad i,j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ and $R(\lambda, A) = \sum_{k=0}^{\infty} \frac{A^k}{\lambda^{k+1}} = \big(r_{ij}(\lambda)\big)_{i,j\in\mathbb{N}}$ for $|\lambda| > 1$, where

$$r_{ij}(\lambda) = \begin{cases} \left(\frac{1}{\lambda}\right)^{j-i+1} & \text{if } j - i \ge 0, \\ 0 & \text{if } j - i < 0. \end{cases}$$

Computing the Cauchy integral from (3.1) (or the power series) one obtains

$$\mathrm{e}^{tA} = \big(e_{ij}(t)\big) \quad \text{with } e_{ij}(t) = \begin{cases} \frac{t^{j-i}}{(j-i)!} & \text{if } j - i \ge 0, \\ 0 & \text{if } j - i < 0 \end{cases}$$

for all $t \in \mathbb{C}$.

Such an explicit representation formula can be used to deduce properties of the semigroup $\big(\mathrm{e}^{tA}\big)_{t\ge0}$ generated by some operator $A$. Since, however, such formulas are seldom available, we pursue the idea that already successfully led to the Liapunov Stability Theorem 2.10 in the matrix case, i.e., we try to characterize (stability) properties of $\big(\mathrm{e}^{tA}\big)_{t\ge0}$ through (spectral) properties of $A$. Before doing so, we define and discuss the basic stability property in the Banach space setting.

**3.11 Definition.** *A semigroup $\big(T(t)\big)_{t\ge0}$ on a Banach space $X$ is called uniformly exponentially stable if there exist constants $\varepsilon > 0$, $M \ge 1$ such that*

$$\|T(t)\| \le M\mathrm{e}^{-\varepsilon t}$$

*for all $t \ge 0$.*

The following proposition contains a surprising characterization of stability by much weaker properties.

**3.12 Proposition.** *For a uniformly continuous semigroup $\big(T(t)\big)_{t\ge0}$, the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\ge0}$ is uniformly exponentially stable.*
(b) $\lim_{t\to\infty} \|T(t)\| = 0$.
(c) *There exists $t_0 > 0$ such that $\|T(t_0)\| < 1$.*
(d) *There exists $t_1 > 0$ such that $\mathrm{r}\big(T(t_1)\big) < 1$, where $\mathrm{r}\big(T(t)\big)$ denotes the spectral radius of $T(t)$.*

Proof. The implications (a) ⇒ (b) ⇒ (c) are trivial, while (c) implies (d), since the norm dominates the spectral radius. Moreover, (d) ⇒ (c) holds, since $\mathrm{r}\big(T(t_1)\big) = \lim_{k\to\infty} \|T(kt_1)\|^{1/k}$.

It remains to show (c) $\Rightarrow$ (a): To this end, we set $q := \|T(t_0)\| < 1$ and $M := \sup_{0 \le s \le t_0} \|T(s)\|$, which exists, since $t \mapsto \|T(t)\|$ is continuous. If we decompose $t = kt_0 + s \in \mathbb{R}_+$ with $s \in [0, t_0)$, we obtain

$$\begin{aligned} \|T(t)\| &\le \|T(s)\| \cdot \|T(kt_0)\| \le M \left\|T(t_0)^k\right\| \\ &\le Mq^k = M\mathrm{e}^{k \log q} \le \tfrac{M}{q}\, \mathrm{e}^{-\varepsilon t} \end{aligned}$$

with $\varepsilon := -{}^{\log q}/_{t_0} > 0$. □

In order to arrive at a Liapunov stability theorem for uniformly continuous semigroups analogous to Theorem 2.10 in the matrix case, we need the following lemma. For the proof we refer to [DS58, Thm. VII.3.11].

**3.13 Lemma.** (*Spectral Mapping Theorem*). *For every uniformly continuous semigroup* $\left(\mathrm{e}^{tA}\right)_{t \ge 0}$ *and its generator* $A$ *one has*

$$\sigma\left(\mathrm{e}^{tA}\right) = \mathrm{e}^{t\sigma(A)} := \left\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\right\} \tag{3.2}$$

*for all* $t \ge 0$.

With this spectral mapping theorem in hand, the characterization of uniform exponential stability becomes easy. Due to property (d) in Proposition 3.12, it suffices to show that $\sigma\left(\mathrm{e}^{tA}\right)$ is properly inside the unit circle, which means, by Lemma 3.13, that $\sigma(A)$ is contained in the open left halfplane.

**3.14 Theorem.** *For a uniformly continuous semigroup* $\left(T(t)\right)_{t \ge 0}$ *with generator* $A$*, the statements* (a)–(d) *in Proposition* 3.12 *are equivalent to*

(e) $\operatorname{Re} \lambda < 0$ *for all* $\lambda \in \sigma(A)$.

With this result we have achieved a theory for uniformly continuous semigroups that is completely parallel to the one for matrix semigroups. In particular, assuming uniform continuity, Cauchy's problem allows a precise answer. However, we will see in the subsequent sections that many new phenomena appear in the infinite-dimensional setting.

Before starting this new discussion, we want to add some comments on (semi) groups on Hilbert spaces. This may serve as a useful exercise for the beginner, but also reflects the historical process. In fact, it was in this context, and with applications to quantum mechanics in mind, that Stone and von Neumann (see [Sto29], [Sto30], [Sto32b], [Neu32a], [Neu32b]) gave the first precise definition of a one-parameter group of linear operators on infinite-dimensional spaces.

**3.15 Semigroups on Hilbert Spaces.** Let $H$ be a Hilbert space and for $T \in \mathcal{L}(H)$ denote by $T^*$ its Hilbert space adjoint, i.e., the unique operator satisfying $(Tx \,|\, y) = (x \,|\, T^*y)$ for all $x, y \in H$. Now take a uniformly continuous group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $H$ and denote its generator by $A$, i.e.,

$$T(t) = \mathrm{e}^{tA} \quad \text{for all } t \in \mathbb{R}.$$

Since $T \mapsto T^*$ is continuous on $\mathcal{L}(H)$, it follows that the adjoint group $\big(T(t)^*\big)_{t\in\mathbb{R}}$ is again uniformly continuous and that

$$T(t)^* = \mathrm{e}^{tA^*} \quad \text{for all } t \in \mathbb{R}.$$

The groups for which all operators $T(t)$ are unitary, i.e., satisfy $T(t)^{-1} = T(t)^*$ for all $t \in \mathbb{R}$, are particularly important and can be characterized as follows.

**Proposition.** *The group $\big(\mathrm{e}^{tA}\big)_{t\in\mathbb{R}}$ is unitary if and only if $A$ is skew-adjoint, i.e., $A^* = -A$.*

Proof. By definition, an operator $\mathrm{e}^{tA}$ is unitary if

$$\mathrm{e}^{tA^*} = \big(\mathrm{e}^{tA}\big)^* = \big(\mathrm{e}^{tA}\big)^{-1} = \mathrm{e}^{-tA} = \mathrm{e}^{t(-A)}.$$

Since a (semi) group always determines uniquely its generator (see Comment 3.9.(i)), we obtain $A^* = -A$. On the other hand, if $A$ is skew-adjoint, the two groups

$$\big(\mathrm{e}^{tA^*}\big)_{t\in\mathbb{R}} \quad \text{and} \quad \big(\mathrm{e}^{-tA}\big)_{t\in\mathbb{R}}$$

coincide. This implies

$$(\mathrm{e}^{tA})^{-1} = \mathrm{e}^{-tA} = \mathrm{e}^{tA^*} = (\mathrm{e}^{tA})^* \quad \text{for all } t \in \mathbb{R};$$

hence $\big(\mathrm{e}^{tA}\big)_{t\in\mathbb{R}}$ is unitary. □

It is one of the key ideas of quantum mechanics to use such unitary groups on a Hilbert space $H$ to "implement" new groups on the operator algebra $\mathcal{L}(H)$ (see [BR79, Sec. 3.2]). We briefly indicate this construction, but will need concepts from the theory of Banach algebras.

**3.16 Semigroups on Operator Algebras $\mathcal{L}(H)$.** We start from a unitary group $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ generated by a skew-adjoint operator $A \in \mathcal{L}(H)$. Then each $\mathrm{e}^{tA}$ defines an *implemented* operator $\mathfrak{U}(t)$ acting on the operator algebra $\mathcal{L}(H)$.

**Definition.** *The implemented operator $\mathfrak{U}(t) : \mathcal{L}(H) \to \mathcal{L}(H)$ is defined by*

$$\mathfrak{U}(t)T := \mathrm{e}^{tA} \cdot T \cdot \mathrm{e}^{tA^*} \tag{3.3}$$

*for each $T \in \mathcal{L}(H)$.*

*It is now simple to check that*

- *each $\mathfrak{U}(t)$ is a $^*$-automorphism on the Banach$^*$-algebra $\mathfrak{L}(H)$,*
- *$(\mathfrak{U}(t))_{t\in\mathbb{R}}$ is a one-parameter operator group on $\mathfrak{L}(H)$, and*
- *this group is uniformly continuous.*

By Theorem 3.7, there exists an operator $\mathfrak{G} : \mathfrak{L}(H) \to \mathfrak{L}(H)$ such that

$$\mathfrak{U}(t) = \mathrm{e}^{t\mathfrak{G}} \qquad \text{for all } t \in \mathbb{R}.$$

On the other hand, differentiation of the map

$$t \mapsto \mathrm{e}^{tA} \cdot T \cdot \mathrm{e}^{tA^*}$$

at $t = 0$ shows that

$$\mathfrak{G}(T) = A \cdot T - T \cdot A$$

for each $T \in \mathfrak{L}(H)$. We state this information in the following proposition.

**Proposition.** *The uniformly continuous group $(\mathfrak{U}(t))_{t\in\mathbb{R}}$ of $^*$-automorphisms on $\mathfrak{L}(H)$ implemented by the unitary group $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ has generator $\mathfrak{G}$ given by*

$$\mathfrak{G}(T) = A \cdot T - T \cdot A \qquad \text{for all } T \in \mathfrak{L}(H). \tag{3.4}$$

It is now a pleasant surprise that each uniformly continuous group of $^*$-automorphisms on $\mathfrak{L}(H)$ is of this form, i.e., it is implemented by a unitary group on $H$ yielding the generator as in (3.4).

To show this nontrivial result, we first characterize uniformly continuous groups consisting of $^*$-automorphisms by an algebraic property of their generators. We formulate this result in the context of Banach$^*$-algebras (see, e.g., [BD73, Chap. I, §12, Def. 8]).

**Lemma 1.** *Let $(\mathrm{e}^{t\mathfrak{D}})_{t\in\mathbb{R}}$ be a uniformly continuous group on a Banach$^*$-algebra $\mathfrak{A}$ with unit $e \in \mathfrak{A}$. The following assertions are equivalent.*

(a) *$(\mathrm{e}^{t\mathfrak{D}})_{t\in\mathbb{R}}$ is a group of $^*$-automorphisms.*

(b) *$\mathfrak{D}$ is a $^*$-derivation, i.e.,*

$$\mathfrak{D}(ab^*) = (\mathfrak{D}a)b^* + a(\mathfrak{D}b)^* \qquad \text{for all } a, b \in \mathfrak{A}. \tag{3.5}$$

PROOF. (a) $\Rightarrow$ (b). For $a, b \in \mathfrak{A}$ we consider the differentiable function

$$t \mapsto \xi_{a,b}(t) := \mathrm{e}^{t\mathfrak{D}}(ab^*).$$

Since each $\mathrm{e}^{t\mathfrak{D}}$ is a $^*$-automorphism, we obtain

$$\xi_{a,b}(t) = (\mathrm{e}^{t\mathfrak{D}}a) \cdot (\mathrm{e}^{t\mathfrak{D}}b)^*;$$

hence its derivative at $t = 0$ is

$$\mathfrak{D}(ab^*) = \dot{\xi}_{a,b}(0) = (\mathfrak{D}a)b^* + a(\mathfrak{D}b)^*,$$

and $\mathfrak{D}$ is a $^*$-derivation.

(b) ⇒ (a). Again for $a, b \in \mathfrak{A}$ and $0 \le s \le t$, we define

$$s \mapsto \eta_{a,b}(s) := e^{(t-s)\mathfrak{D}}\big((e^{s\mathfrak{D}}a)(e^{s\mathfrak{D}}b)^*\big).$$

This function is differentiable, and since $\mathfrak{D}$ is a $^*$-derivation, its derivative satisfies

$$\dot{\eta}_{a,b}(s) = 0 \qquad \text{for all } 0 \le s \le t.$$

This implies

$$e^{t\mathfrak{D}}(ab^*) = \eta_{a,b}(0) = \eta_{a,b}(t) = (e^{t\mathfrak{D}}a)(e^{t\mathfrak{D}}b)^*,$$

i.e., $e^{t\mathfrak{D}}$ is a $^*$-homomorphism. An analogous argument on the interval $[-t, 0]$ shows the same property for the inverse operator $e^{-t\mathfrak{D}}$; hence (a) holds. □

This lemma is remarkable not only for its assertions. In fact, the idea in the proof of the implication (b) ⇒ (a) will be used many times in order to obtain properties of the (semi) group from properties of its derivative at zero. Here it allows us to reduce the study of groups of $^*$-automorphisms to the study of $^*$-derivations.

On the particular Banach$^*$-algebra $\mathcal{L}(H)$, things are particularly nice. First, we observe that each skew-adjoint element $A \in \mathcal{L}(H)$ "implements" a $^*$-derivation by

$$\mathfrak{D}(T) := AT - TA, \quad T \in \mathcal{L}(H).$$

This can be checked directly or follows from the proposition and Lemma 1. We now show that also the converse is true, i.e., each $^*$-derivation is of this form.

**Lemma 2.** *Let $\mathfrak{D}$ be a bounded $^*$-derivation on $\mathfrak{A} = \mathcal{L}(H)$. Then there exists a skew-adjoint operator $A \in \mathcal{L}(H)$ such that*

$$\mathfrak{D}(T) = AT - TA \qquad \text{for all } T \in \mathcal{L}(H).$$

PROOF. For each pair of elements $x, y \in H$ we define the rank-one operator $x \otimes y$ by

$$z \mapsto x \otimes y(z) := (x \mid z)y.$$

Take now some fixed $z \in H$, $\|z\| = 1$, and define $A \in \mathcal{L}(H)$ by

$$Ay := \mathfrak{D}(z \otimes y)(z) \qquad \text{for all } y \in H.$$

For $T \in \mathcal{L}(H)$ we obtain

$$\begin{aligned}(AT - TA)(y) &= \mathfrak{D}(z \otimes Ty)(z) - T\big(\mathfrak{D}(z \otimes y)(z)\big) \\ &= \mathfrak{D}\big(T(z \otimes y)\big)(z) - \big(T \cdot \mathfrak{D}(z \otimes y)\big)(z) \\ &= \big(\mathfrak{D}T \cdot (z \otimes y)\big)(z) + \big(T \cdot \mathfrak{D}(z \otimes y)\big)(z) - \big(T \cdot \mathfrak{D}(z \otimes y)\big)(z) \\ &= \mathfrak{D}T(y) \qquad \text{for all } y \in H.\end{aligned}$$

In the final step we show that the operator $A \in \mathcal{L}(H)$ for which

$$\mathfrak{D}(T) = AT - TA \qquad \text{for all } T \in H$$

can be chosen as a skew-adjoint operator. In fact, since $\mathfrak{D}$ is a $^*$-derivation, we have

$$AT^* - T^*A = \mathfrak{D}(T^*) = \mathfrak{D}(T)^* = -A^*T^* + T^*A^*,$$

and hence

$$(A + A^*)T^* - T^*(A + A^*) = 0$$

for all $T \in \mathcal{L}(H)$. Therefore,

$$\widetilde{A} := A - \frac{A^* + A}{2} = \frac{A - A^*}{2}$$

is a skew-adjoint operator still satisfying

$$\widetilde{A}T - T\widetilde{A} = AT - TA \qquad \text{for all } T \in H.$$

□

If we now start from a uniformly continuous group of $^*$-automorphisms on $\mathcal{L}(H)$, we know from Lemma 1 that its generator $\mathfrak{G}$ is a $^*$-derivation and hence, by Lemma 2, is "implemented" by some skew-adjoint operator $A \in \mathcal{L}(H)$. On the other hand, this operator $A$ generates a unitary group $(e^{tA})_{t\in\mathbb{R}}$ that implements an automorphism group whose generator, by the proposition, coincides with $\mathfrak{G}$. Therefore, the original and the implemented automorphism groups coincide, and we can state the final result.

**Theorem.** *Let $H$ be a Hilbert space and take $(\mathfrak{U}(t))_{t\in\mathbb{R}}$ to be a uniformly continuous group on $\mathcal{L}(H)$. Then the following properties are equivalent.*

(a) *$(\mathfrak{U}(t))_{t\in\mathbb{R}}$ is a group of $^*$-automorphisms on the Banach$^*$-algebra $\mathcal{L}(H)$.*

(b) *There exists a skew-adjoint operator $A \in \mathcal{L}(H)$ and a unitary group $(e^{tA})_{t\in\mathbb{R}}$ on $H$ such that*

$$\mathfrak{U}(t)T = e^{tA} \cdot T \cdot e^{tA^*} \qquad \textit{for all } T \in \mathcal{L}(H).$$

**3.17 Exercises.** (1) On the Banach space $X := C_0(\mathbb{R})$ and for a fixed constant $\alpha > 0$, we define an operator $A_\alpha$ by the difference quotients

$$A_\alpha f(s) := {}^1\!/\!_\alpha\big(f(s+\alpha) - f(s)\big), \quad f \in X, s \in \mathbb{R}.$$

Show that $A_\alpha \in \mathcal{L}(X)$ with $\|A_\alpha\| = {}^2\!/\!_\alpha$, and hence one has the estimate

$$\left\|e^{tA_\alpha}\right\| \le e^{2t/\alpha} \qquad \text{for all } t \ge 0.$$

However, $e^{tA_\alpha}$ can be computed explicitly as

$$e^{tA_\alpha} f(s) = e^{-t/\alpha} \sum_{k=0}^{\infty} \frac{(t/\alpha)^k}{k!} f(s + k\alpha), \quad f \in X, s \in \mathbb{R},$$

and hence it satisfies

$$\left\|e^{tA_\alpha}\right\| = 1 \qquad \text{for all } t \ge 0.$$

What happens as $\alpha \downarrow 0$? (Hint: Use the results of Section III.4.)

(2) Which operators $T \in \mathcal{L}(X)$, $X$ a Banach space, can be embedded into a uniformly continuous semigroup, i.e., can we find $A \in \mathcal{L}(X)$ such that $T = e^A$? (Hint: Find (sufficient) conditions on $T$ such that $A := \log T$ can be defined in analogy to Definition 3.4.) Show that such operators $T$ are *infinitely divisible*, i.e., for each $n \in \mathbb{N}$ there exists $S \in \mathcal{L}(X)$ such that $S^n = T$.

(3) Show that for $A, B \in \mathcal{L}(X)$, $X$ a Banach space, the following assertions are equivalent.

(a) $AB = BA$.

(b) $e^{t(A+B)} = e^{tA} \cdot e^{tB}$ for all $t \in \mathbb{R}$.

(Hint: To show that (a) implies (b) proceed as in the proof of Lemma 2.6. For the converse implication, compute the second derivative of the functions appearing in (b).)

(4) As in Exercise 2.12.(5), we call a uniformly continuous semigroup $(e^{tA})_{t\geq 0}$ *hyperbolic* if there exist constants $\varepsilon > 0$, $M \geq 1$ and a direct decomposition $X = X_s \oplus X_u$ into $A$-invariant, closed subspaces $X_s$ and $X_u$ such that for all $t \geq 0$

$$\left\| e^{tA}x \right\| \leq M e^{-\varepsilon t} \|x\| \qquad \text{for all } x \in X_s,\ t \geq 0,$$

and

$$\left\| e^{tA}y \right\| \geq \tfrac{1}{M}\, e^{\varepsilon t} \|y\| \qquad \text{for all } y \in X_u,\ t \geq 0.$$

In a first step, observe that the restrictions of $(e^{tA})_{t\geq 0}$ to $X_s$ and of $(e^{-tA})_{t\geq 0}$ to $X_u$ are uniformly exponentially stable. Then use the spectral mapping theorem (Lemma 3.13) and Theorem 3.14 to prove the equivalence of the following assertions.

(a) $(e^{tA})_{t\geq 0}$ is hyperbolic.

(b) $\sigma(e^{tA}) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$ for one/all $t > 0$.

(c) $\sigma(A) \cap i\mathbb{R} = \emptyset$.

(5) The reader familiar with Banach algebras should reformulate Definition 3.6 and Theorem 3.7 by replacing the operator algebra $\mathcal{L}(X)$ by an arbitrary Banach algebra.

# 4. More Semigroups

In order to convince the reader that new phenomena appear for semigroups on infinite-dimensional Banach spaces, we first discuss several classes of one-parameter semigroups on concrete spaces. These semigroups will not be uniformly continuous and hence unlike those in Section 3, not of the form $\left(e^{tA}\right)_{t\geq 0}$ for some bounded operator $A$. On the other hand, they are not "pathological" in the sense of being completely unrelated to any analytic structure as, e.g., the semigroups mentioned in Comment 1.5.(iii) or in Exercises 1.7.(1) and (2). In addition, these semigroups will accompany us through the further development of the theory and provide a rich source of illuminating examples and counterexamples. The reader eager to pursue the general theory is recommended to skip this section and return to these examples only when they are necessary or useful.

## *a. Multiplication Semigroups on* $C_0(\Omega)$

*Multiplication operators* can be considered as an infinite-dimensional generalization of diagonal matrices. They are extremely simple to construct, and most of their properties are evident. Nevertheless, their value should not be underestimated. They appear, for example, naturally in the context of Fourier analysis or when one applies the spectral theorem for self-adjoint operators on Hilbert spaces (see [Con85, Chap. 10, Thm. 4.19], [Hal63], or [Wei80, Chap. 7, Thm. 7.33]). We therefore strongly recommend that any first attempt to illustrate a result or disprove a conjecture on semigroups should be made using multiplication semigroups.

We start by considering the Banach space (with sup-norm)

$$\mathrm{C}_0(\Omega) := \left\{ f \in \mathrm{C}(\Omega) : \begin{array}{l} \text{for all } \varepsilon > 0 \text{ there exists a compact } K_\varepsilon \subset \Omega \\ \text{such that } |f(s)| < \varepsilon \text{ for all } s \in \Omega \setminus K_\varepsilon \end{array} \right\}$$

of all continuous, complex-valued functions on some locally compact space $\Omega$ that vanish at infinity. As a typical example the reader might always take $\Omega$ to be a bounded or unbounded interval in $\mathbb{R}$. To any continuous function $q : \Omega \to \mathbb{C}$ we now associate a linear operator $M_q$ defined on its "maximal domain" $D(M_q)$ in $\mathrm{C}_0(\Omega)$.

**4.1 Definition.** *The multiplication operator $M_q$ induced on $\mathrm{C}_0(\Omega)$ by some continuous function $q : \Omega \to \mathbb{C}$ is defined by*

$$\begin{aligned} M_q f &:= q \cdot f \quad \textit{for all } f \textit{ in the domain} \\ D(M_q) &:= \left\{ f \in \mathrm{C}_0(\Omega) : q \cdot f \in \mathrm{C}_0(\Omega) \right\}. \end{aligned}$$

The main feature of these multiplication operators is that most operator-theoretic properties of $M_q$ can be characterized by analogous properties of the function $q$. In the following proposition we give some examples for this correspondence.

**4.2 Proposition.** *Let $M_q$ with domain $D(M_q)$ be the multiplication operator induced on $\mathrm{C}_0(\Omega)$ by some continuous function $q$. Then the following assertions hold.*

(i) *The operator $(M_q, D(M_q))$ is closed and densely defined.*

(ii) *The operator $M_q$ is bounded (with $D(M_q) = \mathrm{C}_0(\Omega)$) if and only if the function $q$ is bounded. In that case, one has*

$$\|M_q\| = \|q\| := \sup_{s \in \Omega} |q(s)|.$$

(iii) *The operator $M_q$ has a bounded inverse if and only if the function $q$ has a bounded inverse $1/q$, i.e., $0 \notin \overline{q(\Omega)}$. In that case, one has*

$$M_q^{-1} = M_{1/q}.$$

(iv) *The spectrum of $M_q$ is the closed range of $q$, i.e.,*

$$\sigma(M_q) = \overline{q(\Omega)}.$$

Proof. (i) The domain $D(M_q)$ always contains the space

$$\mathrm{C_c}(\Omega) := \left\{ f \in \mathrm{C}(\Omega) : \operatorname{supp} f \text{ is compact} \right\}$$

of all continuous functions having compact support

$$\operatorname{supp} f := \overline{\{s \in \Omega : f(s) \neq 0\}}.$$

In order to show that these functions form a dense subspace, we first observe that the one-point compactification of $\Omega$ is a normal topological space (cf. [Dug66,

Chap. XI, Thm. 8.4 and Thm. 1.2] or [Kel75, Chap. 5, Thm. 21 and Thm. 10]). Hence, by Urysohn's lemma (cf. [Dug66, Chap. VII, Thm. 4.1] or [Kel75, Chap. 4, Lem. 4]), for every compact subset $K \subseteq \Omega$ we can find a function $h_K \in \mathrm{C}(\Omega)$ still having compact support satisfying[7]

$$0 \le h_K \le \mathbb{1} \qquad \text{and} \qquad h_K(s) = 1 \text{ for all } s \in K.$$

Then, for each $f \in \mathrm{C}_0(\Omega)$, the function $f \cdot h_K$ has compact support, and

$$\begin{aligned}\|f - f \cdot h_K\| &= \sup_{s \in \Omega \setminus K} \big| f(s)\big(1 - h_K(s)\big)\big| \\ &\le 2 \sup_{s \in \Omega \setminus K} |f(s)|.\end{aligned}$$

This implies that the continuous functions with compact support are dense in $\mathrm{C}_0(\Omega)$; hence $M_q$ is densely defined.

To show the closedness of $M_q$, we take a sequence $(f_n)_{n \in \mathbb{N}} \subset D(M_q)$ converging to $f \in \mathrm{C}_0(\Omega)$ such that $\lim_{n\to\infty} q f_n =: g \in \mathrm{C}_0(\Omega)$ exists. Clearly, this implies $g = qf$ and hence $f \in D(M_q)$ and $M_q f = g$.

(ii) If $q$ is bounded, we have

$$\|M_q f\| = \sup_{s \in \Omega} |q(s) f(s)| \le \|q\| \cdot \|f\|$$

for any $f \in \mathrm{C}_0(\Omega)$; hence $M_q$ is bounded with $\|M_q\| \le \|q\|$. On the other hand, if $M_q$ is bounded, for every $s \in \Omega$ we choose, again using Urysohn's lemma, a continuous function $f_s$ with compact support satisfying $\|f_s\| = 1 = f_s(s)$. This implies

$$\|M_q\| \ge \|M_q f_s\| \ge |q(s) f_s(s)| = |q(s)| \qquad \text{for all } s \in \Omega;$$

hence $q$ is bounded with $\|M_q\| \ge \|q\|$.

(iii) If $0 \notin \overline{q(\Omega)}$, then $1/q$ is a bounded continuous function and $M_{1/q}$ is the bounded inverse of $M_q$. Conversely, assume $M_q$ to have a bounded inverse $M_q^{-1}$. Then we obtain

$$\|f\| \le \|M_q^{-1}\| \cdot \|M_q f\| \qquad \text{for all } f \in D(M_q),$$

whence

$$\delta := \frac{1}{\|M_q^{-1}\|} \le \sup_{s \in \Omega} |q(s) f(s)| \qquad \text{for all } f \in D(M_q),\ \|f\| = 1. \tag{4.1}$$

Now assume $\inf_{s \in \Omega} |q(s)| < \delta/2$. Then there exists an open set $\mathcal{O} \subset \Omega$ such that $|q(s)| < \delta/2$ for all $s \in \mathcal{O}$. On the other hand, by Urysohn's lemma we find a function $f_0 \in \mathrm{C}_0(\Omega)$ such that $\|f_0\| = 1$ and $f_0(s) = 0$ for all $s \in \Omega \setminus \mathcal{O}$. This implies $\sup_{s \in \Omega} |q(s) f_0(s)| \le \delta/2$, contradicting (4.1). Hence $0 < \delta/2 \le |q(s)|$ for all $s \in \Omega$, i.e., $M_{1/q}$ is bounded, and one easily verifies that it yields the inverse of the operator $M_q$.

[7] Here $\mathbb{1}$ denotes the constant function with $\mathbb{1}(s) = 1$ for all $s \in \Omega$.

(iv) By definition, one has $\lambda \in \sigma(M_q)$ if and only if $\lambda - M_q = M_{\lambda-q}$ is not invertible. Thus (iii) applied to the function $\lambda - q$ yields the assertion. $\square$

To any continuous function $q : \Omega \to \mathbb{C}$ we now associate the exponential function

$$\mathrm{e}^{tq} : s \mapsto \mathrm{e}^{tq(s)} \qquad \text{for } s \in \Omega,\ t \geq 0.$$

It is then immediate that the corresponding multiplication operators

$$T_q(t)f := \mathrm{e}^{tq} f, \quad f \in \mathrm{C}_0(\Omega),$$

formally satisfy the semigroup law (FE) from Problem 3.1. So, in order to obtain a one-parameter semigroup on $\mathrm{C}_0(\Omega)$, we have only to make sure that these multiplication operators $T_q(t)$ are bounded operators on $\mathrm{C}_0(\Omega)$. Using Proposition 4.2.(ii), we see that this is the case if and only if

$$\begin{aligned}\sup_{s\in\Omega} |\mathrm{e}^{tq(s)}| &= \sup_{s\in\Omega} \mathrm{e}^{t\,\mathrm{Re}\, q(s)} \\ &= \mathrm{e}^{t \sup_{s\in\Omega} \mathrm{Re}\, q(s)} < \infty.\end{aligned}$$

This observation leads to the following definition.

**4.3 Definition.** *Let $q : \Omega \to \mathbb{C}$ be a continuous function such that*

$$\sup_{s\in\Omega} \mathrm{Re}\, q(s) < \infty.$$

*Then the semigroup $(T_q(t))_{t\geq 0}$ defined by*

$$T_q(t)f := \mathrm{e}^{tq} f$$

*for $t \geq 0$ and $f \in \mathrm{C}_0(\Omega)$ is called the multiplication semigroup generated by the multiplication operator $M_q$ (or, the function $q$) on $\mathrm{C}_0(\Omega)$.*

By Proposition 3.5.(i) and Theorem 3.7 the semigroup $(T_q(t))_{t\geq 0}$ is uniformly continuous if and only if it is of the form $(\mathrm{e}^{tA})_{t\geq 0}$ for some bounded operator $A$. As predicted, this can already be read off from the function $q$.

**4.4 Proposition.** *The multiplication semigroup $(T_q(t))_{t\geq 0}$ generated by $q : \Omega \to \mathbb{C}$ is uniformly continuous if and only if $q$ is bounded.*

Proof. If $q$ and hence $M_q$ are bounded, it is easy to see that $T_q(t)$ coincides with the exponential $\mathrm{e}^{tM_q}$; hence is uniformly continuous by Proposition 3.5.(i).

Now let $q$ be unbounded and choose $(s_n)_{n\in\mathbb{N}} \subset \Omega$ such that $|q(s_n)| \to \infty$ for $n \to \infty$. Then we take $t_n := {}^1\!/_{|q(s_n)|} \to 0$. Since $\mathrm{e}^z \neq 1$ for all $|z| = 1$, there exists $\delta > 0$ such that

$$\left|1 - \mathrm{e}^{t_n q(s_n)}\right| \geq \delta$$

for all $n \in \mathbb{N}$. With functions $f_n \in \mathrm{C}_0(\Omega)$ satisfying $\|f_n\| = 1 = f_n(s_n)$, we finally obtain

$$\begin{aligned}\|T_q(0) - T_q(t_n)\| &\geq \left\|f_n - \mathrm{e}^{t_n q} f_n\right\| \\ &\geq \left|1 - \mathrm{e}^{t_n q(s_n)}\right| \geq \delta > 0\end{aligned}$$

for all $n \in \mathbb{N}$, i.e., $(T_q(t))_{t\geq 0}$ is not uniformly continuous. $\square$

This means that for every unbounded continuous function $q : \Omega \to \mathbb{C}$ satisfying

$$\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty,$$

we obtain a one-parameter semigroup that is *not* uniformly continuous, hence to which Theorem 3.7 does not apply. In order to prepare for later developments, we now show that these multiplication semigroups, while not being uniformly continuous in general, still enjoy a nice continuity property.

**4.5 Proposition.** *Let $(T_q(t))_{t\geq 0}$ be the multiplication semigroup generated by a continuous function $q : \Omega \to \mathbb{C}$ satisfying*

$$w := \sup_{s\in\Omega} \operatorname{Re} q(s) < \infty.$$

*Then the mappings*

$$\mathbb{R}_+ \ni t \mapsto T_q(t)f = \mathrm{e}^{tq} f \in \mathrm{C}_0(\Omega)$$

*are continuous for every $f \in \mathrm{C}_0(\Omega)$.*

Proof. Let $f \in \mathrm{C}_0(\Omega)$ with $\|f\| \leq 1$. For $\varepsilon > 0$ take a compact subset $K$ of $\Omega$ such that $|f(s)| \leq \varepsilon/(\mathrm{e}^{|w|}+1)$ for all $s \in \Omega \setminus K$. Since the exponential function is uniformly continuous on compact sets, there exists $t_0 \in (0, 1]$ such that

$$\left|\mathrm{e}^{tq(s)} - 1\right| \leq \varepsilon$$

for all $s \in K$ and $0 \leq t \leq t_0$. Hence, we obtain

$$\begin{aligned} \left\|\mathrm{e}^{tq} f - f\right\| &\leq \sup_{s\in K} \left(\left|\mathrm{e}^{tq(s)} - 1\right| \cdot |f(s)|\right) + \left(\mathrm{e}^{|w|} + 1\right) \cdot \sup_{s\in\Omega\setminus K} |f(s)| \\ &\leq 2\varepsilon \end{aligned}$$

for all $0 \leq t \leq t_0$. □

Finally, we show that each semigroup consisting of multiplication operators on $\mathrm{C}_0(\Omega)$ and satisfying the continuity property of Proposition 4.5 is a *multiplication semigroup* in the sense of Definition 4.3.

**4.6 Proposition.** *For $t \geq 0$, let $m_t : \Omega \to \mathbb{C}$ be bounded continuous functions and assume that*

(i) *the corresponding multiplication operators*

$$T(t)f := m_t \cdot f$$

*form a semigroup $(T(t))_{t\geq 0}$ of bounded operators on $\mathrm{C}_0(\Omega)$, and*

(ii) *the mappings*

$$\mathbb{R}_+ \ni t \mapsto T(t)f \in \mathrm{C}_0(\Omega)$$

*are continuous for every $f \in \mathrm{C}_0(\Omega)$.*

*Then there exists a continuous function $q : \Omega \to \mathbb{C}$ satisfying*

$$\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty$$

*such that $m_t(s) = \mathrm{e}^{tq(s)}$ for every $s \in \Omega$, $t \geq 0$.*

Proof. For fixed $s \in \Omega$ choose $f \in \mathrm{C}_0(\Omega)$ such that $f \equiv 1$ in some neighborhood of $s$. Then, by assumption (ii),

$$\mathbb{R}_+ \ni t \mapsto \big(T(t)f\big)(s) = m_t(s) \in \mathbb{C}$$

is a continuous function satisfying the functional equation (FE) from Problem 1.1. Therefore, by Theorem 1.4, there exists a unique $q(s) \in \mathbb{C}$ such that $m_t(s) = \mathrm{e}^{tq(s)}$ for all $t \geq 0$. Since the map $s \mapsto m_t(s)$ in a neighborhood of $s$ coincides with $s \mapsto (T(t)f)(s) \in \mathrm{C}_0(\Omega)$, the functions $\Omega \ni s \mapsto \mathrm{e}^{tq(s)} \in \mathbb{C}$ are continuous for all $t \geq 0$. In order to show that $q$ is continuous, we first observe that $q$ is bounded on compact subsets of $\Omega$. In fact, if $K \subset \Omega$ is compact, then $(T(t))_{t\geq 0}$ induces a uniformly continuous semigroup $(T_K(t))_{t\geq 0}$ on $\mathrm{C}(K)$ given by

$$\big(T_K(t)f\big)(s) = \mathrm{e}^{tq(s)}f(s), \qquad f \in \mathrm{C}(K),\ s \in K,$$

and the same arguments as in the second part of the proof of Proposition 4.4 show that $q$ is bounded on $K$. This implies that the convergence in

$$\lim_{t\downarrow 0} \frac{\mathrm{e}^{tq(s)} - 1}{t} = q(s)$$

is uniform on compact sets in $\Omega$. Since every point in $\Omega$ possesses a compact neighborhood, we conclude that $q$, being the uniform limit (on compact subsets) of the continuous functions $s \mapsto {(\mathrm{e}^{tq(s)}-1)}/{t}$, is continuous as well.

Finally, the multiplication operators $T(t)f = \mathrm{e}^{tq}\cdot f$ are supposed to be bounded; hence the real part of $q$ must be bounded from above. □

We conclude this section with some simple observations and concrete examples.

**4.7 Examples.** (i) On a compact space, every multiplication operator given by a continuous function is already bounded, and hence every multiplication semigroup is uniformly continuous.

(ii) We can choose $\Omega$ and $q$ in such a way that the closed range of $q$ is a given closed subset of $\mathbb{C}$. When $q$ generates a multiplication semigroup $(T_q(t))_{t\geq 0}$, this has obvious consequences for the operators $T_q(t)$: Choose any closed subset $\Omega$ of $\mathbb{C}$ and define

$$q(s) := s$$

for $s \in \Omega$. Then $\sigma(M_q) = \Omega$ and $\sigma\left(T_q(t)\right) = \overline{\mathrm{e}^{t\Omega}} := \overline{\{\mathrm{e}^{ts} : s \in \Omega\}}$ for all $t \geq 0$. In particular, if $\Omega \subseteq \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$ (or $\Omega \subseteq \mathrm{i}\mathbb{R}$), we conclude that $(T_q(t))_{t\geq 0}$ consists of contractions (or isometries, respectively) on $\mathrm{C}_0(\Omega)$.

(iii) For $\Omega := \mathbb{N}$ each complex sequence $(q_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ defines a multiplication operator

$$(x_n)_{n\in\mathbb{N}} \mapsto (q_n \cdot x_n)_{n\in\mathbb{N}}$$

on the space $\mathrm{C}_0(\Omega) = \mathrm{c}_0$. For $q_n := in$ we obtain a group of isometries

$$T(t)(x_n)_{n\in\mathbb{N}} = (\mathrm{e}^{int}x_n)_{n\in\mathbb{N}}, \quad t \in \mathbb{R},$$

and for $q_n := -n^2$ we obtain a semigroup of contractions

$$T(t)(x_n)_{n\in\mathbb{N}} = (\mathrm{e}^{-n^2t}x_n)_{n\in\mathbb{N}}, \quad t \geq 0.$$

(iv) This simple example serves just to explain the first sentence in this subsection. Take $\Omega = \{1, 2, \ldots, m\}$ to be a finite set. Then $\mathrm{C}_0(\Omega)$ is simply $\mathbb{C}^m$, and the multiplication operator $(x_n) \mapsto (q_n \cdot x_n)$ corresponds to the diagonal matrix $A = \operatorname{diag}(q_1, \ldots, q_m)$. The corresponding multiplication semigroup is given by $\mathrm{e}^{tA} = \operatorname{diag}(\mathrm{e}^{tq_1}, \ldots, \mathrm{e}^{tq_m})$ as in Example 2.5.(i).

**4.8 Exercises.** (1) For a sequence $q = (q_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ define the corresponding multiplication operator $M_q$ on $X := c_0$ or $X := \ell^p$, $1 \le p \le \infty$. Show that its point spectrum is given by $P\sigma(M_q) = \{q_n : n \in \mathbb{N}\}$ and that $\sigma(M_q) = \overline{P\sigma(M_q)}$.
(2) Many properties of the multiplication semigroup $(T_q(t))_{t\ge 0}$ generated by a multiplication operator $M_q$ on $X := \mathrm{C}_0(\Omega)$ can be characterized by properties of the function $q : \Omega \to \mathbb{C}$.

(i) $(T_q(t))_{t\ge 0}$ is bounded (contractive) if and only if

$$\operatorname{Re} q(s) \le 0 \qquad \text{for all } s \in \Omega.$$

(ii) $(T_q(t))_{t\ge 0}$ is uniformly exponentially stable (see Definition 3.11) if and only if

$$\operatorname{Re} q(s) \le -\varepsilon \qquad \text{for all } s \in \Omega \text{ and some } \varepsilon > 0.$$

(iii) $(T_q(t))_{t\ge 0}$ is hyperbolic (see Exercise 2.12.(5)) if and only if

$$|\operatorname{Re} q(s)| \ge \varepsilon \qquad \text{for all } s \in \Omega \text{ and some } \varepsilon > 0.$$

(iv) $(T_q(t))_{t\ge 0}$ is periodic with $T_q(2\pi) = I$ (see Paragraph 4.18 and Definition IV.2.23) if and only if

$$q(\Omega) \subseteq \mathrm{i}\mathbb{Z}.$$

(3) Take $X := \mathrm{C}_0(\mathbb{R})$ and $q(s) := \frac{-1}{1+|s|} + \mathrm{i}s$, $s \in \mathbb{R}$. Show that the corresponding multiplication semigroup $(T_q(t))_{t\ge 0}$ is not uniformly exponentially stable but satisfies

$$\lim_{t\to\infty} \|T_q(t)f\| = 0$$

for each $f \in X$.

## *b. Multiplication Semigroups on* $\mathrm{L}^p(\Omega, \mu)$

As mentioned at the beginning of the previous subsection, multiplication operators arise in a natural way in various instances. For example, if one applies the Fourier transform to a linear differential operator on $\mathrm{L}^2(\mathbb{R}^n)$, this operator becomes a multiplication operator on $\mathrm{L}^2(\mathbb{R}^n)$ (see Lemma VI.5.4.(ii)). Moreover, the classical "spectral theorem" asserts that each self-adjoint or, more generally, normal operator on a Hilbert space is (isomorphic to) a multiplication operator on some $\mathrm{L}^2$-space. This view point is emphasized in Halmos's article [Hal63] and motivates our systematic analysis of multiplication operators. We therefore formulate this version of the spectral theorem explicitly (see also [Con85, Chap. 10, Thm. 4.19] or [Wei80, Chap. 7, Thm. 7.33]).

**4.9 Spectral Theorem.** *If $A$ is a normal operator on a separable Hilbert space $H$, then there is a $\sigma$-finite measure space $(\Sigma, \Omega, \mu)$ and a measurable function $q : \Omega \to \mathbb{C}$ such that $A$ is unitarily equivalent to the multiplication operator $M_q$ on $\mathrm{L}^2(\Omega, \mu)$, i.e., there exists a unitary operator $U \in \mathcal{L}(H, \mathrm{L}^2(\Omega, \mu))$ such that the diagram*

$$\begin{array}{ccccc}
H & \supseteq & D(A) & \xrightarrow{\quad A \quad} & H \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} & & \uparrow{\scriptstyle U^* = U^{-1}} \\
\mathrm{L}^2(\Omega,\mu) & \supseteq & D(M_q) & \xrightarrow{\quad M_q \quad} & \mathrm{L}^2(\Omega,\mu)
\end{array}$$

*commutes.*

In order to define what we mean by a multiplication operator, we take some $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$; see, e.g., [Hal74, Chap. II] or [Rao87, Chap. 2]. Then, for fixed $1 \le p < \infty$, we consider the Banach space

$$X := \mathrm{L}^p(\Omega, \mu)$$

of all (equivalence classes of) $p$-integrable complex functions on $\Omega$ endowed with the $p$-norm

$$\|f\|_p := \left( \int_\Omega |f(s)|^p \, d\mu(s) \right)^{1/p}.$$

Next, for a measurable function

$$q : \Omega \to \mathbb{C},$$

we call the set

$$q_{\mathrm{ess}}(\Omega) := \Big\{ \lambda \in \mathbb{C} : \mu\big(\{ s \in \Omega : |q(s) - \lambda| < \varepsilon \}\big) \neq 0 \text{ for all } \varepsilon > 0 \Big\},$$

its *essential range* and define the associated *multiplication operator* $M_q$ by

$$\begin{aligned} M_q f &:= q \cdot f \quad \text{for all } f \text{ in the domain} \\ D(M_q) &:= \big\{ f \in \mathrm{L}^p(\Omega,\mu) : q \cdot f \in \mathrm{L}^p(\Omega,\mu) \big\}. \end{aligned} \tag{4.2}$$

In analogy to Proposition 4.2, we now have the following result.

**4.10 Proposition.** *Let $M_q$ with domain $D(M_q)$ be the multiplication operator induced on $\mathrm{L}^p(\Omega,\mu)$ by some measurable function $q$. Then the following assertions hold.*

(i) *The operator $(M_q, D(M_q))$ is closed and densely defined.*

(ii) *The operator $M_q$ is bounded (with $D(M_q) = \mathrm{L}^p(\Omega,\mu)$) if and only if the function $q$ is essentially bounded, i.e., the set $q_{\mathrm{ess}}(\Omega)$ is bounded in $\mathbb{C}$. In this case, one has*

$$\|M_q\| = \|q\|_\infty := \sup \big\{ |\lambda| : \lambda \in q_{\mathrm{ess}}(\Omega) \big\}.$$

(iii) *The operator $M_q$ has a bounded inverse if and only if $0 \notin q_{\mathrm{ess}}(\Omega)$. In that case, one has*

$$M_q^{-1} = M_r$$

*for $r : \Omega \to \mathbb{C}$ defined by*

$$r(s) := \begin{cases} 1/q(s) & \text{if } q(s) \neq 0, \\ 0 & \text{if } q(s) = 0. \end{cases}$$

(iv) *The spectrum of $M_q$ is the essential range of $q$, i.e.,*

$$\sigma(M_q) = q_{\mathrm{ess}}(\Omega).$$

The proof uses measure theory and is left as Exercise 4.13.(3).

Also, the other results of Section 4.a, after the appropriate changes, remain valid in the $\mathrm{L}^p$-case. For the convenience of the reader and due to their importance for the applications, we state them explicitly. The proofs, however, are left as Exercises 4.13.(4) and (5).

**4.11 Proposition.** *Let $(T_q(t))_{t\geq 0}$ be the multiplication semigroup generated by a measurable function $q : \Omega \to \mathbb{C}$ satisfying*

$$\operatorname*{ess\,sup}_{s\in\Omega} \operatorname{Re} q(s) := \sup_{\lambda\in q_{\mathrm{ess}}(\Omega)} \operatorname{Re}\lambda < \infty,$$

*i.e.,*

$$T_q(t)f := \mathrm{e}^{tq} f \qquad \text{for every } f \in \mathrm{L}^p(\Omega,\mu),\ t \geq 0.$$

*Then the mappings*

$$\mathbb{R}_+ \ni t \mapsto T_q(t)f = \mathrm{e}^{tq} f \in \mathrm{L}^p(\Omega,\mu)$$

*are continuous for every $f \in \mathrm{L}^p(\Omega,\mu)$. Moreover, the semigroup $(T_q(t))_{t\geq 0}$ is uniformly continuous if and only if $q$ is essentially bounded.*

**4.12 Proposition.** *For $t \geq 0$, let $m_t : \Omega \to \mathbb{C}$ be bounded measurable functions and assume that*

(i) *the corresponding multiplication operators*

$$T(t)f := m_t \cdot f$$

*form a semigroup $(T(t))_{t\geq 0}$ of bounded operators on $\mathrm{L}^p(\Omega,\mu)$, and*

(ii) *the mappings*

$$\mathbb{R}_+ \ni t \mapsto T(t)f \in \mathrm{L}^p(\Omega,\mu)$$

*are continuous for every $f \in \mathrm{L}^p(\Omega,\mu)$.*

*Then there exists a measurable function $q : \Omega \to \mathbb{C}$ satisfying*

$$\operatorname*{ess\,sup}_{s\in\Omega} \operatorname{Re} q(s) := \sup_{\lambda\in q_{\mathrm{ess}}(\Omega)} \operatorname{Re}\lambda < \infty$$

*such that $m_t = \mathrm{e}^{tq}$ almost everywhere for every $t \geq 0$.*

**4.13 Exercises.** (1) On the spaces $X := \mathrm{c}_0$ and $X := \ell^p$, $1 \leq p < \infty$, there exist multiplication semigroups $(T_q(t))_{t\geq 0}$ such that each $T_q(t)$, $t > 0$, is a compact operator. Construct concrete examples. Observe that this is not possible if

(i) the function spaces are $X := \mathrm{C}_0(\mathbb{R})$ or $X := \mathrm{L}^p(\mathbb{R})$, or if

(ii) the function $q$ is bounded.

(2) For multiplication semigroups $(T_q(t))_{t\geq 0}$ generated by a multiplication operator $M_q$ on a space $X := \mathrm{C}_0(\Omega)$ or $X := \mathrm{L}^p(\Omega,\mu)$, the *weak spectral mapping theorem*

$$\text{(WSMT)} \qquad \sigma\bigl(T_q(t)\bigr) = \overline{\mathrm{e}^{t\sigma(M_q)}} \qquad \text{for } t \geq 0$$

holds (see Proposition IV.3.13). Find counterexamples to the Spectral Mapping Theorem 3.13.

(3) Prove Proposition 4.10. (Hints: To prove that $M_q$ is closed, use the fact that every convergent sequence in $\mathrm{L}^p(\Omega,\mu)$ has a $\mu$-almost everywhere convergent subsequence; see, e.g., [Rud86, Chap. 3, Thm. 3.12]. In order to show that $M_q$ is densely defined, combine the fact that $\Omega$ is $\sigma$-finite with Lebesgue's convergence theorem (cf. [Rud86, Chap. 1, 1.34]). For the "only if" part of (ii), assume $q$ not to be essentially bounded and choose suitable characteristic functions to conclude that $M_q$ is unbounded. In the "only if" part of (iii), show first that $M_q^{-1}$ is given by a multiplication operator and then apply (ii).)

(4) Prove Proposition 4.11. (Hint: Use Lebesgue's convergence theorem.)

(5) Prove Proposition 4.12.

(6) For every measurable function $q : \Omega \to \mathbb{C}$ we can define the multiplication operator $M_q$ on $\mathrm{L}^\infty(\Omega,\mu)$ as we did for $\mathrm{L}^p(\Omega,\mu)$, $1 \le p < \infty$. Show that $M_q$ is densely defined if and only if $q$ is essentially bounded.

(7) Let $A := M_q$ be a multiplication operator on $\mathrm{L}^p(\Omega,\mu)$, $1 \le p < \infty$. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of $A$ if and only if $\mu(\{s \in \Omega : q(s) = \lambda\}) > 0$.

(8) A bounded linear operator $T : \mathrm{L}^p(\Omega,\mu) \to \mathrm{L}^p(\Omega,\mu)$, $1 \le p \le \infty$, is called *local* if for every measurable subset $S \subset \Omega$ one has $Tf = Tg$ a.e. on $S$ if $f = g$ a.e. on $S$. Show that every local operator is a multiplication operator $M_q$ for some $q \in \mathrm{L}^\infty(\Omega,\mu)$. Extend this characterization to unbounded multiplication operators.

## *c. Translation Semigroups*

The other important class of examples is obtained by "translating," to the left or to the right, complex-valued functions defined on (subsets of) $\mathbb{R}$. We first define these "translation operators" and only then specify the appropriate spaces.

**4.14 Definition.** *For a function $f : \mathbb{R} \to \mathbb{C}$ and $t \ge 0$, we call*

$$\big(T_l(t)f\big)(s) := f(s+t), \quad s \in \mathbb{R},$$

*the left translation (of $f$ by $t$), while*

$$\big(T_r(t)f\big)(s) := f(s-t), \quad s \in \mathbb{R},$$

*is the right translation (of $f$ by $t$).*

It is immediately clear that the operators $T_l(t)$ (and $T_r(t)$) satisfy the semigroup law (FE). We have only to choose appropriate function spaces to produce one-parameter operator semigroups. For that purpose, we start with spaces of continuous or integrable functions and the translation on *all* of $\mathbb{R}$.

**4.15 Translations on $\mathbb{R}$.** As Banach space $X$ we take one of the spaces

- $X_\infty := \mathrm{L}^\infty(\mathbb{R})$ of all bounded, measurable functions on $\mathbb{R}$,
- $X_{\mathrm{b}} := \mathrm{C_b}(\mathbb{R})$ of all bounded, continuous functions on $\mathbb{R}$,
- $X_{\mathrm{ub}} := \mathrm{C_{ub}}(\mathbb{R})$ of all bounded, uniformly continuous functions on $\mathbb{R}$,
- $X_0 := \mathrm{C_0}(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ vanishing at infinity,
- $X_{2\pi} := \mathrm{C}_{2\pi}(\mathbb{R})$ of all $2\pi$-periodic, continuous functions on $\mathbb{R}$,

all endowed with the sup-norm $\|\cdot\|_\infty$, or we take the spaces

- $X_p := \mathrm{L}^p(\mathbb{R})$, $1 \le p < \infty$, of all $p$-integrable functions on $\mathbb{R}$

endowed with the corresponding $p$-norm $\|\cdot\|_p$.

Then each left translation operator $T_l(t)$ is an isometry on each of these spaces, having as inverse the right translation operator $T_r(t)$. This means that $(T_l(t))_{t\in\mathbb{R}}$ and $(T_r(t))_{t\in\mathbb{R}}$ form one-parameter groups on $X$, called the (*left* or *right*) *translation group*.

For our purposes, the following continuity properties of these translation groups on the various function spaces are fundamental.

**Proposition.** *The translation group $(T_l(t))_{t\in\mathbb{R}}$ is not uniformly continuous on any of the above spaces, while*

$$\mathbb{R} \ni t \mapsto T_l(t)f \in \mathrm{L}^\infty(\mathbb{R})$$

*is continuous for the sup-norm only for $f \in X_{\mathrm{ub}}$. Finally,*

$$\mathbb{R} \ni t \mapsto \big(T_l(t)f\big)(s) \in \mathbb{C}$$

*is continuous for each $f \in X_{\mathrm{b}}$ and $s \in \mathbb{R}$.*

The proof is left as Exercise 4.19.(4), while the translation group on $\mathrm{L}^p(\mathbb{R})$ is discussed in Example 5.4.

We now modify the spaces on which translation takes place. As a first case, we consider functions defined on $\mathbb{R}_+$ only.

**4.16 Translations on $\mathbb{R}_+$.** In analogy to Paragraph 4.15, let $X$ denote one of the spaces

- $X_\infty := \mathrm{L}^\infty(\mathbb{R}_+)$ of all bounded, measurable functions on $\mathbb{R}_+$,
- $X_{\mathrm{b}} := \mathrm{C_b}(\mathbb{R}_+)$ of all bounded, continuous functions on $\mathbb{R}_+$,
- $X_{\mathrm{ub}} := \mathrm{C_{ub}}(\mathbb{R}_+)$ of all bounded, uniformly continuous functions on $\mathbb{R}_+$,
- $X_0 := \mathrm{C_0}(\mathbb{R}_+)$ of all continuous functions on $\mathbb{R}_+$ vanishing at infinity,
- $X_p := \mathrm{L}^p(\mathbb{R}_+)$, $1 \le p < \infty$, of all $p$-integrable functions on $\mathbb{R}_+$,

and observe that the left translations $T_l(t)$ are well-defined contractions on these spaces, but now yield a *semi*group only, called the *left translation semigroup* $(T_l(t))_{t\ge 0}$ on $\mathbb{R}_+$.

For the right translations $T_r(t)$, however, the value $(T_r(t)f)(s) = f(s-t)$ is not defined if $s-t<0$. To overcome this obstacle, we put

$$\big(T_r(t)f\big)(s) := \begin{cases} f(s-t) & \text{for } s-t \ge 0,\\ f(0) & \text{for } s-t < 0\end{cases}$$

for $f \in X = X_{\mathrm{b}}$, $X_{\mathrm{ub}}$, $X_0$, and

$$\big(T_r(t)f\big)(s) := \begin{cases} f(s-t) & \text{for } s-t \ge 0,\\ 0 & \text{for } s-t < 0\end{cases}$$

for $f \in X_p$. In this way, we again obtain semigroups of contractions on $X$ called the *right translation semigroups* $(T_r(t))_{t\ge 0}$ on $\mathbb{R}_+$. Clearly, the continuity properties stated in the proposition in 4.15 prevail. Moreover, it is not difficult to see that on $X_p$, for $1<p<\infty$, the semigroups $(T_l(t))_{t\ge 0}$ and $(T_r(t))_{t\ge 0}$ are adjoint to each other, i.e., $T_l(t)'$ on $X_p'$ coincides with $T_r(t)$ on $X_{p'}$ where $1/p + 1/p' = 1$.

Even on function spaces on finite intervals, we can define translation semigroups.

**4.17 Translations on finite intervals.** If we take the Banach space $\mathrm{C}[a,b]$ and look at the left translations, we have to specify the values $(T_l(t)f)(s)$ for $s+t>b$. Imitating the idea above, we put

$$\big(T_l(t)f\big)(s) := \begin{cases} f(s+t) & \text{for } s+t \le b, \\ f(b) & \text{for } s+t > b. \end{cases}$$

We note that this choice is not the only one to extend the translations to a semigroup on $\mathrm{C}[a,b]$ (see, e.g., Paragraph II.3.29). In any case, we still call $(T_l(t))_{t\ge 0}$ a *left translation semigroup* on $\mathrm{C}[a,b]$. By a similar definition, involving fixing the value at the left endpoint, we obtain a *right translation semigroup* $(T_r(t))_{t\ge 0}$ on the space $\mathrm{C}[a,b]$.

On the Banach spaces $\mathrm{L}^p[a,b]$, $1 \le p \le \infty$, we can modify this definition by taking

$$\big(T_l(t)f\big)(s) := \begin{cases} f(s+t) & \text{for } s+t \le b, \\ 0 & \text{for } s+t > b, \end{cases}$$

and again this yields a semigroup. However, now a completely new phenomenon appears: This semigroup, i.e., this "exponential function," vanishes for $t > b-a$.

**Proposition.** *The left translation semigroup* $(T_l(t))_{t\ge 0}$ *is nilpotent on* $\mathrm{L}^p[a,b]$*, that is,*

$$T_l(t) = 0$$

*for all* $t \ge b-a$*.*

**4.18 Rotations on the torus.** Take $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ and $X := \mathrm{C}(\Gamma)$. Then the operators $T(t)$, $t \in \mathbb{R}$, defined by

$$\big(T(t)f\big)(s) := f\big(\mathrm{e}^{\mathrm{i}t} \cdot s\big) \qquad \text{for } f \in \mathrm{C}(\Gamma) \text{ and } s \in \Gamma$$

form the so-called *rotation group*. It enjoys the same continuity properties as the translation group on $X_{\mathrm{ub}}$ in Paragraph 4.16. This can be seen by identifying $\mathrm{C}(\Gamma)$ with the Banach space $\mathrm{C}_{2\pi}(\mathbb{R}) \subset X_{\mathrm{ub}}$ of all $2\pi$-periodic continuous functions on $\mathbb{R}$. After this identification, the above rotation group becomes the translation group $(T_l(t))_{t\in\mathbb{R}}$ on $\mathrm{C}_{2\pi}(\mathbb{R})$ satisfying

$$T(2\pi) = I.$$

We will call such a group *periodic* (of period $2\pi$); see also Definition IV.2.23.

Since the operators $T(t)$ are isometries for the $p$-norm and since $\mathrm{C}(\Gamma)$ is dense in $\mathrm{L}^p(\Gamma,\mu)$, $1 \le p < \infty$, and $\mu$ the Lebesgue measure on $\Gamma$, the above definition can be extended to $f \in \mathrm{L}^p(\Gamma,\mu)$, and we obtain a periodic rotation group on each $\mathrm{L}^p$-space for $1 \le p < \infty$.

**4.19 Exercises.** (1) Show that the space $\mathrm{C}_{\mathrm{ub}}(\mathbb{R})$ of all bounded, uniformly continuous functions on $\mathbb{R}$ is the maximal subspace $X$ of $\mathrm{C}_{\mathrm{b}}(\mathbb{R})$ such that the orbits of the left translation group $(T_l(t))_{t\in\mathbb{R}}$, i.e., the mappings

$$\mathbb{R} \ni t \mapsto T_l(t)f \in \mathrm{C}_{\mathrm{b}}(\mathbb{R}),$$

become continuous for each $f \in X$.

(2) Show that in the context of Paragraphs 4.15 and 4.16 and on the corresponding $L^p$-spaces, the right translation semigroups are the adjoints of the left translation semigroups, i.e.,

$$T_l(t)' = T_r(t) \qquad \text{for } t \geq 0.$$

(3) Construct more (left) translation semigroups on $L^p[a,b]$ by defining $(T_l(t)f)(s)$ for $s+t > b$ in an appropriate way. For example, take $\alpha \in \mathbb{C}$ and put[8]

$$\big(T_l(t)f\big)(s) := \alpha^k f\big(s+t-k(b-a)\big) \quad \text{for } s+t-a \in [k(b-a),(k+1)(b-a)],\ k \in \mathbb{N}_0.$$

This semigroup becomes nilpotent for $\alpha = 0$, while it is periodic for $\alpha = 1$. For which $\alpha$ is this semigroup contractive?

(4) Prove the proposition in Paragraph 4.15.

(5) Define translation semigroups on the vector-valued function spaces $C_0(\mathbb{R}, X)$ or $L^p(\mathbb{R}, X)$ (see Appendix A and Appendix C.a) and show that continuity properties as in the scalar-valued case hold.

# 5. Strongly Continuous Semigroups

As we have seen by the previous examples, uniform continuity is too strong a requirement for many natural semigroups defined on concrete function spaces. Instead, "strong" continuity as in Proposition 4.5 and in the proposition in 4.15 holds in (most of) these examples. We take this as a motivation for a systematic investigation of such semigroups on abstract Banach spaces.

**5.1 Definition.** *A family $\big(T(t)\big)_{t\geq 0}$ of bounded linear operators on a Banach space $X$ is called a strongly continuous (one-parameter) semigroup (or $C_0$-semigroup[9]) if it satisfies the functional equation* (FE) *and is strongly continuous.*

For the convenience of the reader, we repeat this definition in a more explicit form. Hence, $\big(T(t)\big)_{t\geq 0}$ is a *strongly continuous semigroup* if the functional equation

$$\text{(FE)} \qquad \begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = I \end{cases}$$

holds and the *orbit maps*

$$\text{(SC)} \qquad \xi_x : t \mapsto \xi_x(t) := T(t)x$$

are continuous from $\mathbb{R}_+$ into $X$ for every $x \in X$.

[8] Here $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

[9] While we prefer the terminology "strongly continuous," we point out that the symbol $C_0$ abbreviates "Cesàro summable of order 0."

The property (SC) can also be expressed by saying that the map

$$t \mapsto T(t)$$

is continuous from $\mathbb{R}_+$ into the space $\mathcal{L}_s(X)$ of all bounded operators on $X$ endowed with the strong operator topology (see Appendix A, (A.3)).

Finally, if these properties hold for $\mathbb{R}$ instead of $\mathbb{R}_+$, we call $\big(T(t)\big)_{t\in\mathbb{R}}$ a *strongly continuous (one-parameter) group* (or *$C_0$-group*) on $X$.

These strongly continuous (semi) groups are the main objects in this book, and we are going to show how rich a theory and how many applications arise from the interplay of the functional equation (FE) and the requirement of strong continuity (SC).

## a. Basic Properties

Our first goal is to facilitate the verification of the strong continuity (SC) required in Definition 5.1. This is possible thanks to the uniform boundedness principle, which implies the following frequently used equivalence. (See also the more abstract version in Proposition A.3 and Exercise 5.9.(1).)

**5.2 Lemma.** *Let $X$ be a Banach space and let $F$ be a function from a compact set $K \subset \mathbb{R}$ into $\mathcal{L}(X)$. Then the following assertions are equivalent.*

(a) *$F$ is continuous for the strong operator topology, i.e., the mappings $K \ni t \mapsto F(t)x \in X$ are continuous for every $x \in X$.*

(b) *$F$ is uniformly bounded on $K$, and the mappings $K \ni t \mapsto F(t)x \in X$ are continuous for all $x$ in some dense subset $D$ of $X$.*

(c) *$F$ is continuous for the topology of uniform convergence on compact subsets of $X$, i.e., the map*

$$K \times C \ni (t,x) \mapsto F(t)x \in X$$

*is uniformly continuous for every compact set $C$ in $X$.*

Proof. The implication (c) $\Rightarrow$ (a) is trivial, while (a) $\Rightarrow$ (b) follows from the uniform boundedness principle, since the mappings $t \mapsto F(t)x$ are continuous, hence bounded, on the compact set $K$.

To show (b) $\Rightarrow$ (c), we assume $\|F(t)\| \le M$ for all $t \in K$ and fix some $\varepsilon > 0$ and a compact set $C$ in $X$. Then there exist finitely many $x_1, \dots, x_m \in D$ such that $C \subset \bigcup_{i=1}^m (x_i + {}^{\varepsilon}\!/_{M}\, U)$, where $U$ denotes the unit ball of $X$. Now choose $\delta > 0$ such that $\|F(t)x_i - F(s)x_i\| \le \varepsilon$ for all $i = 1, \dots, m$, and for all $t, s \in K$, such that $|t-s| \le \delta$. For arbitrary $x$, $y \in C$ and $t, s \in K$ with $\|x-y\| \le {}^{\varepsilon}\!/_{M}$ $|t-s| \le \delta$, this yields

$$\begin{aligned}\|F(t)x - F(s)y\| \le \|F(t)(x-x_j)\| + \big\|\big(F(t)-F(s)\big)x_j\big\| \\ + \|F(s)(x_j - x)\| + \|F(s)(x-y)\| \le 4\,\varepsilon,\end{aligned}$$

where we choose $j \in \{1, \dots, m\}$ such that $\|x - x_j\| \le {}^{\varepsilon}\!/_{M}$. This estimate proves that $(t,x) \mapsto F(t)x$ is uniformly continuous with respect to $t \in K$ and $x \in C$. □

As an easy consequence of this lemma, in combination with the functional equation (FE), we obtain that the continuity of the orbit maps

$$\xi_x : t \mapsto T(t)x$$

at each $t \geq 0$ and for each $x \in X$ is already implied by much weaker properties.

**5.3 Proposition.** *For a semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$, the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\geq 0}$ is strongly continuous.*

(b) *$\lim_{t\downarrow 0} T(t)x = x$ for all $x \in X$.*

(c) *There exist $\delta > 0$, $M \geq 1$, and a dense subset $D \subset X$ such that*

  (i) *$\|T(t)\| \leq M$ for all $t \in [0, \delta]$,*

  (ii) *$\lim_{t\downarrow 0} T(t)x = x$ for all $x \in D$.*

PROOF. The implication (a) ⇒ (c.ii) is trivial. In order to prove that (a) ⇒ (c.i), we assume, by contradiction, that there exists a sequence $(\delta_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ converging to zero such that $\|T(\delta_n)\| \to \infty$ as $n \to \infty$. Then, by the uniform boundedness principle, there exists $x \in X$ such that $\big(\|T(\delta_n)x\|\big)_{n\in\mathbb{N}}$ is unbounded, contradicting the fact that $T(\cdot)$ is continuous at $t = 0$.

In order to verify that (c) ⇒ (b), we put $K := \{t_n : n \in \mathbb{N}\} \cup \{0\}$ for an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \subset [0, \infty)$ converging to $t = 0$. Then $K \subset [0, \infty)$ is compact, $T(\cdot)_{|K}$ is bounded, and $T(\cdot)_{|K}x$ is continuous for all $x \in D$. Hence, we can apply Lemma 5.2.(b) to obtain

$$\lim_{n\to\infty} T(t_n)x = x$$

for all $x \in X$. Since $(t_n)_{n\in\mathbb{N}}$ was chosen arbitrarily, this proves (b).

To show that (b) ⇒ (a), let $t_0 > 0$ and let $x \in X$. Then

$$\lim_{h\downarrow 0} \|T(t_0 + h)x - T(t_0)x\| \leq \|T(t_0)\| \cdot \lim_{h\downarrow 0} \|T(h)x - x\| = 0,$$

which proves right continuity. If $h < 0$, the estimate

$$\|T(t_0 + h)x - T(t_0)x\| \leq \|T(t_0 + h)\| \cdot \|x - T(-h)x\|$$

implies left continuity whenever $\|T(t)\|$ remains uniformly bounded for $t \in [0, t_0]$. This, however, follows as above first for some small interval $[0, \delta]$ by the uniform boundedness principle and then on each compact interval using (FE). □

Since in many cases the uniform boundedness of the operators $T(t)$ for $t \in [0, t_0]$ is obvious, one obtains strong continuity by checking (right) continuity of the orbit maps $\xi_x$ at $t = 0$ for a dense set of "nice" elements $x \in X$ only.

We demonstrate the advantage of this procedure for the translation semigroup on $\mathrm{L}^p(\mathbb{R})$.

**5.4 Example.** The (left) translation group is strongly continuous on $\mathrm{L}^p(\mathbb{R})$ for all $1 \le p < \infty$.

PROOF. It is evident that each $T(t)$ is a contraction, so $\big(T(t)\big)_{t\ge 0}$ is uniformly bounded on $\mathbb{R}$. Now take a continuous function $f$ on $\mathbb{R}$ with compact support and observe that it is uniformly continuous. Therefore,

$$\lim_{t\downarrow 0} \|T(t)f - f\|_\infty = \lim_{t\downarrow 0} \sup_{s\in\mathbb{R}} \big|f(t+s) - f(s)\big| = 0,$$

and since the $p$-norm (for functions on bounded intervals) is weaker,

$$\lim_{t\downarrow 0} \|T(t)f - f\|_p = 0.$$

Since the continuous functions with compact support are dense in $\mathrm{L}^p(\mathbb{R})$ for all $1 \le p < \infty$, the assertion now follows from the adaptation of Proposition 5.3 to groups (see Exercise 5.9.(5)). □

We repeat that for a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ the finite orbits

$$\big\{T(t)x : t \in [0, t_0]\big\}$$

are continuous images of a compact interval, hence compact and therefore bounded for each $x \in X$. So by the uniform boundedness principle, each strongly continuous semigroup is uniformly bounded on each compact interval, a fact that implies exponential boundedness on $\mathbb{R}_+$.

**5.5 Proposition.** *For every strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$, there exist constants $w \in \mathbb{R}$ and $M \ge 1$ such that*

$$\|T(t)\| \le M\mathrm{e}^{wt} \tag{5.1}$$

*for all $t \ge 0$.*

PROOF. Choose $M \ge 1$ such that $\|T(s)\| \le M$ for all $0 \le s \le 1$ and write $t \ge 0$ as $t = s + n$ for $n \in \mathbb{N}$ and $0 \le s < 1$. Then

$$\begin{aligned} \|T(t)\| &\le \|T(s)\| \cdot \|T(1)\|^n \le M^{n+1} \\ &= M\mathrm{e}^{n \log M} \le M\mathrm{e}^{wt} \end{aligned}$$

holds for $w := \log M$ and each $t \ge 0$. □

The infimum of all exponents $w$ for which an estimate of the form (5.1) holds for a given strongly continuous semigroup will play an important role in the sequel. We therefore reserve a name for it.

**5.6 Definition.** *For a strongly continuous semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$, we call*

$$\omega_0 := \omega_0(\mathcal{T}) := \inf\left\{ w \in \mathbb{R} : \begin{array}{l} \text{there exists } M_w \geq 1 \text{ such that} \\ \|T(t)\| \leq M_w \mathrm{e}^{wt} \text{ for all } t \geq 0 \end{array} \right\}$$

*its growth bound (or type). Moreover, a semigroup is called bounded if we can take $w = 0$ in (5.1), and contractive if $w = 0$ and $M = 1$ is possible. Finally, the semigroup $\big(T(t)\big)_{t\geq 0}$ is called isometric if $\|T(t)x\| = \|x\|$ for all $t \geq 0$ and $x \in X$.*

In the following examples, we show that

- $\omega_0 = -\infty$ may occur,
- the infimum in (5.1) may not be attained, and
- constants $M > 1$ may be necessary.

**5.7 Examples.** (i) By Proposition 5.5, we always have $\omega_0 < \infty$, while $\omega_0 = -\infty$ holds for each nilpotent strongly continuous semigroup, e.g., the translation semigroup on $\mathrm{L}^1[0,1]$ from Paragraph 4.17.

(ii) For the semigroup defined by the matrices

$$T(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on $X := \mathbb{C}^2$ one has $\omega_0 = 0$, but $\lim_{t\to\infty} \|T(t)\| = \infty$, i.e., $\big(T(t)\big)_{t\geq 0}$ is not bounded.

(iii) Take $X := \mathrm{L}^1(\mathbb{R})$ and define a translation semigroup "with jump" by

$$\big(T(t)f\big)(s) := \begin{cases} 2f(s+t) & \text{if } s \in [-t, 0], \\ f(s+t) & \text{otherwise.} \end{cases}$$

Then $\big(T(t)\big)_{t\geq 0}$ is a strongly continuous semigroup with $\|T(t)\| = 2$ for each $t > 0$ (since $\left\|T(t)\mathbb{1}_{[0,t]}\right\| = 2 \left\|\mathbb{1}_{[0,t]}\right\|$). Hence $\big(T(t)\big)_{t\geq 0}$ is bounded, but no matter how big we choose $w$ in (5.1) it is not possible to take $M = 1$.

We close this subsection by showing that using the weak operator topology instead of the strong operator topology in Definition 5.1 will not change our class of semigroups.

This is a surprising result, and its proof needs more sophisticated tools from functional analysis than we have used up to this point.

**5.8 Theorem.** *A semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ is strongly continuous if and only if it is weakly continuous, i.e., if the mappings*

$$\mathbb{R}_+ \ni t \mapsto \big\langle T(t)x, x' \big\rangle \in \mathbb{C}$$

*are continuous for each $x \in X$, $x' \in X'$.*

Proof. We have only to show that weak implies strong continuity. As a first step, we use the principle of uniform boundedness twice to conclude that on compact intervals, $(T(t))_{t\geq 0}$ is pointwise and then uniformly bounded. Therefore (use Proposition 5.3.(c)), it suffices to show that

$$E := \left\{ x \in X : \lim_{t\downarrow 0} \|T(t)x - x\| = 0 \right\}$$

is a (strongly) dense subspace of $X$.

To this end, we define for $x \in X$ and $r > 0$ a linear form $x_r$ on $X'$ by

$$\left\langle x_r, x' \right\rangle := \frac{1}{r} \int_0^r \left\langle T(s)x, x' \right\rangle ds \qquad \text{for } x' \in X'.$$

Then $x_r$ is bounded and hence $x_r \in X''$. On the other hand, the set

$$\left\{ T(s)x : s \in [0, r] \right\}$$

is the continuous image of $[0, r]$ in the space $X$ endowed with the weak topology, hence is weakly compact in $X$. Kreĭn's theorem (see Proposition A.1.(ii)) implies that its closed convex hull

$$\overline{\mathrm{co}} \left\{ T(s)x : s \in [0, r] \right\}$$

is still weakly compact in $X$. Since $x_r$ is a $\sigma(X'', X')$-limit of Riemann sums, it follows that

$$x_r \in \overline{\mathrm{co}} \left\{ T(s)x : s \in [0, r] \right\},$$

whence $x_r \in X$. (See also [Rud73, Thm. 3.27].)

It is clear from the definition that the set

$$D := \left\{ x_r : r > 0, x \in X \right\}$$

is weakly dense in $X$. On the other hand, for $x_r \in D$ we obtain

$$\begin{aligned} \|T(t)x_r - x_r\| &= \sup_{\|x'\|\leq 1} \left| \frac{1}{r} \int_t^{t+r} \left\langle T(s)x, x' \right\rangle ds - \frac{1}{r} \int_0^r \left\langle T(s)x, x' \right\rangle ds \right| \\ &\leq \sup_{\|x'\|\leq 1} \left| \frac{1}{r} \int_r^{r+t} \left\langle T(s)x, x' \right\rangle ds \right| + \left| \frac{1}{r} \int_0^t \left\langle T(s)x, x' \right\rangle ds \right| \\ &\leq \frac{2t}{r} \|x\| \sup_{0\leq s\leq r+t} \|T(s)\| \to 0 \end{aligned}$$

as $t \downarrow 0$, i.e., $D \subset E$. We conclude that $E$ is weakly, hence by Proposition A.1.(i) strongly, dense in $X$. □

**5.9 Exercises.** (1) Let $X$ be a Banach space and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X)$. Then the following assertions are equivalent.

(a) $(T_n x)_{n\in\mathbb{N}}$ converges for all $x \in X$.

(b) $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ is bounded and $(T_n x)_{n\in\mathbb{N}}$ converges for all $x$ in some dense subset $D$ of $X$.

(c) $(T_n x)_{n\in\mathbb{N}}$ converges uniformly for all $x \in C$ and every compact set $C$ in $X$.

(2) Show that the left translation semigroup $(T_l(t))_{t\geq 0}$ is strongly continuous on the Banach space

$$C_0^1(\mathbb{R}_+) := \left\{ f \in C_0(\mathbb{R}_+) \cap C^1(\mathbb{R}_+) : \lim_{s\to\infty} f'(s) = 0 \right\}$$

endowed with the norm $\|f\| := \sup_{s\geq 0} |f(s)| + \sup_{s\geq 0} |f'(s)|$.

(3) Define

$$\big(T(t)f\big)(s) := f(se^t), \quad s, t \geq 0,$$

and show that $(T(t))_{t\geq 0}$ yields strongly continuous semigroups on $X_\infty := C_0[1,\infty)$ and $X_p := L^p[1,\infty)$ for $1 \leq p < \infty$. Show that their growth bounds satisfy $\omega_p = {}^{-1}/_p$. (Hint: See the proof of the proposition in IV.3.3.)

(4) There are semigroups $(T(t))_{t\geq 0}$ of bounded operators on a Banach space $X$ satisfying condition (ii), but not condition (i), in Proposition 5.3.(c). (Hint: On $C_0(\mathbb{R}_+)$ consider the multiplication operator $M_q$ for $q(s) := -s + is^2$ and the associated semigroup $(T_q(t))_{t\geq 0}$. Then $T(t) := \left(\begin{smallmatrix} T_q(t) & tM_qT_q(t) \\ 0 & T_q(t) \end{smallmatrix}\right)$ defines a semigroup $(T(t))_{t\geq 0}$ on $X := C_0(\mathbb{R}_+) \times C_0(\mathbb{R}_+)$ with the desired property.)

(5) For a group $(T(t))_{t\in\mathbb{R}}$ on a Banach space $X$, the following conditions are equivalent.

(a) The group $(T(t))_{t\geq 0}$ is strongly continuous, i.e., the map $\mathbb{R} \ni t \mapsto T(t)x \in X$ is continuous for all $x \in X$.

(b) $\lim_{t\to t_0} T(t)x = T(t_0)x$ for some $t_0 \in \mathbb{R}$ and all $x \in X$.

(c) There exist constants $t_0 \in \mathbb{R}$, $\delta > 0$, $M \geq 0$ and a dense subset $D \subset X$ such that

(i) $\|T(t)\| \leq M$ for all $t \in [t_0, t_0 + \delta]$,

(ii) $\lim_{t\downarrow t_0} T(t)x = T(t_0)x$ for all $x \in D$.

(6) If the strongly continuous semigroup $(T(t))_{t\geq 0}$ contains an invertible operator $T(t_0)$ for some $t_0 > 0$, then the semigroup can be extended to a strongly continuous group $(T(t))_{t\in\mathbb{R}}$.

(7) On $X := C[0,1]$, define bounded operators $T(t)$, $t > 0$, by

$$\big(T(t)f\big)(s) := \begin{cases} e^{t\log s}[f(s) - f(0)\log s] & \text{if } s \in (0,1], \\ 0 & \text{if } s = 0 \end{cases}$$

for $f \in X$ and put $T(0) := I$. Prove the following assertions.

(i) $(T(t))_{t\geq 0}$ is a semigroup that is strongly continuous only on $(0,\infty)$.

(ii) $\lim_{t\downarrow 0} \|T(t)\| = \infty$.

(8) Construct a strongly continuous semigroup that is not nilpotent, but has growth bound $\omega_0 = -\infty$. (Hint: Take $(T(t)f)(s) := e^{-t^2+2st} f(s-t)$ on $C_0(-\infty, 0]$.)

## b. Standard Constructions

We now explain how one can construct in various ways new strongly continuous semigroups from a given one. This might seem trivial and/or boring, but there will be many occasions to appreciate the toolbox provided by these considerations. Clearly, this subsection might be skipped by the impatient reader. However, these constructions might give the beginner the necessary exercise to familiarize him- or herself with the concept of a strongly continuous semigroup.

In the following, we always assume that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on a Banach space $X$.

**5.10 Similar Semigroups.** Given another Banach space $Y$ and an isomorphism $V$ from $Y$ onto $X$, we obtain (as in Lemma 2.6) a new strongly continuous semigroup $(S(t))_{t\geq 0}$ on $Y$, called *similar* to $(T(t))_{t\geq 0}$, by defining

$$S(t) := V^{-1}T(t)V \qquad \text{for } t \geq 0.$$

Without explicit reference to the isomorphism $V$, we call the two semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ *isomorphic.*

**5.11 Rescaled Semigroups.** For any numbers $\mu \in \mathbb{C}$ and $\alpha > 0$, we define the *rescaled semigroup* $(S(t))_{t\geq 0}$ by

$$S(t) := \mathrm{e}^{\mu t}T(\alpha t)$$

for $t \geq 0$.

For example, taking $\mu = -\omega_0$ (or $\mu < -\omega_0$) and $\alpha = 1$ the rescaled semigroup will have growth bound equal to (or less than) zero. This is an assumption we will make without loss of generality in many situations.

**5.12 Subspace Semigroups.** If $Y$ is a closed subspace of $X$ such that $T(t)Y \subseteq Y$ for all $t \geq 0$, i.e., if $Y$ is $(T(t))_{t\geq 0}$-*invariant*, then the restrictions

$$T(t)_| := T(t)_{|Y}$$

form a strongly continuous semigroup $(T(t)_|)_{t\geq 0}$, called the *subspace semigroup*, on the Banach space $Y$.

**5.13 Quotient Semigroups.** For a closed $(T(t))_{t\geq 0}$-invariant subspace $Y$ of $X$, we consider the quotient space $X_/ := {}^X\!/_Y$ with canonical quotient map $q : X \to X_/$. The quotient operators $T(t)_/$ given by

$$T(t)_/ q(x) := q\big(T(t)x\big) \qquad \text{for } x \in X \text{ and } t \geq 0$$

are well-defined and form a strongly continuous semigroup, called the *quotient semigroup* $(T(t)_/)_{t\geq 0}$ on the Banach space $X_/$.

**5.14 Adjoint Semigroups.** The *adjoint semigroup* $(T(t)')_{t\geq 0}$ consisting of all adjoint operators $T(t)'$ on the dual space $X'$ is, in general, *not* strongly continuous.

An example is provided by the (left) translation group on $\mathrm{L}^1(\mathbb{R})$. Its adjoint is the (right) translation group on $\mathrm{L}^\infty(\mathbb{R})$, which is not strongly continuous (see the proposition in 4.15). However, it is easy to see that $(T(t)')_{t\geq 0}$ is always *weak**-*continuous* in the sense that the maps

$$t \mapsto \xi_{x,x'}(t) := \big\langle x, T(t)'x' \big\rangle = \big\langle T(t)x, x' \big\rangle$$

are continuous for all $x \in X$ and $x' \in X'$.

Since on a reflexive Banach space weak and weak$^*$ topology coincide, in this case the adjoint semigroup is weakly, and hence by Theorem 5.8 strongly, continuous.

**Proposition.** *The adjoint semigroup of a strongly continuous semigroup on a reflexive Banach space is again strongly continuous.*

**5.15 Product Semigroups.** Let $(S(t))_{t\geq 0}$ be another strongly continuous semigroup commuting with $(T(t))_{t\geq 0}$, i.e., $S(t)T(t) = T(t)S(t)$ for all $t \geq 0$. Then the operators

$$U(t) := S(t)T(t)$$

form a strongly continuous semigroup $(U(t))_{t\geq 0}$, called the *product semigroup* of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$.

PROOF. Clearly, $U(0) = I$. In order to show the semigroup property for $(U(t))_{t\geq 0}$, we first show that $T(s)$ and $S(r)$ commute for all $s, r \geq 0$. To this end, we first take $r = {p_1}/{q}$ and $s = {p_2}/{q} \in \mathbb{Q}_+$. Then

$$\begin{aligned} S(r)T(s) &= S\left({1}/{q}\right)^{p_1} \cdot T\left({1}/{q}\right)^{p_2} \\ &= T\left({1}/{q}\right)^{p_2} \cdot S\left({1}/{q}\right)^{p_1} = T(s)S(r), \end{aligned}$$

i.e., $F(r, s) = G(r, s)$ for all $r, s \in \mathbb{Q}_+$, where

$$F : [0, \infty) \times [0, \infty) \to \mathcal{L}(X), \qquad F(r, s) := S(r)T(s),$$

and

$$G : [0, \infty) \times [0, \infty) \to \mathcal{L}(X), \qquad G(r, s) := T(s)S(r).$$

Now, for fixed $x \in X$, the functions $F(\cdot,\cdot)x$ and $G(\cdot,\cdot)x$ are continuous in each coordinate and coincide on $\mathbb{Q}_+ \times \mathbb{Q}_+$; hence we conclude that $F = G$. This shows that

$$S(r)T(s) = T(s)S(r)$$

for all $s, r \geq 0$, and the semigroup property $U(r + s) = U(r)U(s)$ for $s, r \geq 0$ follows immediately. Finally, the strong continuity of $(U(t))_{t\geq 0}$ follows from Lemma B.15. □

**5.16 Exercises.** (1) Let $(T_l(t))_{t\geq 0}$ be the left translation semigroup on $\mathrm{C}_0(\mathbb{R}_+)$ and take a nonvanishing, continuous function $q$ on $\mathbb{R}_+$ such that $q$ and ${1}/{q}$ are bounded. The multiplication operator $M_q$ yields a similarity transformation. Determine the semigroup $(S(t))_{t\geq 0}$ defined by

$$S(t) := M_q T_l(t) M_{{1}/{q}}, \quad t \geq 0.$$

(2) On $X := \mathrm{C}_0(\mathbb{R}^2)$, consider the strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ defined by

$$\big(S(t)f\big)(x, y) := f(x + t, y) \qquad \text{and} \qquad \big(T(t)f\big)(x, y) := f(x, y + t)$$

for $f \in X$, $t \geq 0$. What is their product semigroup?

(3) In this exercise we introduce another, more sophisticated, "standard construction."

Let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space $X$. Consider the new Banach space $\ell^\infty(X)$ of all bounded $X$-valued sequences and its closed subspace $c_0(X)$ of all null sequences in $X$. Finally, consider the quotient space

$$\widehat{X} := \ell^\infty(X) \big/ c_0(X)$$

and the semigroup thereon defined by

$$\widehat{T}(t)\big((x_n)_{n\in\mathbb{N}} + c_0(X)\big) := \big(T(t)x_n\big)_{n\in\mathbb{N}} + c_0(X).$$

(i) Show that this semigroup is strongly continuous if and only if $(T(t))_{t\geq 0}$ is uniformly continuous.

(ii) Replace $\ell^\infty(X)$ by the closed subspace

$$\ell^\infty_{\mathcal{T}}(X) := \Big\{(x_n)_{n\in\mathbb{N}} \in \ell^\infty(X) : \lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} \|T(t)x_n - x_n\| = 0\Big\}$$

and show that now the quotient semigroup in

$$\widehat{X}_{\mathcal{T}} := \ell^\infty_{\mathcal{T}}(X) \big/ c_0(X)$$

becomes strongly continuous.

(iii) Modify this construction by replacing the Fréchet filter by some free ultrafilter on $\mathbb{N}$ (Hint: See [Nag86, A-I, Sec. 3.6].)

For a continuation see Exercises II.2.8.(3) and IV.2.22.(5).

(4) Consider the function space

$$Y := \big\{f : [0,1] \to \mathbb{C} : |f(s)| \leq ns \text{ for all } s \in [0,1] \text{ and some } n \in \mathbb{N}\big\},$$

which becomes a Banach space for the norm

$$\|f\| := \inf\big\{c \geq 0 : |f(s)| \leq cs \text{ for all } s \in [0,1]\big\}.$$

On $X := \mathbb{C} \oplus Y$, we define a "translation" semigroup $(T(t))_{t\geq 0}$ by $T(0) := I$ and

$$T(t)\tbinom{\alpha}{f} := \tbinom{0}{g} \qquad \text{for } t > 0,$$

where

$$g(s) := \begin{cases} 0 & \text{for } s < t, \\ \alpha & \text{for } s = t, \\ f(s-t) & \text{for } s > t. \end{cases}$$

(i) Show that $\|T(t)\| = t^{-1}$ for $t > 0$, and hence $(T(t))_{t\geq 0}$ is not exponentially bounded.

(ii) Find the largest $(T(t))_{t\geq 0}$-invariant closed subspace of $X$ on which the restriction of $(T(t))_{t\geq 0}$ becomes strongly continuous for $t > 0$ ($t \geq 0$, respectively).

# Notes to Chapter I

The material treated in this chapter is quite standard; hence we make only some brief historical comments or give references for further reading.

**Section 1.** Functional equations more complicated than Cauchy's equation (FE) are still an area of active research. We refer to the book of Aczél [Acz66] or to his survey article [Acz89]. The historical ties with semigroup theory are indicated in [Hof92] and [Law96].

**Section 2.** The matrix-valued exponential function as the "fundamental solution" of a system of linear differential equations with constant coefficients and its Liapunov stability appears in every text on ordinary differential equations (e.g., [CL55], [Har64], [Bra75], [Ama90]), or in many books on matrix analysis (see [Gan59], [Gan90], [HJ91, Chap. 2], [Hor91]). For numerical computations as well as explicit formulas for the exponential of a matrix, we refer to [ML78], [BS93], and [CY97].

**Section 3.** The first concrete definition of an exponential function with values in an infinite-dimensional space can be found in a rather unknown paper by Maria Gramegna [Gra10], a student of G. Peano. Using the new concepts of functional analysis, Nagumo [Nag36] characterized uniformly continuous one-parameter groups with values in a Banach algebra. A nice and more recent treatment can be found in the booklet [Sin82].

For the functional calculus used in Definition 3.4, we refer to [DS58], [TL80], or any other book on spectral theory.

The implementation of one-parameter groups on the operator algebra $\mathcal{L}(H)$ indicated in Paragraph 3.16 is an important construction in quantum mechanics. The proofs presented are due to M. Mathieu. We refer to [BR79] for further reading.

**Section 4.** Multiplication operators (and semigroups) occur frequently in applications via the spectral theorem for normal operators on Hilbert spaces (see Halmos's article [Hal63]), or via the Fourier transform of differential operators. On Banach lattices such as $\mathrm{L}^p(\Omega,\mu)$- or $\mathrm{C}_0(\Omega)$-spaces, they can be characterized abstractly as *local* or as *central* operators (see Exercise 4.13.(8), [Nag86, C-I.9] and [Nag86, C-II.5.15]).

Translation semigroups are special cases of semigroups induced by a continuous or measure-preserving (semi)flow or dynamical system (see Paragraph II.3.28). This aspect is explained in [Nag91] and [Ves96b], and treated extensively in [LM94]. For an abstract characterization of such semigroups see [Nag86, B-II.3.13].

**Section 5.** The still unsurpassed classic on one-parameter semigroups is E. Hille's *Functional Analysis and Semigroups* [Hil48], with its second edition coauthored by R.S. Phillips. After that, the most widely diffused references were [DS58, Chap. XIII], [Yos65, Chap. IX] and [Kre71]. The series of books entirely devoted to semigroups started with [BM79], [Dav80], [Paz83], and then [Gol85], which contains an exhaustive list of books and papers on semigroup theory up to 1985. Other and more recent books on the general theory of semigroups are, e.g., [Cas85], [McB87], [CHA$^+$87], [Ves96a]. The literature on special types of semigroups or on abstract evolution equations will be mentioned in the notes of the relevant sections.

# Chapter II

# Semigroups, Generators, and Resolvents

In this chapter it is our aim to achieve what we obtained, without too much effort, for uniformly continuous semigroups in Section I.3. There, we characterized every uniformly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ as an operator-valued exponential function, i.e., we found an operator $A \in \mathcal{L}(X)$ such that

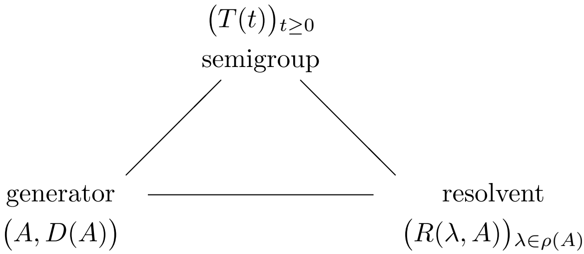
$$T(t) = \mathrm{e}^{tA}$$

for all $t \geq 0$ (see Theorem I.3.7). For *strongly continuous semigroups*, we will succeed in defining an analogue of $A$, called the *generator* of the semigroup. It will be a linear, but generally unbounded, operator defined only on a dense subspace $D(A)$ of the Banach space $X$. In order to retrieve the semigroup $\big(T(t)\big)_{t\geq 0}$ from its generator $\big(A, D(A)\big)$, we will need a third object, namely the *resolvent operator*

$$R(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{L}(X)$$

of $A$, which is defined for all complex numbers in the *resolvent set* $\rho(A)$ (see Definition IV.1.1).

To find and discuss the various relations between these objects is the theme of this chapter, which can be illustrated by the following triangle.




## 1. Generators of Semigroups and Their Resolvents

We recall that for a one-parameter semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ *uniform* continuity implies differentiability of the map $t \mapsto T(t) \in \mathcal{L}(X)$. The right derivative of $T(\cdot)$ at $t = 0$ then yields a bounded operator $A$ with $T(t) = \mathrm{e}^{tA}$ for all $t \geq 0$.

Following the suggestion expressed in Paragraph I.1.6, we hope that *strong* continuity of a semigroup $(T(t))_{t\geq 0}$ might still imply some differentiability of the *orbit maps*

$$\xi_x : t \mapsto T(t)x \in X.$$

In order to pursue this idea we first show, in analogy to Proposition I.5.3, that differentiability of $\xi_x$ is already implied by right differentiability at $t = 0$.

**1.1 Lemma.** *Take a strongly continuous semigroup $(T(t))_{t\geq 0}$ and an element $x \in X$. For the orbit map $\xi_x : t \mapsto T(t)x$, the following properties are equivalent.*

(a) *$\xi_x(\cdot)$ is differentiable on $\mathbb{R}_+$.*

(b) *$\xi_x(\cdot)$ is right differentiable at $t = 0$.*

Proof. We have only to show that (b) implies (a). For $h > 0$, one has

$$\begin{aligned}\lim_{h\downarrow 0} \tfrac{1}{h}\big(T(t+h)x - T(t)x\big) &= T(t) \lim_{h\downarrow 0} \tfrac{1}{h}\big(T(h)x - x\big)\\ &= T(t)\,\dot{\xi}_x(0),\end{aligned}$$

and hence $\xi_x(\cdot)$ is right differentiable on $\mathbb{R}_+$.

On the other hand, for $-t \leq h < 0$, we write

$$\begin{aligned}\tfrac{1}{h}\big(T(t+h)x - T(t)x\big) - T(t)\dot{\xi}_x(0) &= T(t+h)\left(\tfrac{1}{h}\big(x - T(-h)x\big) - \dot{\xi}_x(0)\right)\\ &\quad + T(t+h)\dot{\xi}_x(0) - T(t)\dot{\xi}_x(0).\end{aligned}$$

As $h \uparrow 0$, the first term on the right-hand side converges to zero, since $\|T(t+h)\|$ remains bounded. The remaining part converges to zero by the strong continuity of $\big(T(t)\big)_{t\geq 0}$. Hence, $\xi_x$ is also left differentiable, and its derivative is

$$\dot{\xi}_x(t) = T(t)\,\dot{\xi}_x(0) \tag{1.1}$$

for each $t \geq 0$. □

On the subspace of $X$ consisting of all those $x \in X$ for which the orbit maps $\xi_x$ are differentiable, the right derivative at $t = 0$ then yields an operator $A$ from which, in a sense to be specified later, we can hope to obtain the operators $T(t)$ as the "exponentials $\mathrm{e}^{tA}$." This hope is expressed in the choice of the term "generator" in the following definition.

**1.2 Definition.** *The* generator $A : D(A) \subseteq X \to X$ *of a strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$ *on a Banach space* $X$ *is the operator*

$$Ax := \dot{\xi}_x(0) = \lim_{h\downarrow 0} \tfrac{1}{h}\big(T(h)x - x\big) \tag{1.2}$$

*defined for every* $x$ *in its domain*

$$D(A) := \{x \in X : \xi_x \text{ is differentiable}\}. \tag{1.3}$$

We observe from Lemma 1.1 that the domain $D(A)$ is also given as the set of all elements $x \in X$ for which $\xi_x(\cdot)$ is right differentiable in $t = 0$, i.e.,

$$D(A) = \Big\{x \in X : \lim_{h\downarrow 0} \tfrac{1}{h}\big(T(h)x - x\big) \text{ exists}\Big\}. \tag{1.4}$$

The domain $D(A)$, which is a linear subspace, is an essential part of the definition of the generator $A$. Accordingly, we should always denote it by the *pair* $\big(A, D(A)\big)$, but for convenience, we will often only write $A$ and assume implicitly that its domain is given by (1.4).

To ensure that the operator $\big(A, D(A)\big)$ has reasonable properties, we proceed as in Chapter I. There we used the "smoothing operators" $V(t) := \int_0^t T(s)\,ds$ to prove differentiability of the semigroup $\big(T(t)\big)_{t\geq 0}$ (see the proof of Theorem I.3.7). Since we now assume that the orbit maps $\xi_x$ are only continuous, we need to look at "smoothed" elements of the form

$$y_t := \frac{1}{t}\int_0^t \xi_x(s)\,ds = \frac{1}{t}\int_0^t T(s)x\,ds \qquad \text{for } x \in X, t > 0.$$

It is a simple consequence of the definition of the integral as a limit of Riemann sums that the vectors $y_t$ converge to $x$ as $t \downarrow 0$. In addition, they always belong to the domain $D(A)$. This and other elementary facts are collected in the following result.

**1.3 Lemma.** *For the generator $(A, D(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$, the following properties hold.*

(i) *$A : D(A) \subseteq X \to X$ is a linear operator.*

(ii) *If $x \in D(A)$, then $T(t)x \in D(A)$ and*

$$\tfrac{d}{dt}T(t)x = T(t)Ax = AT(t)x \quad \text{for all } t \geq 0. \tag{1.5}$$

(iii) *For every $t \geq 0$ and $x \in X$, one has*

$$\int_0^t T(s)x\,ds \in D(A).$$

(iv) *For every $t \geq 0$, one has*

$$T(t)x - x = A\int_0^t T(s)x\,ds \quad \text{if } x \in X, \tag{1.6}$$

$$= \int_0^t T(s)Ax\,ds \quad \text{if } x \in D(A). \tag{1.7}$$

PROOF. Assertion (i) is trivial. To prove (ii) take $x \in D(A)$. Then it follows from (1.1) that $1/h\big(T(t+h)x - T(t)x\big)$ converges to $T(t)Ax$ as $h \downarrow 0$. Therefore,

$$\lim_{h\downarrow 0} \frac{1}{h}\big(T(h)T(t)x - T(t)x\big)$$

exists, and hence $T(t)x \in D(A)$ by (1.4) with $AT(t)x = T(t)Ax$.

The Proof of assertion (iii) is included in the following proof of (iv). For $x \in X$ and $t \geq 0$, one has

$$\begin{aligned}
&\frac{1}{h}\left(T(h)\int_0^t T(s)x\,ds - \int_0^t T(s)x\,ds\right)\\
&\qquad = \frac{1}{h}\int_0^t T(s+h)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds\\
&\qquad = \frac{1}{h}\int_h^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^t T(s)x\,ds\\
&\qquad = \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds,
\end{aligned}$$

which converges to $T(t)x - x$ as $h \downarrow 0$. Hence (1.6) holds.

If $x \in D(A)$, then the functions $s \mapsto T(s)\frac{T(h)x-x}{h}$ converge uniformly on $[0,t]$ to the function $s \mapsto T(s)Ax$ as $h \downarrow 0$. Therefore,

$$\begin{aligned}\lim_{h\downarrow 0} \frac{1}{h}\big(T(h)-I\big)\int_0^t T(s)x\,ds &= \lim_{h\downarrow 0}\int_0^t T(s)\frac{1}{h}\big(T(h)-I\big)x\,ds\\ &= \int_0^t T(s)Ax\,ds.\end{aligned}$$

□

With the help of this lemma we now show that the generator introduced in Definition 1.2, although unbounded in general, has nice properties.

**1.4 Theorem.** *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Proof. Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$. As already noted, its generator $\big(A, D(A)\big)$ is a linear operator. To show that $A$ is closed, consider a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ for which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ exist. By (1.7) in the previous lemma, we have

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,ds$$

for $t > 0$. The uniform convergence of $T(\cdot)Ax_n$ on $[0,t]$ for $n \to \infty$ implies that

$$T(t)x - x = \int_0^t T(s)y\,ds.$$

Multiplying both sides by $1/t$ and taking the limit as $t \downarrow 0$, we see that $x \in D(A)$ and $Ax = y$, i.e., $A$ is closed.

By Lemma 1.3.(iii) the elements $1/t \int_0^t T(s)x\,ds$ always belong to $D(A)$. Since the strong continuity of $\big(T(t)\big)_{t\geq 0}$ implies

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T(s)x\,ds = x$$

for every $x \in X$, we conclude that $D(A)$ is dense in $X$.

Finally, let $\big(S(t)\big)_{t\geq 0}$ be another strongly continuous semigroup having the same generator $\big(A, D(A)\big)$. For $x \in D(A)$ and $t > 0$, we consider the map

$$s \mapsto \eta_x(s) := T(t-s)S(s)x$$

for $0 \leq s \leq t$. Since for fixed $s$ the set

$$\left\{\frac{S(s+h)x - S(s)x}{h} : h \in (0,1]\right\} \cup \{AS(s)x\}$$

is compact, the difference quotients

$$\frac{1}{h}\big(\eta_x(s+h)-\eta_x(s)\big) = T(t-s-h)\frac{1}{h}\big(S(s+h)x - S(s)x\big) + \frac{1}{h}\big(T(t-s-h)-T(t-s)\big)S(s)x$$

converge by Lemma I.5.2 and Lemma 1.3.(ii) to

$$\tfrac{d}{ds}\eta_x(s) = T(t-s)AS(s)x - AT(t-s)S(s)x = 0.$$

From $\eta_x(0) = T(t)x$ and $\eta_x(t) = S(t)x$ we obtain

$$T(t)x = S(t)x$$

for all $x$ in the dense domain $D(A)$. Hence, $T(t) = S(t)$ for each $t \geq 0$. □

Combining these properties of the generator with the closed graph theorem gives a new characterization of uniformly continuous semigroups, thus complementing Theorem I.3.7.

**1.5 Corollary.** *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ with generator $(A, D(A))$, the following assertions are equivalent.*

(a) *The generator $A$ is bounded, i.e., there exists $M > 0$ such that $\|Ax\| \leq M\,\|x\|$ for all $x \in D(A)$.*

(b) *The domain $D(A)$ is all of $X$.*

(c) *The domain $D(A)$ is closed in $X$.*

(d) *The semigroup $(T(t))_{t\geq 0}$ is uniformly continuous.*

*In each case, the semigroup is given by*

$$T(t) = \mathrm{e}^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}, \qquad t \geq 0.$$

The proof of this corollary and of some more equivalences is left as Exercise 1.15.(1).

Property (b) indicates that the domain of the generator contains important information about the semigroup and therefore has to be taken into account carefully. However, in many examples (see, e.g., Paragraph 2.7 and Example 4.10 below) it is often routine to compute the expression $Ax$ for some or even many elements in the domain $D(A)$, while it is difficult to identify $D(A)$ precisely. In these situations, the following concept helps to distinguish between "small" and "large" subspaces of $D(A)$.

**1.6 Definition.** *A subspace $D$ of the domain $D(A)$ of a linear operator $A : D(A) \subseteq X \to X$ is called a core for $A$ if $D$ is dense in $D(A)$ for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|.$$

We now state a useful criterion for subspaces to be a core for the generator.

**1.7 Proposition.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$. A subspace $D$ of $D(A)$ that is $\|\cdot\|$-dense in $X$ and invariant under the semigroup $\big(T(t)\big)_{t\geq 0}$ is always a core for $A$.*

PROOF. For every $x \in D(A)$ we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $\lim_{n\to\infty} x_n = x$. Since for each $n$ the map $s \mapsto T(s)x_n \in D$ is continuous for the graph norm $\|\cdot\|_A$ (use (1.5)), it follows that

$$\int_0^t T(s)x_n\, ds,$$

being a Riemann integral, belongs to the $\|\cdot\|_A$-closure of $D$. Similarly, the $\|\cdot\|_A$-continuity of $s \mapsto T(s)x$ for $x \in D(A)$ implies that

$$\left\| \frac{1}{t}\int_0^t T(s)x\, ds - x \right\|_A \to 0 \qquad \text{as } t \downarrow 0 \text{ and}$$

$$\left\| \frac{1}{t}\int_0^t T(s)x_n\, ds - \frac{1}{t}\int_0^t T(s)x\, ds \right\|_A \to 0 \qquad \text{as } n \to \infty \text{ and for each } t > 0.$$

This proves that for every $\varepsilon > 0$ we can find $t > 0$ and $n \in \mathbb{N}$ such that

$$\left\| \frac{1}{t}\int_0^t T(s)x_n\, ds - x \right\|_A < \varepsilon.$$

Hence, $x \in \overline{D}^{\|\cdot\|_A}$. $\square$

Important examples of cores are given by the domains $D(A^n)$ of the powers $A^n$ of a generator $A$.

**1.8 Proposition.** *For the generator $\big(A, D(A)\big)$ of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ the space*

$$D(A^\infty) := \bigcap_{n\in\mathbb{N}} D(A^n),$$

*hence each $D(A^n) := \big\{x \in D(A^{n-1}) : A^{n-1}x \in D(A)\big\}$, is a core for $A$.*

PROOF. Since the space $D(A^\infty)$ is a $\big(T(t)\big)_{t\geq0}$-invariant subspace of $D(A)$, it remains to show that it is dense in $X$. To that purpose, we prove that for each function $\varphi \in \mathrm{C}^\infty(-\infty,\infty)$ with compact support in $(0,\infty)$ and each $x \in X$ the element

$$x_\varphi := \int_0^\infty \varphi(s)T(s)x\,ds$$

belongs to $D(A^\infty)$. In fact, if we set

$$\mathcal{D} := \big\{\varphi \in \mathrm{C}^\infty(-\infty,\infty) : \operatorname{supp}\varphi \text{ is compact in } (0,\infty)\big\},$$

then for $x \in X$, $\varphi \in \mathcal{D}$, and $h > 0$ sufficiently small we have

$$\begin{aligned}\frac{T(h)-I}{h}\,x_\varphi &= \frac{1}{h}\int_0^\infty \varphi(s)\big(T(s+h)-T(s)\big)x\,ds\\ &= \frac{1}{h}\int_h^\infty \big(\varphi(s-h)-\varphi(s)\big)T(s)x\,ds - \frac{1}{h}\int_0^h \varphi(s)T(s)x\,ds\\ &= \int_0^\infty \frac{1}{h}\big(\varphi(s-h)-\varphi(s)\big)T(s)x\,ds. \qquad (1.8)\end{aligned}$$

The integrand in (1.8) converges uniformly on $[0,\infty)$ to $-\varphi'(s)T(s)x$ as $h \downarrow 0$. This shows that $x_\varphi \in D(A)$ and

$$Ax_\varphi = -\int_0^\infty \varphi'(s)T(s)x\,ds.$$

Since $\varphi^{(n)} \in \mathcal{D}$ for all $n \in \mathbb{N}$, we conclude by induction that $x_\varphi \in D(A^n)$ for all $n \in \mathbb{N}$, i.e., $x_\varphi \in D(A^\infty)$. Assume that the linear span

$$D := \operatorname{lin}\big\{x_\varphi : x \in X,\ \varphi \in \mathcal{D}\big\}$$

is not dense in $X$. By the Hahn–Banach theorem there is a linear functional $0 \neq x' \in X'$ such that $\langle y, x'\rangle = 0$ for all $y \in D$, that is,

$$\int_0^\infty \varphi(s)\,\big\langle T(s)x, x'\big\rangle\,ds = \Big\langle \int_0^\infty \varphi(s)\,T(s)x\,ds, x'\Big\rangle = 0 \qquad (1.9)$$

for all $x \in X$ and $\varphi \in \mathcal{D}$. This implies that the continuous functions $s \mapsto \big\langle T(s)x, x'\big\rangle$ vanish on $[0,\infty)$ for all $x \in X$. Otherwise there would exist $\varphi \in \mathcal{D}$ such that the left-hand side of (1.9) does not vanish. Choosing $s = 0$, we obtain $\langle x, x'\rangle = 0$ for all $x \in X$; hence $x' = 0$. This contradicts the choice of $x' \neq 0$, and therefore $D \subset X$ is dense.

Since we have seen in the first step that $D \subset D(A^\infty)$, and since $D(A^\infty)$ is invariant under $\big(T(t)\big)_{t\geq0}$, the assertion follows from Proposition 1.7. □

In the remaining part of this section we introduce some basic spectral properties for generators of strongly continuous semigroups. Recall from Section IV.1 the notions

*spectrum* $\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not bijective}\}$,
*resolvent set* $\rho(A) := \mathbb{C} \setminus \sigma(A)$, and
*resolvent* $R(\lambda, A) := (\lambda - A)^{-1}$ at $\lambda \in \rho(A)$

for a closed operator $(A, D(A))$ on a Banach space $X$.

Our starting point are the following two identities, which are easily derived from their predecessors in Lemma 1.3.(iv). We stress that these identities will be continually used throughout the book.

**1.9 Lemma.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then, for every $\lambda \in \mathbb{C}$ and $t > 0$, the following identities hold:*

$$\text{(1.10)} \quad e^{-\lambda t}T(t)x - x = (A - \lambda)\int_0^t e^{-\lambda s}T(s)x\, ds \qquad \text{if } x \in X,$$

$$\text{(1.11)} \quad = \int_0^t e^{-\lambda s}T(s)(A - \lambda)x\, ds \qquad \text{if } x \in D(A).$$

PROOF. It suffices to apply Lemma 1.3.(iv) to the rescaled semigroup

$$S(t) := e^{-\lambda t}T(t), \quad t \geq 0,$$

whose generator is $B := A - \lambda$ with domain $D(B) = D(A)$. ☐

Next, we give an important formula relating the semigroup to the resolvent of its generator.

**1.10 Theorem.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space $X$ and take constants $w \in \mathbb{R}$, $M \geq 1$ (see Proposition* I.5.5*) such that*

$$\text{(1.12)} \qquad \|T(t)\| \leq Me^{wt}$$

*for $t \geq 0$. For the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ the following properties hold.*

(i) *If $\lambda \in \mathbb{C}$ such that $R(\lambda)x := \int_0^\infty e^{-\lambda s}T(s)x\, ds$ exists for all $x \in X$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$.*

(ii) *If $\operatorname{Re}\lambda > w$, then $\lambda \in \rho(A)$, and the resolvent is given by the integral expression in* (i).

(iii) $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda - w}$ *for all* $\operatorname{Re}\lambda > w$.

The formula for $R(\lambda, A)$ in (i) is called the *integral representation of the resolvent*. Of course, the integral has to be understood as an improper Riemann integral, i.e.,

$$\text{(1.13)} \qquad R(\lambda, A)x = \lim_{t\to\infty}\int_0^t e^{-\lambda s}T(s)x\, ds$$

for all $x \in X$.

Having in mind this interpretation, we will frequently write

$$R(\lambda, A) = \int_0^\infty e^{-\lambda s} T(s)\, ds. \tag{1.14}$$

PROOF OF THEOREM 1.10. (i) By a simple rescaling argument (cf. Paragraph I.5.11) we may assume that $\lambda = 0$. Then, for arbitrary $x \in X$ and $h > 0$, we have

$$\begin{aligned}
\frac{T(h) - I}{h} R(0)x &= \frac{T(h) - I}{h} \int_0^\infty T(s)x\, ds \\
&= \frac{1}{h} \int_0^\infty T(s + h)x\, ds - \frac{1}{h} \int_0^\infty T(s)x\, ds \\
&= \frac{1}{h} \int_h^\infty T(s)x\, ds - \frac{1}{h} \int_0^\infty T(s)x\, ds \\
&= -\frac{1}{h} \int_0^h T(s)x\, ds.
\end{aligned}$$

By taking the limit as $h \downarrow 0$, we conclude that $\operatorname{rg} R(0) \subseteq D(A)$ and $AR(0) = -I$. On the other hand, for $x \in D(A)$ we have

$$\lim_{t \to \infty} \int_0^t T(s)x\, ds = R(0)x,$$

and

$$\lim_{t \to \infty} A \int_0^t T(s)x\, ds = \lim_{t \to \infty} \int_0^t T(s)Ax\, ds = R(0)Ax,$$

where we have used Lemma 1.3.(iv) for the second equality. Since by Theorem 1.4 $A$ is closed, this implies $R(0)Ax = AR(0)x = -x$ and therefore $R(0) = (-A)^{-1}$ as claimed.

Parts (ii) and (iii) then follow easily from (i) and the estimate

$$\left\| \int_0^t e^{-\lambda s} T(s)\, ds \right\| \le M \int_0^t e^{(w - \operatorname{Re} \lambda)s}\, ds,$$

since for $\operatorname{Re} \lambda > w$ the right-hand side converges to $M/(\operatorname{Re} \lambda - w)$ as $t \to \infty$. □

The above integral representation can now be used to represent and estimate the powers of $R(\lambda, A)$.

**1.11 Corollary.** *For the generator $\big(A, D(A)\big)$ of a strongly continuous semigroup $\big(T(t)\big)_{t \ge 0}$ satisfying*

$$\|T(t)\| \le M e^{wt} \qquad \text{for all } t \ge 0,$$

*one has, for* $\operatorname{Re}\lambda > w$ *and* $n \in \mathbb{N}$, *that*

$$R(\lambda, A)^n x = \frac{(-1)^{n-1}}{(n-1)!} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)x \tag{1.15}$$

$$= \frac{1}{(n-1)!} \int_0^\infty s^{n-1} \mathrm{e}^{-\lambda s} T(s)x \, ds \tag{1.16}$$

*for all* $x \in X$. *In particular, the estimates*

$$\|R(\lambda, A)^n\| \le \frac{M}{(\operatorname{Re}\lambda - w)^n} \tag{1.17}$$

*hold for all* $n \in \mathbb{N}$ *and* $\operatorname{Re}\lambda > w$.

Proof. Equation (1.15) is actually valid for every operator with nonempty resolvent set. We postpone its proof until Chapter IV, Proposition 1.3. On the other hand, by Theorem 1.10.(i), one has

$$\begin{aligned} \frac{d}{d\lambda} R(\lambda, A)x &= \frac{d}{d\lambda} \int_0^\infty \mathrm{e}^{-\lambda s} T(s)x \, ds \\ &= - \int_0^\infty s\mathrm{e}^{-\lambda s} T(s)x \, ds \end{aligned}$$

for $\operatorname{Re}\lambda > w$ and all $x \in X$. Proceeding by induction, we deduce (1.16). Finally, the estimate (1.17) follows from

$$\begin{aligned} \|R(\lambda, A)^n x\| &= \frac{1}{(n-1)!} \cdot \left\| \int_0^\infty s^{n-1} \mathrm{e}^{-\lambda s} T(s)x \, ds \right\| \\ &\le \frac{M}{(n-1)!} \cdot \int_0^\infty s^{n-1} \mathrm{e}^{(w-\operatorname{Re}\lambda)s} \, ds \cdot \|x\| \\ &= \frac{M}{(\operatorname{Re}\lambda - w)^n} \cdot \|x\| \end{aligned}$$

for all $x \in X$. □

Property (ii) in Theorem 1.10 says that the spectrum of a semigroup generator is always contained in a left half-plane. The number determining the smallest such half-plane is an important characteristic of any linear operator and is defined as follows.

**1.12 Definition.** *To any linear operator* $A$ *we associate its spectral bound defined by*

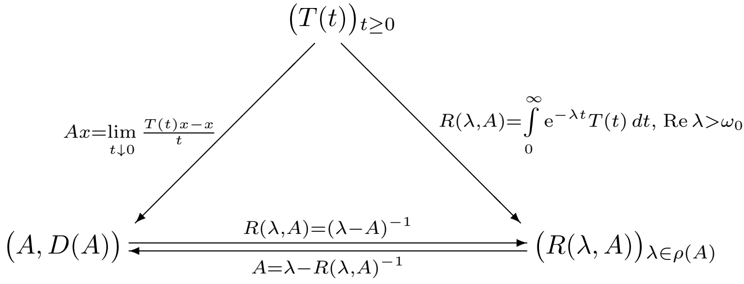
$$\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

As an immediate consequence of Theorem 1.10.(ii) the following relation holds between the growth bound of a strongly continuous semigroup (see Definition I.5.6) and the spectral bound of its generator.

**1.13 Corollary.** *For a strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$ *with generator* $A$*, one has*

$$-\infty \leq \mathrm{s}(A) \leq \omega_0 < +\infty.$$

**1.14 Diagram.** To conclude this section, we collect in a diagram the information obtained so far on the relations between a semigroup, its generator, and its resolvent.




By developing our theory further, we will be able to add one of the missing links in this diagram (see Diagram III.5.6).

**1.15 Exercises.** (1) Prove that the statements (a)–(d) in Corollary 1.5 are equivalent to each of the follows conditions.

(e) $\|T(t) - I\| \leq ct \qquad$ for $0 \leq t \leq 1$ and some $c > 0$.

(f) $\overline{\lim}_{\lambda\to\infty} \|\lambda A R(\lambda, A)\| < \infty$.

(2) Show that for a closed linear operator $(A, D(A))$ on a Banach space $X$ and a linear subspace $Y \subset D(A)$ the following assertions are equivalent.

(a) $Y$ is a core for $(A, D(A))$.

(b) $\overline{A_{|Y}} = A$.

If, in addition, $\rho(A) \neq \emptyset$, then these assertions are equivalent to

(c) $(\lambda - A)Y$ is dense in $X$ for one/all $\lambda \in \rho(A)$.

(3) Show that the space of all continuous functions with compact support forms a core for each multiplication operator $M_q$ on $\mathrm{C}_0(\Omega)$.

(4) Decide whether $\mathcal{D} := \big\{f \in \mathrm{C}^\infty(\mathbb{R}_+) : f'(0) = 0 \text{ and } \operatorname{supp} f \text{ is compact}\big\}$ is a core for

(i) the generator of the left translation semigroup on $\mathrm{C}_0(\mathbb{R}_+)$, and

(ii) the generator of the right translation semigroup on $\mathrm{C}_0(\mathbb{R}_+)$,

as defined in Paragraph I.4.16. (Hint: Compare the hint in (Exercise 6.iii).)

(5) Consider the Banach space $X := \mathrm{C}_0(\Omega)$ for some locally compact space $\Omega$. Show that for a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on $X$ the following statements are equivalent.

(a) $(T(t))_{t\geq 0}$ is a semigroup of *algebra homomorphisms* on $X$, i.e., $T(t)(f\cdot g) = T(t)f \cdot T(t)g$ for $f, g \in X$ and $t \geq 0$.

(b) $(A, D(A))$ is a *derivation*, i.e., $D(A)$ is a subalgebra of $X$ and

$$A(f \cdot g) = (Af)\cdot g + f \cdot Ag$$

for $f, g \in D(A)$.

(Hint: For the implication (b) $\Rightarrow$ (a) consider the maps $s \mapsto T(t-s)[T(s)f\cdot T(s)g]$ for each $0 \leq s \leq t$ and $f, g \in D(A)$.)

(6) Let $(A, D(A))$ be the generator of a contraction semigroup $(T(t))_{t\geq 0}$ on some Banach space $X$. Establish the following assertions.

(i) The *Landau–Kolmogorov Inequality*, which states that

$$\|Ax\|^2 \leq 4 \left\|A^2 x\right\| \cdot \|x\|$$

for each $x \in D(A^2)$. (Hint: As a first step, verify Taylor's formula

$$T(t)x = x + tAx + \int_0^t (t-s)T(s)A^2 x \, ds$$

for $x \in D(A^2)$.)

(ii) If $(T(t))_{t\geq 0}$ is a group of isometries, then (i) can be improved to

$$\|Ax\|^2 \leq 2 \left\|A^2 x\right\| \cdot \|x\|$$

for $x \in D(A^2)$.

(iii) Apply (i) and (ii) to the various translation semigroups of Section I.4.c, in particular to the left translation semigroup on $\mathrm{L}^p(\mathbb{R}_+)$. (Hint: The generator of the (left) translation semigroup is the differentiation operator with appropriate domain; see Paragraph 2.10.)

# 2. Examples Revisited

Before proceeding with the abstract theory, we pause for a moment and examine the concrete semigroups from Section I.4 and the semigroup constructions established in Section I.5.b. In each case, we try to identify the corresponding

*generator*, its *spectrum* and *resolvent*,

so that our abstract definitions gain a more concrete meaning. However, the impatient reader might skip these examples and look at them only later.

## a. Standard Constructions

Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on a Banach space $X$. For each of the semigroups constructed in Section I.5.b, we now characterize its generator and its resolvent.

**2.1 Similar Semigroups.** If $V$ is an isomorphism from a Banach space $Y$ onto $X$ and $(S(t))_{t\geq 0}$ is the strongly continuous semigroup on $Y$ given by $S(t) := V^{-1}T(t)V$, then its generator is

$$B = V^{-1}AV \quad \text{with domain} \quad D(B) = \big\{y \in Y : Vy \in D(A)\big\}.$$

Equality of the spectra

$$\sigma(A) = \sigma(B)$$

is clear, and the resolvent of $B$ is $R(\lambda, B) = V^{-1}R(\lambda, A)V$ for $\lambda \in \rho(A)$.

A particularly important example of this situation is given by the Spectral Theorem I.4.9, which states that every normal or self-adjoint operator on a Hilbert space is similar to a multiplication operator on an $\mathrm{L}^2$-space.

**2.2 Rescaled Semigroups.** The rescaled semigroup $(\mathrm{e}^{\mu t}T(\alpha t))_{t\geq 0}$ for some fixed $\mu \in \mathbb{C}$ and $\alpha > 0$ has generator

$$B = \alpha A + \mu I \quad \text{with domain} \quad D(A) = D(B).$$

Moreover, $\sigma(B) = \alpha\sigma(A) + \mu$ and $R(\lambda, B) = {}^1\!/\!_\alpha R\left({}^{\lambda-\mu}\!/\!_\alpha, A\right)$ for $\lambda \in \rho(B)$.

This shows that we can switch quite easily between the original and the rescaled objects.

**2.3 Subspace Semigroups.** While we considered in Paragraph I.5.12 the subspace semigroup $\big(T(t)_{|Y}\big)_{t\geq 0}$ only for *closed* subspaces $Y$ in $X$, we begin here with a more general situation.

Let $Y$ be a Banach space that is continuously embedded in $X$ (in symbols: $Y \hookrightarrow X$). Assume also that the restrictions $T(t)_|$ leave $Y$ invariant and form a strongly continuous semigroup $(T(t)_|)_{t\geq 0}$ on $Y$. In order to be able to identify the generator of $\big(T(t)_{|Y}\big)_{t\geq 0}$, we introduce the following concept.

**Definition.** *The part of $A$ in $Y$ is the operator $A_|$ defined by*

$$A_| y := Ay$$

*with domain*

$$D(A_|) := \big\{y \in D(A) \cap Y : Ay \in Y\big\}.$$

In other words, $A_|$ is the "maximal" operator induced by $A$ on $Y$ and, as will be seen, coincides with the generator of the semigroup $(T(t)_|)_{t\geq 0}$ on $Y$.

**Proposition.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X$ and assume that the restricted semigroup $(T(t)_|)_{t\geq 0}$ is strongly continuous on some $(T(t))_{t\geq 0}$-invariant Banach space $Y \hookrightarrow X$. Then the generator of $(T(t)_|)_{t\geq 0}$ is the part $(A_|, D(A_|))$ of $A$ in $Y$.*

PROOF. Let $(C, D(C))$ denote the generator of $(T(t)_|)_{t\geq 0}$. Since $Y$ is continuously embedded in $X$, we immediately have that $C$ is a restriction of $A_|$. For the converse inclusion, choose $\lambda \in \mathbb{R}$ large enough such that both $R(\lambda, C)$ and $R(\lambda, A)$ are given by the integral representation from Theorem 1.10.(i). Then

$$R(\lambda, C)y = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)y\, ds = R(\lambda, A)y \qquad \text{for all} \qquad y \in Y.$$

For $x \in D(A_|)$, we obtain that

$$x = R(\lambda, A)(\lambda - A)x = R(\lambda, C)(\lambda - A)x \in D(C),$$

and hence $D(A_|) = D(C)$. □

If $Y$ is a $(T(t))_{t\geq 0}$-invariant closed subspace of $X$, then the strong continuity of $(T(t)_|)_{t\geq 0}$ is automatic. Moreover, the existence of

$$z := \lim_{t\downarrow 0} \frac{1}{t}\big(T(t)y - y\big) \in X$$

for some $y \in Y$ implies that $z \in Y$. Therefore, the part $A_|$ simply becomes the "restriction" of $A$.

**Corollary.** *If $Y$ is a $(T(t))_{t\geq 0}$-invariant closed subspace of $X$, then the generator of $(T(t)_|)_{t\geq 0}$ is*

$$A_| y = Ay,$$

*with domain*

$$D(A_|) = D(A) \cap Y.$$

**Example.** A typical example for the situation considered here occurs when we take $X := \mathrm{L}^1(\Gamma, m)$ and $Y := \mathrm{C}(\Gamma)$. The rotation group from I.4.18 is strongly continuous on both spaces; hence its generator on $\mathrm{C}(\Gamma)$ is the part of its generator on $\mathrm{L}^1(\Gamma, m)$. The generator on $\mathrm{L}^1(\Gamma, m)$ can now be obtained by modifying the arguments from the proposition in Paragraph 2.10.(ii) below.

**2.4 Quotient Semigroup.** Let $Y$ be a $(T(t))_{t\geq 0}$-invariant closed subspace of $X$. Then the generator $(A_/, D(A_/))$ of the quotient semigroup $\big(T(t)_{/Y}\big)_{t\geq 0}$ on the quotient space $X_/ := {}^X\!/_Y$ is given (with the notation from Paragraph I.5.13) by

$$A_/ q(x) = q(Ax) \qquad \text{with domain } D(A_/) = q\big(D(A)\big).$$

This follows from the fact that each element $\widehat{x} := q(x) \in D(A_/)$ can be written as

$$\widehat{x} = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)_/ \widehat{y}\, ds$$

for some $\widehat{y} = q(y) \in {}^X\!/_Y$ and some $\lambda > \omega_0$ (use 1.10.(i)). Therefore,

$$\widehat{x} = \int_0^\infty \mathrm{e}^{-\lambda s} q\big(T(s)y\big)\, ds = q\left(\int_0^\infty \mathrm{e}^{-\lambda s} T(s) y\, ds\right) = q(z)$$

with $z \in D(A)$. This means that for every $\widehat{x} \in D(A_/)$ there exists a representative $z \in X$ belonging to $D(A)$.

For a concrete example, we refer to IV.2.14.

**2.5 Adjoint Semigroups.** Even though the adjoint semigroup $(T(t)')_{t\geq 0}$ is not necessarily strongly continuous on $X'$, it is still possible to associate a "generator" to it. In fact, defining

$$A^\sigma x' := \sigma(X', X)\text{-}\lim_{h\downarrow 0} \frac{1}{h}\big(T(h)'x' - x'\big)$$

on the domain

$$D(A^\sigma) := \left\{ x' \in X' : \sigma(X', X)\text{-}\lim_{h\downarrow 0} \frac{1}{h}\big(T(h)'x' - x'\big) \text{ exists} \right\},$$

one obtains a linear operator called the *weak* generator* of $(T(t)')_{t\geq 0}$. It is a $\sigma(X', X)$-closed and $\sigma(X', X)$-densely defined operator and coincides with the *adjoint* $A'$ of $A$ (see Definition B.8), i.e.,

$$D(A^\sigma) = \left\{ x' \in X' : \begin{array}{l} \text{there exists } y' \in X' \text{ such that} \\ \langle x, y' \rangle = \langle Ax, x' \rangle \text{ for all } x \in D(A) \end{array} \right\}$$

and

$$A^\sigma x' = A' x'.$$

(See Exercise 2.8.(1).) By Corollary B.12 it then follows that $\sigma(A^\sigma) = \sigma(A) = \sigma(A')$ and $R(\lambda, A^\sigma) = R(\lambda, A') = R(\lambda, A)'$ for $\lambda \in \rho(A)$.

**2.6 Sun Dual Semigroups.** To overcome the handicap that the adjoint semigroup $(T(t)')_{t\geq 0}$ may not be strongly continuous on $X'$, we restrict it to its so-called (closed) *subspace of strong continuity*.

**Definition.** *Corresponding to a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* $X$, *we define its sun dual (or semigroup dual) by*

$$X^{\odot} := \left\{ x' \in X' : \lim_{t\downarrow 0} \left\| T(t)'x' - x' \right\| = 0 \right\}$$

*and call the semigroup given by the restricted operators*

$$T(t)^{\odot} := T(t)'_{|_{X^{\odot}}}, \quad t \geq 0,$$

*the sun dual semigroup.*

Without loss of generality, we now assume $(T(t))_{t\geq 0}$ to be bounded (use Paragraph 2.2), i.e.,

$$\left\| T(t)' \right\| = \| T(t) \| \leq M \qquad \text{for all } t \geq 0.$$

Then it follows from Proposition I.5.3 that $X^{\odot}$ is a closed subspace of $X'$ and that the sun dual semigroup $(T(t)^{\odot})_{t\geq 0}$ is in fact strongly continuous. As a first step, we show that $X^{\odot}$ is reasonably large.

**Lemma.** *One always has* $D(A') \subset X^{\odot}$.

PROOF. Take $y' \in D(A')$. Then, by (1.6), we have

$$\begin{aligned} \left| \left\langle x, T(t)'y' - y' \right\rangle \right| &= \left| \left\langle T(t)x - x, y' \right\rangle \right| \\ &= \left| \left\langle A \int_0^t T(s)x \, ds, y' \right\rangle \right| \\ &\leq tM \|x\| \cdot \left\| A'y' \right\| \end{aligned}$$

for all $x \in X$. As $t \downarrow 0$ this expression converges to zero uniformly for $\|x\| \leq 1$. Hence,

$$\lim_{t\downarrow 0} \left\| T(t)'y' - y' \right\| = 0,$$

and we obtain $y' \in X^{\odot}$. □

A consequence of this lemma are the estimates

$$\sup_{\substack{y' \in D(A') \\ \|y'\| \leq 1}} \left| \left\langle x, y' \right\rangle \right| \leq \|x\| \leq \sup_{\substack{y' \in D(A') \\ \|y'\| \leq M}} \left| \left\langle x, y' \right\rangle \right| \tag{2.1}$$

for $x \in X$ and

$$\left\| T(t)^{\odot} \right\| \leq \left\| T(t)' \right\| = \|T(t)\| \leq M \left\| T(t)^{\odot} \right\| \tag{2.2}$$

for all $t \geq 0$. To establish (2.1) we use the fact that $\lim_{n\to\infty} nR(n,A)x = x$ for each $x \in X$ (see Lemma 3.4.(i)). Then, for arbitrary $x' \in X'$, we obtain that

$$\left\langle x, x' \right\rangle = \lim_{n\to\infty} \left\langle nR(n,A)x, x' \right\rangle = \lim_{n\to\infty} \left\langle x, nR(n,A')x' \right\rangle.$$

This proves (2.1), since $nR(n,A')x' \in D(A')$ and $\|nR(n,A')x'\| \le M\|x'\|$ by Theorem 1.10.(iii). Again by Lemma 3.4.(i) one has that

$$\left\langle x, T(t)'x' \right\rangle = \lim_{n\to\infty} \left\langle x, T(t)^{\odot} nR(n,A')x' \right\rangle$$

for $x \in X$ and $x' \in X'$. This implies first that

$$\begin{aligned} \left| \left\langle x, T(t)'x' \right\rangle \right| &\le \overline{\lim_{n\to\infty}} \, \|x\| \cdot \left\|nR(n,A')x'\right\| \cdot \left\|T(t)^{\odot}\right\| \\ &\le M \, \|x\| \cdot \|x'\| \cdot \left\|T(t)^{\odot}\right\| \end{aligned}$$

and then (2.2). □

Next we relate the generator $(A^{\odot}, D(A^{\odot}))$ of the sun dual semigroup $(T(t)^{\odot})_{t\ge 0}$ to the adjoint operator $(A', D(A'))$.

**Proposition.** *The generator $(A^{\odot}, D(A^{\odot}))$ of the strongly continuous semigroup $(T(t)^{\odot})_{t\ge 0}$ is the part of $(A', D(A'))$ in $X^{\odot}$, i.e.,*

$$A^{\odot}x' = A'x' \qquad \textit{for} \qquad x' \in D(A^{\odot}) = \left\{ x' \in D(A') : A'x' \in X^{\odot} \right\}.$$

Proof. Since the weak* topology on $X'$ is weaker than the norm topology, it is clear that $A'$ is an extension of $A^{\odot}$. Now take $x' \in D(A')$ such that $A'x' \in X^{\odot}$. Since $A'$ is a weak*-closed operator, it follows as in Lemma 1.3 that

$$T(t)^{\odot}x' - x' = A' \int_0^t T(s)^{\odot}x' \, ds = \int_0^t T(s)^{\odot} A'x' \, ds$$

for each $t > 0$. From the norm continuity of $s \mapsto T(s)^{\odot}A'x'$, we obtain

$$\|\cdot\|\text{-}\lim_{t\downarrow 0} \frac{1}{t}\left(T(t)^{\odot}x' - x'\right) = A'x'.$$

□

Since $D(A^{\odot})$ is dense in $X^{\odot}$ (by Theorem 1.4), it follows that

$$X^{\odot} = \overline{D(A')}.$$

Accordingly, the strongly continuous semigroup $(T(t)^{\odot})_{t\ge 0}$ is obtained by restricting $(T(t)')_{t\ge 0}$ to the closure of the domain $D(A')$ of the operator $A'$. We will encounter such a situation again, without involving dual spaces and adjoint operators, in Corollary 3.21 below. Here we conclude with two examples in which we can identify the sun duals $X^{\odot}$.

**Examples.** (i) It is easy to see that the right translations $T_r(t)$ on $L^{\infty}(\mathbb{R})$ are the adjoints of the left translations $T_l(t)$ on $X := L^1(\mathbb{R})$ (see Definition I.4.14). The largest subspace of $L^{\infty}(\mathbb{R})$ on which these translation operators form a strongly continuous semigroup for the sup-norm is the space $C_{ub}(\mathbb{R})$ of all bounded, uniformly continuous functions on $\mathbb{R}$ (cf. Exercise I.4.19.(1)), i.e.,

$$X^{\odot} = C_{ub}(\mathbb{R}).$$

In addition, one obtains from the generator $(A, D(A))$ of $(T(t))_{t\geq 0}$ as identified in the proposition in 2.10.(ii) below and Example B.9 that

$$D(A') = \big\{f \in \mathrm{L}^\infty(\mathbb{R}) : f \text{ absolutely continuous, } f' \in \mathrm{L}^\infty(\mathbb{R})\big\}$$

and

$$D(A^\odot) = \big\{f \in \mathrm{L}^\infty(\mathbb{R}) : f \in \mathrm{C}^1(\mathbb{R}), f' \in \mathrm{C_{ub}}(\mathbb{R})\big\}.$$

Recall that a function $f$ is *absolutely continuous* if and only if for every $\varepsilon > 0$ there corresponds $\delta > 0$ such that for arbitrary disjoint intervals $I_k = [a_k, b_k]$, $k = 1, \ldots, n$,

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \qquad \text{implies} \qquad \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon;$$

see [Tay85, 9-8]. The function $s \mapsto |\sin s|$ serves as an example showing that $D(A^\odot) \neq D(A')$ in general.

(ii) Now consider the left translation (semi) group on $\mathrm{C}_0(\mathbb{R})$. By the Riesz–Markov theorem (see [Rud86, Thm. 6.19]), the dual of $\mathrm{C}_0(\mathbb{R})$ is the space $\mathrm{M_b}(\mathbb{R})$ of all bounded, regular (signed or complex) Borel measures. The dual operators form the right translation semigroup on $\mathrm{M_b}(\mathbb{R})$, and the sun dual semigroup is the right translation semigroup on $X^\odot = \mathrm{L}^1(\mathbb{R})$ as a subspace of $\mathrm{M_b}(\mathbb{R})$.

The details are left as Exercise 2.8.(2), and for a continuation see Example 5.22.

**2.7 Product Semigroups.** Let $(B, D(B))$ be the generator of a second strongly continuous semigroup $(S(t))_{t\geq 0}$ commuting with $(T(t))_{t\geq 0}$. It is easy to deduce some information on the generator $(C, D(C))$ of the product semigroup $(U(t))_{t\geq 0}$, defined by $U(t) := S(t)T(t)$ for $t \geq 0$; see Paragraph I.5.15.

We first show that $D(A) \cap D(B)$ satisfies the conditions of Proposition 1.7 and so is a core for $C$.

Since $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ commute, each domain $D(A)$ and $D(B)$ is invariant under both semigroups. Hence $D(A)\cap D(B)$ is $(U(t))_{t\geq 0}$-invariant. Take $\lambda$ large enough such that $R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)\, ds$ and $R(\lambda, B) = \int_0^\infty \mathrm{e}^{-\lambda s} S(s)\, ds$. From these representations we deduce that the resolvent operators commute, i.e., $R(\lambda, A)R(\lambda, B) = R(\lambda, B)R(\lambda, A)$. Therefore, $R(\lambda, B)$ maps $D(A)$ into $D(A)$, and so $R(\lambda, B)R(\lambda, A)X$ is contained in $D(A) \cap D(B)$. Since both $R(\lambda, A)$ and $R(\lambda, B)$ are continuous and have dense range, we conclude that $D(A) \cap D(B)$ is dense in $X$, i.e., is a core for $C$.

Now, by Lemma B.16, the map $\mathbb{R}_+ \ni t \mapsto U(t)x$ is differentiable for all elements $x \in D(A) \cap D(B)$. Moreover, its derivative at $t = 0$ is

$$\left[\tfrac{d}{dt} U(t)x\right](0) = Cx = Ax + Bx,$$

which determines the generator $C$ of $(U(t))_{t\geq 0}$ on the core $D(A) \cap D(B)$.

**2.8 Exercises.** (1) Show that the operator $A^\sigma$ defined in Paragraph 2.5 is $\sigma(X', X)$-closed, $\sigma(X', X)$-densely defined, and that it coincides with the adjoint $A'$ of $A$.

(2) Work out the details for the examples in Paragraph 2.6.

(3) Let $(A, D(A))$ be the generator of the strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on the Banach space $X$ and take the semigroup $\widehat{\mathcal{T}} = (\widehat{T}(t))_{t\geq 0}$ on $\widehat{X}_{\mathcal{T}}$ from Exercise I.5.16.(3). Show that the generator $(\widehat{A}, D(\widehat{A}))$ of $\widehat{\mathcal{T}}$ is given by

$$\widehat{A}\big((x_n) + c_0(X)\big) = (Ax_n) + c_0(X),$$

$$D(\widehat{A}) = \big\{(x_n) + c_0(X) : x_n \in D(A) \text{ and } (x_n),\ (Ax_n) \in \ell^\infty_{\mathcal{T}}(X)\big\}.$$

## *b. Standard Examples*

In this subsection we return to the examples of strongly continuous semigroups introduced in Chapter I, Section 4, and identify the corresponding generators and resolvent operators. We start with multiplication semigroups for which all operators involved can be computed explicitly.

**2.9 Multiplication Semigroups.** We saw in Proposition I.4.6 (or Proposition I.4.12) that strongly continuous multiplication semigroups on spaces $\mathrm{C}_0(\Omega)$ (or $\mathrm{L}^p(\Omega,\mu)$) are multiplications by $\mathrm{e}^{tq}$, $t \geq 0$, for some continuous (or measurable) function $q : \Omega \to \mathbb{C}$ with real part (essentially) bounded above. It should be no surprise that this function also yields the generator of the semigroup.

**Lemma.** *The generator $(A, D(A))$ of a strongly continuous multiplication semigroup $(T(t))_{t\geq 0}$ on $X := \mathrm{C}_0(\Omega)$ or $X := \mathrm{L}^p(\Omega,\mu)$ defined by*

$$T_q(t)f := \mathrm{e}^{tq} \cdot f, \quad f \in X \text{ and } t \geq 0,$$

*is given by the multiplication operator*

$$Af = M_q f := q \cdot f$$

*with domain $D(A) = D(M_q) := \{f \in X : qf \in X\}$.*

Proof. Let $X := \mathrm{C}_0(\Omega)$ and take $f \in D(A)$. Then

$$\lim_{t\downarrow 0} \frac{\mathrm{e}^{tq}f - f}{t}(s) = \lim_{t\downarrow 0} \frac{\mathrm{e}^{tq(s)}f(s) - f(s)}{t} = q(s)f(s)$$

exists for all $s \in \Omega$, and we obtain $qf \in \mathrm{C}_0(\Omega)$. This shows that $D(A) \subseteq D(M_q)$ and $Af = M_q f$. Since by Theorem 1.10.(ii) and Proposition I.4.2.(iv), respectively, $A - \lambda$ and $M_q - \lambda$ are both invertible for $\lambda$ sufficiently large. This implies $A = M_q$ (use Exercise IV.1.21.(5)). The proof for $X := \mathrm{L}^p(\Omega,\mu)$ is left as Exercise 2.14.(2). ☐

This lemma, in combination with Propositions I.4.5 and I.4.6 (or Propositions I.4.11 and I.4.12), completely characterizes the generators of strongly continuous multiplication semigroups. We restate this in the following result by identifying the closed (or the essential) range of $q$ with the spectrum of $M_q$; see Proposition I.4.2.(iv) (or Proposition I.4.10.(iv)).

**Proposition.** *For an operator $(A, D(A))$ on the Banach space $\mathrm{C}_0(\Omega)$ or $\mathrm{L}^p(\Omega,\mu)$, $1 \leq p < \infty$, the following assertions are equivalent.*

(a) *$(A, D(A))$ is the generator of a strongly continuous multiplication semigroup.*

(b) *$(A, D(A))$ is a multiplication operator such that*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > w\} \subseteq \rho(A) \qquad \textit{for some } w \in \mathbb{R}.$$

The remarkable feature of this proposition is the fact that condition (b), which corresponds to the spectral condition (ii) from Theorem 1.10, already guarantees the existence of a corresponding semigroup. This is in sharp contrast to the situation for general semigroups (see Generation Theorems 3.5 and 3.8 below).

**2.10 Translation Semigroups.** As seen in Section I.4.c and Example I.5.4, the (left) translation operators

$$T_l(t)f(s) := f(s+t), \quad s, t \in \mathbb{R},$$

define a strongly continuous (semi) group on the spaces $\mathrm{C_{ub}}(\mathbb{R})$ and $\mathrm{L}^p(\mathbb{R})$, $1 \leq p < \infty$. In each case, the generator $(A, D(A))$ is given by differentiation, but we have to adapt its domain to the underlying space.

**Proposition 1.** *The generator of the (left) translation semigroup $(T_l(t))_{t\geq 0}$ on the space $X$ is given by*

$$Af := f'$$

*with domain:*

(i)

$$D(A) = \big\{ f \in \mathrm{C_{ub}}(\mathbb{R}) : f \text{ differentiable and } f' \in \mathrm{C_{ub}}(\mathbb{R}) \big\},$$

*if $X := \mathrm{C_{ub}}(\mathbb{R})$, and*

(ii)

$$D(A) = \big\{ f \in \mathrm{L}^p(\mathbb{R}) : f \text{ absolutely continuous and } f' \in \mathrm{L}^p(\mathbb{R}) \big\},$$

*if $X := \mathrm{L}^p(\mathbb{R})$, $1 \leq p < \infty$.*

Proof. It suffices to show that the generator $(B, D(B))$ of $(T_l(t))_{t\geq 0}$ is a restriction of the operator $(A, D(A))$ defined above. In fact, since $(T_l(t))_{t\geq 0}$ is a contraction semigroup on $X$, Theorem 1.10.(ii) implies $1 \in \rho(B)$. On the other hand, by Proposition 2 below, we know that $1 \in \rho(A)$, and therefore the inclusion $B \subseteq A$ will imply $A = B$ by Exercise IV.1.21.(5).

(i) Fix $f \in D(B)$. Since $\delta_0$ is a continuous linear form on $\mathrm{C_{ub}}(\mathbb{R})$, the function

$$\mathbb{R}_+ \ni t \mapsto \delta_0\big(T_l(t)f\big) = f(t)$$

is differentiable by Lemma 1.1 and Definition 1.2, and

$$Bf = \left[\tfrac{d}{dt} T_l(t) f\right]_{t=0} = \left[\tfrac{d}{dt} f(t+\cdot)\right]_{t=0} = f'.$$

This proves $D(B) \subseteq D(A)$ and $A_{|D(B)} = B$. Hence, $A = B$ as mentioned above.

(ii) Take $f \in D(B)$ and set $g := Bf \in \mathrm{L}^p(\mathbb{R})$. Since integration over compact intervals is continuous in $\mathrm{L}^p(\mathbb{R})$, we obtain for every $a, b \in \mathbb{R}$ that

$$\frac{1}{h}\int_b^{b+h} f(s)\,ds - \frac{1}{h}\int_a^{a+h} f(s)\,ds = \int_a^b \frac{f(s+h) - f(s)}{h}\,ds$$

converges to $\int_a^b g(s)\,ds$ as $h \downarrow 0$. However, the left-hand side converges to $f(b) - f(a)$ for almost all $a, b$; see [Tay85, Thm. 9-8 VI]. By redefining $f$ on a null set we obtain

$$f(b) = \int_a^b g(s)\,ds + f(a), \qquad b \in \mathbb{R},$$

which is an absolutely continuous function with derivative (almost everywhere) equal to $g$. Again this shows that $D(B) \subseteq D(A)$ and $A_{|D(B)} = B$. It follows that $A = B$ as above. ☐

In order to finish this proof, we give an explicit formula for the resolvent of the differentiation operator $A$ with "maximal" domain $D(A)$ as specified in the previous result. The simple proof is left to Exercise 2.14.(1).

**Proposition 2.** *The resolvent $R(\lambda, A)$ for $\operatorname{Re}\lambda > 0$ of the differentiation operator $A$ with maximal domain $D(A)$ (i.e., of the generator of the left translation semigroup) on any of the above spaces $X$ is given by*

$$\big(R(\lambda, A)f\big)(s) = \int_s^\infty \mathrm{e}^{-\lambda(\tau - s)} f(\tau)\, d\tau \quad \textit{for } f \in X,\ s \in \mathbb{R}. \tag{2.3}$$

Clearly, there are many other function spaces on which the translations define a strongly continuous semigroup. As soon as they are contained in $\mathrm{L}^p(\mathbb{R})$ or $\mathrm{C_{ub}}(\mathbb{R})$, for example, Proposition 2.3 allows us to identify the generator as the part of the differentiation operator. This, and the quotient construction from Paragraph 2.4, yields the generators of the translation semigroups on $\mathbb{R}_+$ and on finite intervals (see Paragraphs I.4.16 and I.4.17).

We present an example of this argument.

**2.11 Translation Semigroups (Continued).** Consider the (left) translation (semi) group from Paragraph 2.10 on the space $X := \mathrm{L}^1(\mathbb{R})$. Then the closed subspace

$$Y := \big\{ f \in \mathrm{L}^1(\mathbb{R}) : f(s) = 0 \text{ for } s \geq 1 \big\},$$

which is isomorphic to $\mathrm{L}^1(-\infty, 1)$, is $(T(t))_{t\geq 0}$-invariant. The generator of the subspace semigroup $(T(t)_|)_{t\geq 0}$ is

$$A_| f = f'$$

with domain

$$D(A_|) = \left\{ f \in \mathrm{L}^1(\mathbb{R}) : \begin{array}{l} f \text{ is absolutely continuous,} \\ f' \in \mathrm{L}^1(\mathbb{R}) \text{ and } f(s) = 0 \text{ for } s \geq 1 \end{array} \right\}.$$

In $Y$ and for the subspace semigroup $(T(t)_|)_{t\geq 0}$, the space

$$Z := \big\{ f \in Y : f(s) = 0 \text{ for } 0 \leq s \leq 1 \big\}$$

is again closed and invariant. The quotient space ${}^Y\!/_Z$ is isomorphic to $\mathrm{L}^1[0,1]$, and the quotient semigroup is isomorphic to the nilpotent (left) translation semigroup from Paragraph I.4.17. By Paragraph 2.4, we obtain for its generator $A_{|/}$ that

$$A_{|/} f = f'$$

with domain

$$D(A_{|/}) = \left\{ f \in \mathrm{L}^1[0,1] : \begin{array}{l} f \text{ is absolutely continuous,} \\ f' \in \mathrm{L}^1[0,1] \text{ and } f(1) = 0 \end{array} \right\}.$$

As above, its resolvent can be determined explicitly using (1.13). We obtain for every $\lambda \in \mathbb{C}$ that

$$\big(R(\lambda, A_{|/})f\big)(s) = \int_s^1 \mathrm{e}^{-\lambda(\tau - s)} f(\tau)\, d\tau \quad \text{for } f \in \mathrm{L}^1[0,1],\ s \in [0,1]. \tag{2.4}$$

In the previous examples we always started with an explicit semigroup and then identified its generator. In the final two examples we look at (second-order) differential operators and show by direct computation that they generate strongly continuous semigroups.

**2.12 Diffusion Semigroups (one-dimensional).** Consider the Banach space $X := \mathrm{C}[0,1]$ and the differential operator

$$Af := f''$$

with domain

$$D(A) := \big\{ f \in \mathrm{C}^2[0,1] : f'(0) = f'(1) = 0 \big\}.$$

This domain is a dense subspace of $X$ that is complete for the graph norm; hence $(A, D(A))$ is a closed, densely defined operator. Each function

$$s \mapsto \mathrm{e}_n(s) := \begin{cases} 1 & \text{if } n = 0, \\ \sqrt{2}\cos(\pi n s) & \text{if } n \geq 1, \end{cases}$$

belongs to $D(A)$ and satisfies

$$A\mathrm{e}_n = -\pi^2 n^2 \mathrm{e}_n. \tag{2.5}$$

By the Stone–Weierstrass theorem and elementary trigonometric identities we conclude that

$$Y := \operatorname{lin}\{\mathrm{e}_n : n \geq 0\} \tag{2.6}$$

is a dense subalgebra of $X$. Consider the rank-one operators

$$\mathrm{e}_n \otimes \mathrm{e}_n : f \mapsto \langle f, \mathrm{e}_n \rangle\, \mathrm{e}_n := \left( \int_0^1 f(s)\mathrm{e}_n(s)\, ds \right) \mathrm{e}_n,$$

which satisfy

$$\|\mathrm{e}_n \otimes \mathrm{e}_n\| \leq 2$$

and

$$(\mathrm{e}_n \otimes \mathrm{e}_n)\, \mathrm{e}_m = \delta_{nm} \mathrm{e}_m \tag{2.7}$$

for all $n, m \geq 0$. They can be used to define, for $t > 0$, the operators

$$T(t) := \sum_{n=0}^{\infty} \mathrm{e}^{-\pi^2 n^2 t} \cdot \mathrm{e}_n \otimes \mathrm{e}_n. \tag{2.8}$$

For $f \in \mathrm{C}[0,1]$ and $s \in [0,1]$, this means that

$$\big(T(t)f\big)(s) = \int_0^1 k_t(s,r) f(r)\, dr, \tag{2.9}$$

where

$$k_t(s,r) := 1 + 2 \sum_{n \in \mathbb{N}} \mathrm{e}^{-\pi^2 n^2 t} \cos(\pi n s) \cdot \cos(\pi n r).$$

The Jacobi identity

$$w_t(s) := \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\frac{(s+2n)^2}{4t}} = \frac{1}{2} + \sum_{n \in \mathbb{N}} \mathrm{e}^{-\pi^2 n^2 t} \cos(\pi n s)$$

(see [SD80, Kap. I, Satz 10.4]) and various trigonometric relations imply that for each $t > 0$, the kernel $k_t(\cdot,\cdot)$ satisfies

$$k_t(s,r) = w_t(s+r) + w_t(s-r).$$

Hence, $k_t(\cdot,\cdot)$ is a positive, continuous function on $[0,1]^2$, and we obtain

$$\|T(t)\| = \|T(t)\mathbb{1}\| = \sup_{s \in [0,1]} \int_0^1 k_t(s,r)\, dr = 1.$$

Using the identity (2.7), one easily verifies that on the one-dimensional subspaces generated by $\mathrm{e}_n$, $n \geq 0$, the operators $T(t)$ satisfy the semigroup law (FE), which by continuity then holds on all of $X$. Similarly, the strong continuity holds on $Y$ and hence, by Proposition I.5.3, on $X$.

These considerations already prove most of the following result.

**Proposition.** *The above operators $T(t)$, $t \geq 0$, with $T(0) = I$ form a strongly continuous semigroup on $X := \mathrm{C}[0,1]$ whose generator is given by*

$$Af = f'',$$

*with domain*

$$D(A) = \big\{ f \in \mathrm{C}^2[0,1] : f'(0) = f'(1) = 0 \big\}.$$

PROOF. It remains only to show that the generator $B$ of $(T(t))_{t\geq 0}$ coincides with $A$. To this end, we first observe that the subspace $Y$ defined by (2.6) is dense in $X$, contained in $D(B)$, and $(T(t))_{t\geq 0}$-invariant. Hence, by Proposition 1.7, it is a core for $B$. Next, using the definition of $T(t)$ and formula (2.7), it follows that $A$ and $B$ coincide on $Y$. Therefore, we obtain that $B = \overline{A_{|Y}}$ and, in particular, that $B$ is a restriction of $A$. From the theory of linear ordinary differential equations it follows that $1 \in \rho(A)$. Moreover, by Theorem 1.10.(ii), we know that $1 \in \rho(B)$, and therefore $A = B$. □

**2.13 Diffusion Semigroups ($n$-dimensional).** The following classical example was one of the main sources for the development of semigroup theory. It describes heat flow, diffusion processes, or Brownian motion and bears names like *heat semigroup*, *Gaussian semigroup*, or *diffusion semigroup*. We consider it on $X := \mathrm{L}^p(\mathbb{R}^n)$, $1 \leq p < \infty$, where it is defined explicitly by

$$T(t)f(s) := (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|s-r|^2/4t} f(r)\, dr \tag{2.10}$$

for $t > 0$, $s \in \mathbb{R}^n$, and $f \in X$. By putting

$$\mu_t(s) := (4\pi t)^{-n/2} \mathrm{e}^{-|s|^2/4t},$$

this can be written as

$$T(t)f(s) = \mu_t * f(s).$$

**Proposition.** *The above operators $T(t)$, for $t > 0$ and with $T(0) = I$, form a strongly continuous semigroup on $\mathrm{L}^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and its generator $A$ coincides with the closure of the Laplace operator*

$$\Delta f(s) := \sum_{i=1}^{n} \frac{\partial^2}{\partial s_i^2} f(s_1, \ldots, s_n)$$

*defined for every $f$ in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ (see Definition* VI.5.1*).*

PROOF. The integral defining $T(t)f(s)$ exists for every $f \in \mathrm{L}^p(\mathbb{R}^n)$, since $\mu_t \in \mathscr{S}(\mathbb{R}^n)$. Moreover,

$$\|T(t)f\|_p \leq \|\mu_t\|_1 \cdot \|f\|_p \leq \|f\|_p$$

by Young's inequality (see [RS75, p. 28]). Hence, each $T(t)$ is a contraction on $\mathrm{L}^p$. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in $\mathrm{L}^p$ and invariant under $T(t)$, it suffices to study $T(t)_{|\mathscr{S}(\mathbb{R}^n)}$. This is done using the Fourier transformation $\mathfrak{F}$, which leaves $\mathscr{S}(\mathbb{R}^n)$ invariant. By the usual properties of $\mathfrak{F}$ (see Lemma C.12 or [Rud73, Thm. 7.2]) one obtains

$$\mathfrak{F}(\mu_t * f) = \mathfrak{F}(\mu_t) \cdot \mathfrak{F}(f)$$

for each $f \in \mathscr{S}(\mathbb{R}^n)$. Since

$$\mathfrak{F}(\mu_t)(\xi) = \mathrm{e}^{-|\xi|^2 t}$$

for $\xi \in \mathbb{R}^n$, where $|\xi| := \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}$ (cf. Example VI.5.3), we see that $\mathfrak{F}$ transforms $(T(t)_{|\mathscr{S}(\mathbb{R}^n)})_{t\geq 0}$ into a multiplication semigroup on $\mathscr{S}(\mathbb{R}^n)$, which is pointwise continuous for the usual topology on $\mathscr{S}(\mathbb{R}^n)$. Moreover, direct computations as in Lemma 2.9 show that the right derivative at $t = 0$ is the multiplication operator

$$Bg(\xi) := -|\xi|^2 g(\xi)$$

for $\xi \in \mathbb{R}^n$, $g \in \mathscr{S}(\mathbb{R}^n)$. Pulling this information back via the inverse Fourier transformation shows that $(T(t))_{t\geq 0}$ satisfies the semigroup law. Since the topology of $\mathscr{S}(\mathbb{R}^n)$ is finer than the one induced from $\mathrm{L}^p(\mathbb{R}^n)$, we also obtain strong continuity on $\mathscr{S}(\mathbb{R}^n)$, hence on $\mathrm{L}^p(\mathbb{R}^n)$. Finally, we observe that the inverse Fourier transformation of the multiplication operator $B$ is the Laplace operator. Since $\mathscr{S}(\mathbb{R}^n)$ is dense and $(T(t))_{t\geq 0}$-invariant, by Proposition 1.7 we have therefore determined the generator $A$ of $(T(t))_{t\geq 0}$ on a core of its domain. □

For generalizations of this example we refer to Section VI.5.

**2.14 Exercises.** (1) Compute the resolvent operators of the generators of the various translation semigroups on $\mathbb{R}$, $\mathbb{R}_+$, or on finite intervals. In particular, deduce the resolvent representation (2.3). (Hint: Use the integral representation 1.14.) Determine from this the generator and its domain as already found in Paragraph 2.10 and Paragraph 2.11.

(2) Prove the lemma in Paragraph 2.9 for $X := \mathrm{L}^p(\Omega, \mu)$.

(3) Let $X := \mathrm{L}^\infty(\mathbb{R})$. Show that

  (i) a multiplication semigroup on $X$ is strongly continuous if and only if it is uniformly continuous, and

  (ii) the translation (semi) group is not strongly continuous.

Remark that Lotz in [Lot85] showed that a strongly continuous semigroup on a class of Banach spaces containing all $\mathrm{L}^\infty$-spaces is necessarily uniformly continuous. See also [Nag86, A-II.3].

(4*) Consider the translation (semi) group $(T(t))_{t\in\mathbb{R}}$ on $X := \mathrm{L}^\infty(\mathbb{R})$ and the closed, $(T(t))_{t\in\mathbb{R}}$-invariant subspace $Y := \mathrm{C}_{\mathrm{ub}}(\mathbb{R})$. The quotient operators $T(t)_/$ define a contraction (semi) group on ${}^X/_Y$ whose orbits $t \mapsto T(t)_/$ are continuous (differentiable) only if $\widehat{f} = 0$. Note that in this way we obtained a noncontinuous, but not pathological, solution of Problem I.3.1. (Hint: See [NP94].)

# 3. Hille–Yosida Generation Theorems

We now turn to the fundamental problem of semigroup theory, which is to find arrows in Diagram 1.14 leading from the generator (or its resolvent) to the semigroup. More precisely, this means that we will discuss the following problem.

**3.1 Problem.** *Characterize those linear operators that are the generator of some strongly continuous semigroup.*

## a. Generation of Groups and Semigroups

In Theorems 1.4 and 1.10, we already saw that generators

- are necessarily closed operators,
- have dense domain, and
- have their spectrum contained in some proper left half-plane.

These conditions, however, are not sufficient.

**3.2 Example.** On the space

$$X := \big\{ f \in \mathrm{C}_0(\mathbb{R}_+) : f \text{ continuously differentiable on } [0,1] \big\}$$

endowed with the norm

$$\|f\| := \sup_{s\in\mathbb{R}_+} |f(s)| + \sup_{s\in[0,1]} |f'(s)|,$$

we consider the operator $\big(A, D(A)\big)$ defined by

$$Af := f' \qquad \text{for } f \in D(A) := \big\{ f \in \mathrm{C}_0^1(\mathbb{R}_+) : f' \in X \big\}.$$

Then $A$ is closed and densely defined, its resolvent exists for $\operatorname{Re}\lambda > 0$, and can be expressed by

$$\big(R(\lambda, A)f\big)(s) = \int_s^\infty \mathrm{e}^{-\lambda(\tau - s)} f(\tau)\, d\tau \qquad \text{for } f \in X,\ s \geq 0$$

(compare (2.3)). Assume now that $A$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$. For $f \in D(A)$ and $0 \leq s, t$ we define

$$\xi(\tau) := \big(T(t-\tau)f\big)(s+\tau), \qquad 0 \leq \tau \leq t,$$

which is a differentiable function. Its derivative satisfies

$$\dot{\xi}(\tau) := -\big(T(t-\tau)Af\big)(s+\tau) + \big(T(t-\tau)f'\big)(s+\tau) = 0,$$

and hence

$$\big(T(t)f\big)(s) = \xi(0) = \xi(t) = f(s+t).$$

This proves that $\big(T(t)\big)_{t\geq 0}$ must be the (left) translation semigroup. The translation operators, however, do not map $X$ into itself.

This indicates that we need more assumptions on $A$, and the norm estimates

- $\|R(\lambda, A)\| \leq \frac{M}{\operatorname{Re}\lambda - w}, \quad \operatorname{Re}\lambda > w,$

proved in Theorem 1.10.(iii) may serve for this purpose.

To tackle the above problem, it is helpful to recall the results from the introductory Sections I.1–I.3 and to think of the semigroup generated by an operator $A$ as an "exponential function"

$$t \mapsto e^{tA}.$$

**3.3 Exponential Formulas.** We pursue this idea by recalling the various ways by which we can define "exponential functions." Each of these formulas and each method will then be checked for a possible generalization to infinite-dimensional Banach spaces and, in particular, to unbounded operators. Here are some more or less promising formulas for "$e^{tA}$."

**Formula (i)** As in the matrix case (see Section I.2) we might use the power series and define

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n. \tag{3.1}$$

**Comment.** For unbounded $A$, it is unrealistic to expect convergence of this series. In fact, there exist strongly continuous semigroups such that for its generator $A$ the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$$

converges only for $t = 0$ or $x = 0$. See Exercise 3.12.(2).

**Formula (ii)** As in Section I.3, for uniformly continuous semigroups we might use the Cauchy integral formula and define

$$e^{tA} := \frac{1}{2\pi i} \int_{+\partial U} e^{\lambda t} R(\lambda, A)\, d\lambda. \tag{3.2}$$

**Comment.** As already noted, the generator $A$, hence also its spectrum $\sigma(A)$, may be unbounded. Therefore, the path $+\partial U$ surrounding $\sigma(A)$ will be unbounded, and so we need extra conditions to make the integral converge. See Section 4.a for a class of semigroups for which this approach does work.

**Formula (iii)** At least in the one-dimensional case, the formulas

$$e^{tA} = \lim_{n\to\infty} \left(1 + \frac{t}{n} A\right)^n = \lim_{n\to\infty} \left(1 - \frac{t}{n} A\right)^{-n}$$

are well known (indeed Euler used them; see Section VII.3).

**Comment.** While the first formula again involves powers of the unbounded operator $A$ and therefore will rarely converge, we can rewrite the second (using the resolvent operators $R(\lambda, A) := (\lambda - A)^{-1}$) as

$$e^{tA} = \lim_{n\to\infty} \left[ {}^{n}\!/_{t} R\left({}^{n}\!/_{t}, A\right)\right]^n. \tag{3.3}$$

This yields a formula involving only powers of bounded operators. It was Hille's idea to use this formula and to prove that under appropriate conditions, the limit exists and defines a strongly continuous semigroup (cf. also Corollary III.5.5).

**Formula (iv)** Since it is well understood how to define the exponential function for bounded operators (see Section I.3), one can try to approximate $A$ by a sequence $(A_n)_{n\in\mathbb{N}}$ of bounded operators and hope that

$$\mathrm{e}^{tA} := \lim_{n\to\infty} \mathrm{e}^{tA_n} \tag{3.4}$$

exists and is a strongly continuous semigroup.

**Comment.** This was Yosida's idea and will now be examined in detail in order to obtain strongly continuous semigroups and therefore a general solution of Problem I.3.1.

We start with an important convergence property for the resolvent under the assumption that $\|\lambda R(\lambda, A)\|$ remains bounded as $\lambda \to \infty$.

**3.4 Lemma.** *Let $\big(A, D(A)\big)$ be a closed, densely defined operator. Suppose there exist $w \in \mathbb{R}$ and $M > 0$ such that $[w, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \le M$ for all $\lambda \ge w$. Then the following convergence statements hold for $\lambda \to \infty$.*

(i) *$\lambda R(\lambda, A)x \to x$ for all $x \in X$.*

(ii) *$\lambda A R(\lambda, A)x = \lambda R(\lambda, A)Ax \to Ax$ for all $x \in D(A)$.*

Proof. If $y \in D(A)$, then $\lambda R(\lambda, A)y = R(\lambda, A)Ay + y$ by (1.1) in Chapter IV. This expression converges to $y$ as $\lambda \to \infty$, since $\|R(\lambda, A)Ay\| \le {}^{M}\!/\!{}_{\lambda}\, \|Ay\|$. Since $\|\lambda R(\lambda, A)\|$ is uniformly bounded for all $\lambda \ge w$, statement (i) follows by Proposition A.3. The second statement is then an immediate consequence of the first one. □

This lemma suggests immediately which bounded operators $A_n$ should be chosen to approximate the unbounded operator $A$. Since for contraction semigroups the technical details of the subsequent proof become much easier (and since the general case can then be deduced from this one), we first give the characterization theorem for generators in this special case.

**3.5 Generation Theorem.** (*Contraction Case,* Hille, Yosida, 1948). *For a linear operator $\big(A, D(A)\big)$ on a Banach space $X$, the following properties are all equivalent.*

(a) *$\big(A, D(A)\big)$ generates a strongly continuous contraction semigroup.*

(b) *$\big(A, D(A)\big)$ is closed, densely defined, and for every $\lambda > 0$ one has $\lambda \in \rho(A)$ and*

$$\|\lambda R(\lambda, A)\| \le 1. \tag{3.5}$$

(c) $\big(A, D(A)\big)$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > 0$ *one has* $\lambda \in \rho(A)$ *and*

$$\|R(\lambda, A)\| \le \frac{1}{\operatorname{Re}\lambda}. \tag{3.6}$$

PROOF. In view of Theorem 1.4 and Theorem 1.10, it suffices to show (b) $\Rightarrow$ (a). To that purpose, we define the so-called *Yosida approximants*

$$A_n := nAR(n, A) = n^2 R(n, A) - nI, \tag{3.7}$$

which are bounded operators for each $n \in \mathbb{N}$ and commute with one another. Consider then the uniformly continuous semigroups given by

$$T_n(t) := \mathrm{e}^{tA_n}, \quad t \ge 0. \tag{3.8}$$

Since $A_n$ converges to $A$ pointwise on $D(A)$ (by Lemma 3.4.(ii)), we anticipate that the following properties hold.

(i) $T(t)x := \lim_{n\to\infty} T_n(t)x$ exists for each $x \in X$.

(ii) $\big(T(t)\big)_{t\ge 0}$ is a strongly continuous semigroup on $X$.

(iii) This semigroup has generator $\big(A, D(A)\big)$.

By establishing these statements we will complete the proof.

(i) Each $\big(T_n(t)\big)_{t\ge 0}$ is a contraction semigroup, since

$$\|T_n(t)\| \le \mathrm{e}^{-nt}\mathrm{e}^{\|n^2 R(n,A)\|t} \le \mathrm{e}^{-nt}\mathrm{e}^{nt} = 1 \qquad \text{for } t \ge 0.$$

So, again by Proposition A.3, it suffices to prove convergence just on $D(A)$. By (the vector-valued version of) the fundamental theorem of calculus, applied to the functions

$$s \mapsto T_m(t-s)T_n(s)x$$

for $0 \le s \le t$, $x \in D(A)$, and $m, n \in \mathbb{N}$, and using the mutual commutativity of the semigroups $\big(T_n(t)\big)_{t\ge 0}$ for all $n \in \mathbb{N}$, one has

$$\begin{aligned} T_n(t)x - T_m(t)x &= \int_0^t \tfrac{d}{ds}\big(T_m(t-s)T_n(s)x\big)\, ds \\ &= \int_0^t T_m(t-s)T_n(s)(A_n x - A_m x)\, ds. \end{aligned}$$

Accordingly,

$$\|T_n(t)x - T_m(t)x\| \le t\, \|A_n x - A_m x\|. \tag{3.9}$$

By Lemma 3.4.(ii), $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence for each $x \in D(A)$. Therefore, $\big(T_n(t)x\big)_{n\in\mathbb{N}}$ converges uniformly on each interval $[0, t_0]$.

(ii) The pointwise convergence of $\big(T_n(t)x\big)_{n\in\mathbb{N}}$ implies that the limit family $\big(T(t)\big)_{t\geq 0}$ satisfies the functional equation (FE), hence is a semigroup, and consists of contractions. Moreover, for each $x \in D(A)$, the corresponding orbit map

$$\xi : t \mapsto T(t)x, \quad 0 \leq t \leq t_0,$$

is the uniform limit of continuous functions (use (3.9)) and so is continuous itself. This suffices to obtain strong continuity via Proposition I.5.3.

(iii) Denote by $\big(B, D(B)\big)$ the generator of $\big(T(t)\big)_{t\geq 0}$ and fix $x \in D(A)$. On each compact interval $[0, t_0]$, the functions

$$\xi_n : t \mapsto T_n(t)x$$

converge uniformly to $\xi(\cdot)$ by (3.9), while the differentiated functions

$$\dot{\xi}_n : t \mapsto T_n(t)A_n x$$

converge uniformly to

$$\eta : t \mapsto T(t)Ax.$$

This implies differentiability of $\xi$ with $\dot{\xi}(0) = \eta(0)$, i.e., $D(A) \subset D(B)$ and $Ax = Bx$ for $x \in D(A)$.

Now choose $\lambda > 0$. Then $\lambda - A$ is a bijection from $D(A)$ onto $X$, since $\lambda \in \rho(A)$ by assumption. On the other hand, $B$ generates a contraction semigroup, and so $\lambda \in \rho(B)$ by Theorem 1.10. Hence, $\lambda - B$ is also a bijection from $D(B)$ onto $X$. But we have seen that $\lambda - B$ coincides with $\lambda - A$ on $D(A)$. This is possible only if $D(A) = D(B)$ and $A = B$ (see Exercise IV.1.21.(5)). □

If a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with generator $A$ satisfies, for some $w \in \mathbb{R}$, an estimate

$$\|T(t)\| \leq \mathrm{e}^{wt} \qquad \text{for } t \geq 0,$$

then we can apply the above characterization to the rescaled contraction semigroup given by

$$S(t) := \mathrm{e}^{-wt}T(t) \qquad \text{for } t \geq 0.$$

Since the generator of $\big(S(t)\big)_{t\geq 0}$ is $B = A - w$ (see Paragraph 2.2), Generation Theorem 3.5 takes the following form.

**3.6 Corollary.** *Let $w \in \mathbb{R}$. For a linear operator $\big(A, D(A)\big)$ on a Banach space $X$ the following conditions are equivalent.*

(a) *$\big(A, D(A)\big)$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq \mathrm{e}^{wt} \qquad \text{for } t \geq 0. \tag{3.10}$$

(b) *$\big(A, D(A)\big)$ is closed, densely defined, and for each $\lambda > w$ one has $\lambda \in \rho(A)$ and*

$$\|(\lambda - w)R(\lambda, A)\| \leq 1. \tag{3.11}$$

(c) *$\big(A, D(A)\big)$ is closed, densely defined, and for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > w$ one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re}\lambda - w}. \tag{3.12}$$

Semigroups satisfying (3.10) are also called *quasicontractive*.

Note, by Paragraph 3.11 below, that an operator $A$ generates a strongly continuous *group* if and only if both $A$ and $-A$ are generators. Therefore, we can combine the conditions of the Generation Theorem 3.5 for $A$ and $-A$ simultaneously and obtain a characterization of generators of contraction groups, i.e., of groups of isometries.

**3.7 Corollary.** *For a linear operator $\big(A, D(A)\big)$ on a Banach space $X$ the following properties are equivalent.*

(a) *$\big(A, D(A)\big)$ generates a strongly continuous group of isometries.*

(b) *$\big(A, D(A)\big)$ is closed, densely defined, and for every $\lambda \in \mathbb{R} \setminus \{0\}$ one has $\lambda \in \rho(A)$ and*

$$\|\lambda R(\lambda, A)\| \leq 1. \tag{3.13}$$

(c) *$\big(A, D(A)\big)$ is closed, densely defined, and for every $\lambda \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$ one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{|\operatorname{Re}\lambda|}. \tag{3.14}$$

It is now a pleasant surprise that the characterization of generators of arbitrary strongly continuous semigroups can be deduced from the above result for contraction semigroups. However, norm estimates for *all* powers of the resolvent are needed.

**3.8 Generation Theorem.** (*General Case,* Feller, Miyadera, Phillips, 1952). *Let $\big(A, D(A)\big)$ be a linear operator on a Banach space $X$ and let $w \in \mathbb{R}$, $M \geq 1$ be constants. Then the following properties are equivalent.*

(a) *$\big(A, D(A)\big)$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq M\mathrm{e}^{wt} \qquad \text{for } t \geq 0. \tag{3.15}$$

(b) *$\big(A, D(A)\big)$ is closed, densely defined, and for every $\lambda > w$ one has $\lambda \in \rho(A)$ and*

$$\big\|\big[(\lambda - w)R(\lambda, A)\big]^n\big\| \leq M \qquad \text{for all } n \in \mathbb{N}. \tag{3.16}$$

(c) *$\big(A, D(A)\big)$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > w$ one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - w)^n} \qquad \text{for all } n \in \mathbb{N}. \tag{3.17}$$

Proof. The implication (a) ⇒ (c) has been proved in Corollary 1.11, while (c) ⇒ (b) is trivial. To prove (b) ⇒ (a) we use, as for Corollary 3.6, the rescaling technique from Paragraph 2.2. So, without loss of generality, we assume that $w = 0$, i.e.,

$$\|\lambda^n R(\lambda, A)^n\| \leq M \qquad \text{for all } \lambda > 0,\ n \in \mathbb{N}.$$

For every $\mu > 0$, define a new norm on $X$ by

$$\|x\|_\mu := \sup_{n\geq 0} \|\mu^n R(\mu, A)^n x\|.$$

These norms have the following properties.

(i) $\|x\| \leq \|x\|_\mu \leq M\|x\|$, i.e., they are all equivalent to $\|\cdot\|$.

(ii) $\|\mu R(\mu, A)\|_\mu \leq 1$.

(iii) $\|\lambda R(\lambda, A)\|_\mu \leq 1$ for all $0 < \lambda \leq \mu$.

(iv) $\|\lambda^n R(\lambda, A)^n x\| \leq \|\lambda^n R(\lambda, A)^n x\|_\mu \leq \|x\|_\mu$ for all $0 < \lambda \leq \mu$ and $n \in \mathbb{N}$.

(v) $\|x\|_\lambda \leq \|x\|_\mu$ for $0 < \lambda \leq \mu$.

We give the proof only of (iii). Due to the Resolvent Equation IV.1.2, we have that

$$y := R(\lambda, A)x = R(\mu, A)x + (\mu-\lambda)R(\mu, A)R(\lambda, A)x = R(\mu, A)(x + (\mu-\lambda)y).$$

This implies, by using (ii), that

$$\|y\|_\mu \leq \frac{1}{\mu}\|x\|_\mu + \frac{\mu - \lambda}{\mu}\|y\|_\mu, \qquad \text{whence} \qquad \lambda\|y\|_\mu \leq \|x\|_\mu.$$

On the basis of these properties one can define still another norm by

$$\llbracket x \rrbracket := \sup_{\mu>0} \|x\|_\mu\,, \tag{3.18}$$

which evidently satisfies

(vi) $\|x\| \le \llbracket x \rrbracket \le M\,\|x\|$ and

(vii) $\llbracket \lambda R(\lambda, A) \rrbracket \le 1$ for all $\lambda > 0$.

Thus, the operator $\big(A, D(A)\big)$ satisfies condition (3.5) for the equivalent norm $\llbracket\cdot\rrbracket$ and so, by the Generation Theorem 3.5, generates a $\llbracket\cdot\rrbracket$-contraction semigroup $\big(T(t)\big)_{t\ge0}$. Using (vi) again, we obtain $\|T(t)\| \le M$. □

**3.9 Comment.** As a general rule, we point out that for an operator $\big(A, D(A)\big)$ to be a generator one needs

- conditions on the location of the spectrum $\sigma(A)$ in some left half-plane and
- growth estimates of the form
$$\|R(\lambda, A)^n\| \le \frac{M}{(\operatorname{Re}\lambda - w)^n}$$
for *all* powers of the resolvent $R(\lambda, A)$ in some right half-plane (or on some semiaxis $(w, \infty)$).

This last condition is rather complicated and can be checked only for nontrivial examples in the (quasi) contraction case, i.e., only if $n = 1$ is sufficient as in Generation Theorem 3.5 and Corollary 3.6.

On the other hand, every strongly continuous semigroup can be rescaled (see Paragraph I.5.11) to become bounded. For a bounded semigroup, we can find an equivalent norm making it a contraction semigroup. This does not help much in concrete examples, since only in rare cases will it be possible to compute this new norm. However, this fact is extremely helpful in abstract considerations and will be stated explicitly.

**3.10 Lemma.** *Let $\big(T(t)\big)_{t\ge0}$ be a bounded strongly continuous semigroup on a Banach space $X$. Then the norm*

$$|\!|\!| x |\!|\!| := \sup_{t\ge0} \|T(t)x\|, \qquad x \in X,$$

*is equivalent to the original norm on $X$, and $\big(T(t)\big)_{t\ge0}$ becomes a contraction semigroup on $\big(X, |\!|\!|\cdot|\!|\!|\big)$.*

The proof is left as Exercise 3.12.(1).

**3.11 Generators of Groups.** This paragraph is devoted to the question of which operators are generators of strongly continuous *groups* (see the explanation following Definition I.5.1). In order to make this more precise we first adapt Definition 1.2 to this situation.

**Definition.** *The generator* $A : D(A) \subseteq X \to X$ *of a strongly continuous group* $\big(T(t)\big)_{t\in\mathbb{R}}$ *on a Banach space* $X$ *is the operator*

$$Ax := \lim_{h\to 0} \tfrac{1}{h}\big(T(h)x - x\big)$$

*defined for every* $x$ *in its domain*

$$D(A) := \Big\{x \in X : \lim_{h\to 0} \tfrac{1}{h}\big(T(h)x - x\big) \text{ exists}\Big\}.$$

Given a strongly continuous group $\big(T(t)\big)_{t\in\mathbb{R}}$ with generator $\big(A, D(A)\big)$, we can define $T_+(t) := T(t)$ and $T_-(t) := T(-t)$ for $t \geq 0$. Then, from the previous definition, it is clear that $\big(T_+(t)\big)_{t\geq 0}$ and $\big(T_-(t)\big)_{t\geq 0}$ are strongly continuous semigroups with generators $A$ and $-A$, respectively. Therefore, if $A$ is the generator of a group, then both $A$ and $-A$ generate strongly continuous semigroups. The next result shows that the converse of this statement is also true.

**Generation Theorem for Groups.** *Let* $w \in \mathbb{R}$ *and* $M \geq 1$ *be constants. For a linear operator* $\big(A, D(A)\big)$ *on a Banach space* $X$ *the following properties are equivalent.*

(a) $\big(A, D(A)\big)$ *generates a strongly continuous group* $\big(T(t)\big)_{t\in\mathbb{R}}$ *satisfying the growth estimate*

$$\|T(t)\| \leq M\mathrm{e}^{w|t|} \qquad \text{for } t \in \mathbb{R}.$$

(b) $\big(A, D(A)\big)$ *and* $\big(-A, D(A)\big)$ *are the generators of strongly continuous semigroups* $\big(T_+(t)\big)_{t\geq 0}$ *and* $\big(T_-(t)\big)_{t\geq 0}$*, respectively, which satisfy*

$$\|T_+(t)\|,\ \|T_-(t)\| \leq M\mathrm{e}^{wt} \qquad \text{for all } t \geq 0.$$

(c) $\big(A, D(A)\big)$ *is closed, densely defined, and for every* $\lambda \in \mathbb{R}$ *with* $|\lambda| > w$ *one has* $\lambda \in \rho(A)$ *and*

$$\big\|\big[(|\lambda| - w)R(\lambda, A)\big]^n\big\| \leq M \qquad \text{for all } n \in \mathbb{N}.$$

(d) $\big(A, D(A)\big)$ *is closed, densely defined, and for every* $\lambda \in \mathbb{C}$ *with* $|\operatorname{Re}\lambda| > w$ *one has* $\lambda \in \rho(A)$ *and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{\big(|\operatorname{Re}\lambda| - w\big)^n} \qquad \text{for all } n \in \mathbb{N}. \tag{3.19}$$

Proof. (a) implies (b) as already mentioned above.

(b) ⇒ (d). We first recall, by Theorem 1.4, that the generator $\big(A, D(A)\big)$ is closed and densely defined. Moreover, using the assumptions on $A$, we obtain from Generation Theorem 3.8 the estimate (3.19) for the case $\operatorname{Re}\lambda > w$. In order to verify (3.19) for $\operatorname{Re}\lambda < -w$, observe that $R(-\lambda, A) = -R(\lambda, -A)$ for all $\lambda \in -\rho(A) = \rho(-A)$. Then, using the conditions on $-A$, the required estimate follows as above.

Since the implication (d) ⇒ (c) is trivial, it suffices to prove that (c) ⇒ (a). To this end we first note, by Generation Theorem 3.8, that both $A$ and $-A$ are generators of strongly continuous semigroups $\big(T_+(t)\big)_{t\geq 0}$ and $\big(T_-(t)\big)_{t\geq 0}$, respectively, which satisfy $\|T_\pm(t)\| \leq M\mathrm{e}^{wt}$ for $t \geq 0$. Moreover, the Yosida approximants (cf. (3.7)) $A_{+,n}$ and $A_{-,n}$ of $A$ and $-A$, respectively, commute. Since, as in the contractive case (cf. (i)–(iii) in the proof of Generation Theorem 3.5, p. 74), we have

$$T_+(t)x = \lim_{n\to\infty} \exp(tA_{+,n})x \qquad \text{and} \qquad T_-(t)x = \lim_{n\to\infty} \exp(tA_{-,n})x$$

for all $x \in X$, we see that $\big(T_+(t)\big)_{t\geq 0}$ and $\big(T_-(t)\big)_{t\geq 0}$ commute. Hence, by what was shown in Paragraph 2.7, the products

$$U(t) := T_+(t)T_-(t), \quad t \geq 0,$$

define a strongly continuous semigroup with generator $C$ that satisfies

$$Cx = Ax - Ax = 0$$

for all $x \in D(A) \cap D(-A) = D(A) \subset D(C)$. From (1.6) in Lemma 1.3 we then obtain $U(t)x = x$ for all $x \in X$, i.e., $T_-(t) = T_+(t)^{-1}$. Finally, the operators

$$T(t) := \begin{cases} T_+(t) & \text{if } t \geq 0, \\ T_-(-t) & \text{if } t < 0, \end{cases}$$

form a one-parameter group $\big(T(t)\big)_{t\in\mathbb{R}}$ and satisfy the estimate $\|T(t)\| \leq M\mathrm{e}^{w|t|}$. Since the map $\mathbb{R} \ni t \mapsto T(t)$ is strongly continuous if and only if it is strongly continuous at some arbitrary point $t_0 \in \mathbb{R}$, the group $\big(T(t)\big)_{t\in\mathbb{R}}$ is strongly continuous. This completes the proof. □

The following result is quite useful in order to check whether a given semigroup can be embedded in a group.

**Proposition.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$. If there exists some $t_0 > 0$ such that $T(t_0)$ is invertible, then $\big(T(t)\big)_{t\geq 0}$ can be embedded in a group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $X$.*

PROOF. First, we show that $T(t)$ is invertible for all $t \geq 0$. This follows for $t \in [0, t_0]$ from

$$T(t_0) = T(t_0 - t)T(t) = T(t)T(t_0 - t),$$

since by assumption, $T(t_0)$ is bijective. If $t \geq t_0$, we write $t = nt_0 + s$ for $n \in \mathbb{N}$, $s \in [0, t_0)$ and conclude from

$$T(t) = T(t_0)^n T(s)$$

that $T(t)$ is invertible. Hence, we can extend $\big(T(t)\big)_{t\geq 0}$ to all of $\mathbb{R}$ by

$$T(t) := T(-t)^{-1} \qquad \text{for } t \leq 0,$$

thereby obtaining a group $\big(T(t)\big)_{t\in\mathbb{R}}$. Since the map $\mathbb{R} \ni t \mapsto T(t)$ is strongly continuous if and only if it is strongly continuous at some arbitrary point, the proof is complete. □

**3.12 Exercises.** (1) Prove Lemma 3.10 concerning the renorming of bounded semigroups.

(2) For a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$ on a Banach space $X$, we call a vector $x \in D(A^\infty)$ *entire* if the power series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x \tag{3.20}$$

converges for every $t \in \mathbb{R}$. Show the following properties.

(i) If $x$ is an entire vector of $(T(t))_{t\geq 0}$, then $T(t)x$ is given by (3.20) for every $t \geq 0$.

(ii) If $(T(t))_{t\geq 0}$ is nilpotent, then the set of entire vectors consists of $x = 0$ only.

(iii) If $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group, then the set of entire vectors is dense in $X$. Moreover, if $x$ is an entire vector of $(T(t))_{t\in\mathbb{R}}$, then $T(t)x$ is given by (i) for every $t \in \mathbb{R}$. (Hint: For given $x \in X$ consider the sequence $x_n := \left({}^n\!/_{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-ns^2/2} T(s)x\, ds$. See also [Gel39].)

(3) Let $M_q$ be a multiplication operator on $X := \mathrm{C}_0(\mathbb{R}_+)$ and define the operator $\mathcal{A} := \left(\begin{smallmatrix} M_q & M_q \\ 0 & M_q \end{smallmatrix}\right)$ with domain $D(\mathcal{A}) := D(M_q) \times D(M_q)$ on $\mathfrak{X} := X \times X$.

(i) If $q(s) := \mathrm{i}s$, $s \geq 0$, then $\mathcal{A}$ satisfies $\|R(\lambda, \mathcal{A})\| \leq {}^2\!/_\lambda$ for $\lambda > 0$, but is *not* the generator of a strongly continuous semigroup on $\mathfrak{X}$.

(ii) Find an unbounded function $q$ on $\mathbb{R}_+$ such that $\mathcal{A}$ becomes a generator.

(iii) Find necessary and sufficient conditions on $q$ such that $\mathcal{A}$ becomes a generator on $\mathfrak{X}$. (Hint: Compare Exercise 4.12.(7).)

(4) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$. Show that $(T(t))_{t\geq 0}$ can be embedded in a group $(T(t))_{t\in\mathbb{R}}$ if there exists $t_0 > 0$ such that $I - T(t_0)$ is compact. (Hint: By the proposition in Paragraph 3.11 and the compactness assumption, it suffices to show that 0 is not an eigenvalue of $T(t_0)$.)

## b. Dissipative Operators and Contraction Semigroups

Due to their importance, we now return to the study of contraction semigroups and look for a characterization of their generator that does not require explicit knowledge of the resolvent. The following is a key notion towards this goal.

**3.13 Definition.** *A linear operator $\big(A, D(A)\big)$ on a Banach space $X$ is called dissipative if*

$$\|(\lambda - A)x\| \geq \lambda \|x\| \tag{3.21}$$

*for all $\lambda > 0$ and $x \in D(A)$.*

To familiarize ourselves with these operators we state some of their basic properties.

**3.14 Proposition.** *For a dissipative operator $\big(A, D(A)\big)$ the following properties hold.*

(i) *$\lambda - A$ is injective for all $\lambda > 0$ and*

$$\big\|(\lambda - A)^{-1}z\big\| \leq \frac{1}{\lambda}\|z\|$$

*for all $z$ in the range $\operatorname{rg}(\lambda - A) := (\lambda - A)D(A)$.*

(ii) *$\lambda - A$ is surjective for some $\lambda > 0$ if and only if it is surjective for each $\lambda > 0$. In that case, one has $(0, \infty) \subset \rho(A)$.*

(iii) *$A$ is closed if and only if the range $\operatorname{rg}(\lambda - A)$ is closed for some (hence all) $\lambda > 0$.*

(iv) *If $\operatorname{rg}(A) \subseteq \overline{D(A)}$, e.g., if $A$ is densely defined, then $A$ is closable. Its closure $\overline{A}$ is again dissipative and satisfies $\operatorname{rg}(\lambda - \overline{A}) = \overline{\operatorname{rg}(\lambda - A)}$ for all $\lambda > 0$.*

Proof. (i) is just a reformulation of estimate (3.21).

To show (ii) we assume that $(\lambda_0 - A)$ is surjective for some $\lambda_0 > 0$. In combination with (i), this yields $\lambda_0 \in \rho(A)$ and $\|R(\lambda_0, A)\| \leq 1/\lambda_0$. The series expansion for the resolvent (see Proposition IV.1.3.(i)) yields $(0, 2\lambda_0) \subset \rho(A)$, and the dissipativity of $A$ implies that

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}$$

for $0 < \lambda < 2\lambda_0$. Proceeding in this way, we see that $\lambda - A$ is surjective for all $\lambda > 0$, and therefore $(0, \infty) \subset \rho(A)$.

(iii) The operator $A$ is closed if and only if $\lambda - A$ is closed for some (hence all) $\lambda > 0$. This is again equivalent to

$$(\lambda - A)^{-1} : \mathrm{rg}(\lambda - A) \to D(A)$$

being closed. By (i), this operator is bounded. Hence, by Theorem B.6, it is closed if and only if its domain, i.e., $\mathrm{rg}(\lambda - A)$, is closed.

(iv) Take a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ satisfying $x_n \to 0$ and $Ax_n \to y$. By Proposition B.4, we have to show that $y = 0$. The inequality (3.21) implies that

$$\|\lambda(\lambda - A)x_n + (\lambda - A)w\| \ge \lambda\, \|\lambda x_n + w\|$$

for every $w \in D(A)$ and all $\lambda > 0$. Passing to the limit as $n \to \infty$ yields

$$\|-\lambda y + (\lambda - A)w\| \ge \lambda\, \|w\|, \quad \text{and hence} \quad \left\|-y + w - \frac{1}{\lambda}Aw\right\| \ge \|w\|.$$

For $\lambda \to \infty$ we obtain that

$$\|-y + w\| \ge \|w\|,$$

and by choosing $w$ from the domain $D(A)$ arbitrarily close to $y \in \overline{\mathrm{rg}(A)}$, we see that

$$0 \ge \|y\|.$$

Hence $y = 0$.

In order to verify that $\overline{A}$ is dissipative, take $x \in D\left(\overline{A}\right)$. By definition of the closure of a linear operator, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ satisfying $x_n \to x$ and $Ax_n \to \overline{A}x$ when $n \to \infty$. Since $A$ is dissipative and the norm is continuous, this implies that $\|(\lambda - \overline{A})x\| \ge \lambda\|x\|$ for all $\lambda > 0$. Hence $\overline{A}$ is dissipative. Finally, observe that the range $\mathrm{rg}(\lambda - A)$ is dense in $\mathrm{rg}\left(\lambda - \overline{A}\right)$. Since by assertion (iii) $\mathrm{rg}\left(\lambda - \overline{A}\right)$ is closed in $X$, we obtain the final assertion in (iv). □

From the resolvent estimate (3.5) in Generation Theorem 3.5, it is evident that the generator of a contraction semigroup satisfies the estimate (3.21), and hence is dissipative. On the other hand, many operators can be shown directly to be dissipative and densely defined. We therefore reformulate Generation Theorem 3.5 in such a way as to single out the property that ensures that a densely defined, dissipative operator is a generator.

**3.15 Theorem.** (Lumer, Phillips, 1961). *For a densely defined, dissipative operator $\big(A, D(A)\big)$ on a Banach space $X$ the following statements are equivalent.*

(a) *The closure $\overline{A}$ of $A$ generates a contraction semigroup.*

(b) $\mathrm{rg}(\lambda - A)$ *is dense in $X$ for some (hence all) $\lambda > 0$.*

Proof. (a) ⇒ (b). Generation Theorem 3.5 implies that $\mathrm{rg}(\lambda - \overline{A}) = X$ for all $\lambda > 0$. Since $\mathrm{rg}(\lambda - \overline{A}) = \overline{\mathrm{rg}(\lambda - A)}$, by Proposition 3.14.(iv), we obtain (b).

(b) ⇒ (a). By the same argument, the density of the range $\operatorname{rg}(\lambda - A)$ implies that $(\lambda - \overline{A})$ is surjective. Proposition 3.14.(ii) shows that $(0,\infty) \subset \rho(\overline{A})$, and dissipativity of $A$ implies the estimate

$$\|R(\lambda, \overline{A})\| \le \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

This was required in Generation Theorem 3.5 to assure that $\overline{A}$ generated a contraction semigroup. □

The above theorem gains its significance when viewed in the context of the abstract Cauchy problem associated to an operator $A$ (see Section 6).

**3.16 Remark.** Assume that the operator $A$ is known to be closed, densely defined, and dissipative. Then Theorem 3.15 in combination with Proposition 6.2 below yields the following fact.

*In order to solve the (time-dependent) initial value problem*

$$\text{(ACP)} \qquad \dot{x}(t) = Ax(t),\ x(0) = x$$

*for all $x \in D(A)$, it is sufficient to solve the (stationary) resolvent equation*

$$\text{(RE)} \qquad x - Ax = y$$

*for all $y$ in some dense subset in the Banach space $X$.*

In many examples (RE) can be solved explicitly while (ACP) cannot, cf. Paragraph 3.29 or Section VI.6.

The following result, in combination with the characterization of dissipativity in Proposition 3.23 below, gives an even simpler condition for an operator to generate a contraction semigroup.

**3.17 Corollary.** *Let $(A, D(A))$ be a densely defined operator on a Banach space $X$. If both $A$ and its adjoint $A'$ are dissipative, then the closure $\overline{A}$ of $A$ generates a contraction semigroup on $X$.*

Proof. By the Lumer–Phillips Theorem 3.15, it suffices to show that the range $\operatorname{rg}(I - A)$ is dense in $X$. By way of contradiction, assume that $\overline{\operatorname{rg}(I - A)} \neq X$. By the Hahn–Banach theorem there exists $0 \neq x' \in X'$ such that

$$\langle (I - A)x, x' \rangle = 0 \qquad \text{for all } x \in D(A).$$

It follows that $x' \in D(A')$ and

$$\langle x, (I - A')x' \rangle = 0 \qquad \text{for all } x \in D(A).$$

Since $D(A)$ is dense in $X$, we conclude that $(I - A')x' = 0$, thereby contradicting Proposition 3.14.(i). □

At this point we insert various considerations emphasizing the density of the domain, which up to now was a more or less standard assumption in our results. In the next two corollaries we show how dissipativity can be used to get around this hypothesis. However, based on the properties stated in Proposition 3.14, we assume that the dissipative operator $A$ is such that $\lambda - A$ is surjective for some $\lambda > 0$. Hence $(0, \infty) \subset \rho(A)$.

**3.18 Corollary.** *Let $(A, D(A))$ be a dissipative operator on the Banach space $X$ such that $\lambda - A$ is surjective for some $\lambda > 0$. Then the part $A_|$ of $A$ in the subspace $X_0 := \overline{D(A)}$ is densely defined and generates a contraction semigroup in $X_0$.*

PROOF. We recall from Definition 2.3 that

$$A_| x := Ax$$

for $x \in D(A_|) := \{x \in D(A) : Ax \in X_0\} = R(\lambda, A)X_0$. Since $R(\lambda, A)$ exists for $\lambda > 0$, this implies that $R(\lambda, A)_| = R(\lambda, A_|)$. Hence $(0, \infty) \subset \rho(A_|)$. Due to the Generation Theorem 3.5, it remains to show that $D(A_|)$ is dense in $X_0$. Take $x \in D(A)$ and set $x_n := nR(n, A)x$. Then $x_n \in D(A)$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} R(n, A)Ax + x = x$, since $\|R(n, A)\| \le 1/n$ (see Proposition 3.14.(i) and Lemma 3.4). Therefore, the operators $nR(n, A)$ converge pointwise on $D(A)$ to the identity. Since $\|nR(n, A)\| \le 1$ for all $n \in \mathbb{N}$, we obtain convergence of

$$y_n := nR(n, A)y \to y$$

for all $y \in X_0$. Since each $y_n$ is in $D(A_|)$, the density of $D(A_|)$ in $X_0$ is proved. □

We now give two rather typical examples for dissipative operators with non-dense domains, one concrete and one abstract.

**3.19 Examples.** (i) Let $X := \mathrm{C}[0, 1]$ and consider the operator

$$Af := -f'$$

with domain

$$D(A) := \left\{f \in \mathrm{C}^1[0, 1] : f(0) = 0\right\}.$$

It is a closed operator whose domain is not dense. However, it is dissipative, since its resolvent can be computed explicitly as

$$R(\lambda, A)f(t) := \int_0^t \mathrm{e}^{-\lambda(t-s)} f(s)\, ds$$

for $t \in [0, 1]$, $f \in \mathrm{C}[0, 1]$. Moreover,

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}$$

for all $\lambda > 0$. Therefore, $(A, D(A))$ is dissipative.

Let $X_0 := \overline{D(A)} = \{f \in \mathrm{C}[0,1] : f(0) = 0\}$, and consider the part $A_|$ of $A$ in $X_0$, that is

$$\begin{aligned} A_| f &= -f', \\ D(A_|) &= \big\{ f \in \mathrm{C}^1[0,1] : f(0) = f'(0) = 0 \big\}. \end{aligned}$$

By the above corollary, this operator generates a semigroup on $X_0$. In fact, this semigroup $(T_0(t))_{t\geq 0}$ can be identified as the nilpotent right translation semigroup, cf. Paragraph I.4.17, given by

$$T_0(t)f(s) := \begin{cases} f(s-t) & \text{for } t \leq s, \\ 0 & \text{for } t > s. \end{cases}$$

Observe that the same definition applied to an arbitrary function $f \in \mathrm{C}[0,1]$ does not yield necessarily a continuous function again. Therefore, the semigroup $(T_0(t))_{t\geq 0}$ does not extend to the space $\mathrm{C}[0,1]$.

(ii) Consider a strongly continuous contraction semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Its generator $A$ is dissipative with $(0,\infty) \subset \rho(A)$. The same holds for its adjoint $A'$, since $R(\lambda, A') = R(\lambda, A)'$ and $\|R(\lambda, A')\| = \|R(\lambda, A)\|$ for all $\lambda > 0$. The domain $D(A')$ of the adjoint is not dense in $X'$ in general (see the example in Paragraph 2.6). However, taking the part of $A'$ in $X^{\odot} := \overline{D(A')} \subset X'$, we obtain the generator of a contraction semigroup (given by the restrictions of $T(t)'$ to $X^{\odot}$; see Paragraph 2.6 on sun dual semigroups).

In the next corollary we show that the phenomenon discussed in Corollary 3.18 and Example 3.19 cannot occur in reflexive Banach spaces.

**3.20 Corollary.** *Let $(A, D(A))$ be a dissipative operator on a reflexive Banach space such that $\lambda - A$ is surjective for some $\lambda > 0$. Then $A$ is densely defined and generates a contraction semigroup.*

PROOF. We only have to show the density of $D(A)$. Take $x \in X$ and define $x_n := nR(n, A)x \in D(A)$. The element $y := R(1, A)x$ also belongs to $D(A)$. Moreover, by the proof of Corollary 3.18 the operators $nR(n, A)$ converge towards the identity pointwise on $X_0 := \overline{D(A)}$. It follows that

$$y_n := R(1, A)x_n = nR(n, A)R(1, A)x \to y \quad \text{for } n \to \infty.$$

Since $X$ is reflexive and $\{x_n : n \in \mathbb{N}\}$ is bounded, there exists a subsequence, still denoted by $(x_n)_{n\in\mathbb{N}}$, that converges weakly to some $z \in X$. Since $x_n \in D(A)$, Proposition A.1.(i) implies that $z \in \overline{D(A)}$. On the other hand, the elements $x_n = (1 - A)y_n$ converge weakly to $z$, so the weak closedness of $A$ (see Definition B.1) implies that $y \in D(A)$ and $x = (1 - A)y = z \in \overline{D(A)}$. □

In Corollary 3.18 and Corollary 3.20, we considered not necessarily densely defined operators and showed that dissipativity and the range condition $\mathrm{rg}(\lambda - A) = X$ for some $\lambda > 0$ imply certain generation properties. It is now a direct consequence of the renorming trick used in the proof of Generation Theorem 3.8 that these results also hold for all operators satisfying the Hille–Yosida resolvent estimates (3.16). We state this extension of Generation Theorem 3.8.

**3.21 Corollary.** *Let $w \in \mathbb{R}$ and $(A, D(A))$ be an operator on a Banach space $X$. Suppose that $(w, \infty) \subset \rho(A)$ and*

$$\|R(\lambda, A)^n\| \le \frac{M}{(\lambda - w)^n} \tag{3.22}$$

*for all $n \in \mathbb{N}$, $\lambda > w$ and some $M \ge 1$. Then the part $A_|$ of $A$ in $X_0 := \overline{D(A)}$ generates a strongly continuous semigroup $(T_0(t))_{t \ge 0}$ satisfying $\|T_0(t)\| \le M e^{wt}$ for all $t \ge 0$. If in addition the Banach space $X$ is reflexive, then $X_0 = X$.*

Proof. As in many previous cases we may assume that $w = 0$. Then the renorming procedure (3.18) from the proof of the implication (b) ⇒ (a) in Generation Theorem 3.8 yields an equivalent norm for which $A$ a dissipative operator. The assertions then follow from Corollary 3.18 (after returning to the original norm) and, in the reflexive case, from Corollary 3.20. □

It is sometimes convenient to use the following terminology.

**3.22 Definition.** *Operators satisfying the assumptions of Corollary* 3.21 *and, in particular, the resolvent estimate* (3.22) *are called Hille–Yosida operators.*

Observe that Corollary 3.21 states that Hille–Yosida operators satisfy *all* assumptions of the Hille–Yosida Generation Theorem 3.8 on the closure of their domains. Moreover, the Banach space $X$ can always be identified with a closed subspace of the extrapolated Favard space $F_0$. See Section 5.b and Exercise 5.23.(3).

We now return to dissipative operators, which represent, up to renorming, the most general case. When introducing them we had aimed for an easy (or at least more direct) way to characterizing generators. However, up to now, the only way to arrive at the norm inequality (3.21) was explicit computation of the resolvent and then deducing the norm estimate

$$\|R(\lambda, A)\| \le \frac{1}{\lambda} \qquad \text{for } \lambda > 0.$$

This was done in Example 3.19.(i). Fortunately, there is a simpler method that works particularly well in concrete function spaces such as $C_0(\Omega)$ or $L^p(\mu)$.

To introduce this method we start with a Banach space $X$ and its dual space $X'$. By the Hahn–Banach theorem, for every $x \in X$ there exists $x' \in X'$ such that

$$\langle x, x' \rangle = \|x\|^2 = \|x'\|^2.$$

For every $x \in X$, the following set, called its *duality set*,

$$\mathfrak{J}(x) := \left\{ x' \in X' : \langle x, x' \rangle = \|x\|^2 = \|x'\|^2 \right\}, \tag{3.23}$$

is nonempty. Such sets allow a new characterization of dissipativity.

**3.23 Proposition.** *An operator $\big(A, D(A)\big)$ is dissipative if and only if for every $x \in D(A)$ there exists $j(x) \in \mathfrak{J}(x)$ such that*

$$\operatorname{Re}\langle Ax, j(x)\rangle \le 0. \tag{3.24}$$

*If $A$ is the generator of a strongly continuous contraction semigroup, then (3.24) holds for all $x \in D(A)$ and arbitrary $x' \in \mathfrak{J}(x)$.*

Proof. Assume (3.24) is satisfied for $x \in D(A)$, $\|x\| = 1$, and some $j(x) \in \mathfrak{J}(x)$. Then $\langle x, j(x)\rangle = \|j(x)\|^2 = 1$ and

$$\begin{aligned}\|\lambda x - Ax\| &\ge |\langle \lambda x - Ax, j(x)\rangle| \\ &\ge \operatorname{Re}\langle \lambda x - Ax, j(x)\rangle \ge \lambda\end{aligned}$$

for all $\lambda > 0$. This proves one implication.

To show the converse, we take $x \in D(A)$, $\|x\| = 1$, and assume that $\|\lambda x - Ax\| \ge \lambda$ for all $\lambda > 0$. Choose $y'_\lambda \in \mathfrak{J}(\lambda x - Ax)$ and consider the normalized elements

$$z'_\lambda := \frac{y'_\lambda}{\|y'_\lambda\|}.$$

Then the inequalities

$$\begin{aligned}\lambda \le \|\lambda x - Ax\| &= \langle \lambda x - Ax, z'_\lambda\rangle \\ &= \lambda \operatorname{Re}\langle x, z'_\lambda\rangle - \operatorname{Re}\langle Ax, z'_\lambda\rangle \\ &\le \min\big\{\lambda - \operatorname{Re}\langle Ax, z'_\lambda\rangle\,,\ \lambda \operatorname{Re}\langle x, z'_\lambda\rangle + \|Ax\|\big\}\end{aligned}$$

are valid for each $\lambda > 0$. This yields

$$\operatorname{Re}\langle Ax, z'_\lambda\rangle \le 0 \qquad \text{and} \qquad 1 - \frac{1}{\lambda}\|Ax\| \le \operatorname{Re}\langle x, z'_\lambda\rangle\,.$$

Let $z'$ be a weak* accumulation point of $z'_\lambda$ as $\lambda \to \infty$. Then

$$\|z'\| \le 1, \qquad \operatorname{Re}\langle Ax, z'\rangle \le 0, \qquad \text{and} \qquad \operatorname{Re}\langle x, z'\rangle \ge 1.$$

Combining these facts, it follows that $z'$ belongs to $\mathfrak{J}(x)$ and satisfies (3.24).

Finally, assume that $A$ generates a contraction semigroup $\big(T(t)\big)_{t\ge0}$ on $X$. Then, for every $x \in D(A)$ and arbitrary $x' \in \mathfrak{J}(x)$, we have

$$\begin{aligned}\operatorname{Re}\langle Ax, x'\rangle &= \lim_{h\downarrow 0}\Big(\frac{\operatorname{Re}\langle T(h)x, x'\rangle}{h} - \frac{\operatorname{Re}\langle x, x'\rangle}{h}\Big) \\ &\le \overline{\lim_{h\downarrow 0}}\Big(\frac{\|T(h)x\|\cdot\|x'\|}{h} - \frac{\|x\|^2}{h}\Big) \le 0.\end{aligned}$$

This completes the proof. □

Using the previous results we easily arrive at the following generalization of the characterization of unitary groups on Hilbert spaces from Paragraph I.3.15. Its discovery by Stone was one of the major steps towards the construction of the exponential function in infinite dimensions, hence towards the solution of Problem I.3.8; cf. Chapter VII.

**3.24 Theorem.** (Stone, 1932). *Let $\big(A, D(A)\big)$ be a densely defined operator on a Hilbert space $H$. Then $A$ generates a unitary group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $H$ if and only if $A$ is skew-adjoint, i.e., $A^* = -A$.*

Proof. First, assume that $A$ generates a unitary group $\big(T(t)\big)_{t\in\mathbb{R}}$. By Paragraph 3.11, we have

$$T(t)^* = T(t)^{-1} = T(-t) \qquad \text{for all } t \in \mathbb{R}.$$

Moreover, by (the Hilbert space version of) Paragraph 2.6 on sun dual semigroups, the generator of $\big(T(t)^*\big)_{t\in\mathbb{R}}$ is given by $A^*$. This implies that $A^* = -A$.

On the other hand, if $A^* = -A$, then we conclude from

$$(Ax \,|\, x) = (x \,|\, A^*x) = -(x \,|\, Ax) = -\overline{(Ax \,|\, x)} \qquad \text{for all } x \in D(A) = D(A^*)$$

that $(Ax \,|\, x) \in \mathrm{i}\mathbb{R}$. Combining Proposition 3.23 with the identification of the duality set as $\mathfrak{J}(x) = \{x\}$ (see Exercise 3.25.(i) below), this shows that both $\pm A$ are dissipative and closed. From Corollary 3.17 and the characterization of group generators in Paragraph 3.11, it follows that the operator $A$ generates a contraction group $\big(T(t)\big)_{t\in\mathbb{R}}$. Since $T(t)^{-1} = T(-t)$, we conclude that each $T(t)$ is a surjective isometry and therefore unitary (see [Ped89, Sec. 3.2.15]). □

**3.25 Exercise.** Prove the following statements for a Hilbert space $H$.

(i) For every $x \in H$, one has $\mathfrak{J}(x) = \{x\}$.

(ii) If $A$ is a normal operator on $H$, then $A$ is a generator of a strongly continuous semigroup if and only if

$$\mathrm{s}(A) < \infty.$$

(iii) Prove Stone's theorem by arguing via multiplication semigroups.

(Hint: For (ii) and (iii) use the Spectral Theorem I.4.9 and the results of Paragraph 3.11.)

## c. More Examples

We close this section with a rather long discussion of all of these notions and results for concrete examples. We begin by identifying the sets $\mathfrak{J}(x)$ for some standard function spaces.

**3.26 Examples.** (i) Consider $X := \mathrm{C}_0(\Omega)$, $\Omega$ locally compact. For $0 \neq f \in X$, the set $\mathfrak{J}(f) \subset X'$ contains (multiples of) all point measures supported by those points $s_0 \in \Omega$ where $|f|$ reaches its maximum. More precisely,

$$\left\{\overline{f(s_0)} \cdot \delta_{s_0} : s_0 \in \Omega \text{ and } |f(s_0)| = \|f\|\right\} \subset \mathfrak{J}(f). \tag{3.25}$$

(ii) Let $X := \mathrm{L}^p(\Omega, \mu)$ for $1 \leq p < \infty$, and $0 \neq f \in \mathrm{L}^p(\Omega, \mu)$. Then

$$\varphi \in \mathfrak{J}(f) \subset \mathrm{L}^q(\Omega, \mu), \qquad {}^1\!/\!{}_p + {}^1\!/\!{}_q = 1,$$

where $\varphi$ is defined by

$$\varphi(s) := \begin{cases} \overline{f(s)} \cdot |f(s)|^{p-2} \cdot \|f\|^{2-p} & \text{if } f(s) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

Note that for the reflexive $\mathrm{L}^p$-spaces, as for every Banach space with a strictly convex dual, the sets $\mathfrak{J}(f)$ are singletons (see [Bea82]). Hence, for $1 < p < \infty$, one has $\mathfrak{J}(f) = \{\varphi\}$, while for $p = 1$ every function $\varphi \in \mathrm{L}^\infty(\Omega, \mu)$ satisfying

$$\|\varphi\|_\infty \leq \|f\|_1 \qquad \text{and} \qquad \varphi(s)\,|f(s)| = \overline{f(s)}\,\|f\|_1 \quad \text{if } f(s) \neq 0 \tag{3.27}$$

belongs to $\mathfrak{J}(f)$.
(iii) It is easy, but important, to state the result for Hilbert spaces $H$. After the canonical identification of $H$ with its dual $H'$, the duality set of $x \in H$ is

$$\mathfrak{J}(x) = \{x\}; \tag{3.28}$$

cf. Exercise 3.25.(i). Hence, a linear operator on $H$ is dissipative if and only if

$$\mathrm{Re}(Ax \,|\, x) \leq 0 \tag{3.29}$$

for all $x \in D(A)$.

These examples suggest that dissipativity for concrete operators on such function spaces can be verified via the inequality (3.24). In the following examples we do this and establish the dissipativity and generation property for various operators. We start with a concrete version of Theorem 3.24.

**3.27 Example.** (*Self-Adjoint Operators*). On the Hilbert space $H := \mathrm{L}^2(\Omega, \mu)$ consider a multiplication operator $A := M_q$ for some (measurable) function $q : \Omega \to \mathbb{C}$. Since its adjoint is $A^* = M_{\overline{q}}$, this operator is self-adjoint if and only if $q$ is real-valued. In this case, it follows by Theorem 3.24 that the group $(T_{\mathrm{i}q}(t))_{t\in\mathbb{R}}$ generated by $M_{\mathrm{i}q}$ is unitary.
However, this can be seen more directly by inspection of the corresponding multiplication group $(T_{\mathrm{i}q}(t))_{t\in\mathbb{R}}$, for which we have

$$T_{\mathrm{i}q}(t)^* = T_{\overline{\mathrm{i}q}}(t) = T_{-\mathrm{i}q}(t) = T_{\mathrm{i}q}(-t) \qquad \text{for all } t \in \mathbb{R}.$$

It is this argument for multiplication operators and semigroups that can be used to give a simple proof of Stone's Theorem 3.24. In fact, an application of the Spectral Theorem I.4.9 transforms the unitary group $(T(t))_{t\in\mathbb{R}}$ and its (skew-adjoint) generator $A$ on an arbitrary Hilbert space into multiplication operators on some $\mathrm{L}^2$-space. See Exercise 3.25.(iii).

The same argument, i.e., passing from a self-adjoint operator to a (real-valued) multiplication operator, yields the following characterization of self-adjoint semigroups.

**Proposition.** *A self-adjoint operator $(A, D(A))$ on a Hilbert space $H$ generates a strongly continuous semigroup (of self-adjoint operators) if and only if it is bounded above, i.e., there exists $w \in \mathbb{R}$ such that*

$$(Ax \,|\, x) \le w \, \|x\|^2 \qquad \text{for all } x \in D(A).$$

PROOF. It suffices to consider the multiplication operator $M_q$ that is isomorphic, via the Spectral Theorem I.4.9, to $A$. Then the boundedness condition $(Ax \,|\, x) \le w \, \|x\|^2$ for all $x \in D(A)$ means that the real-valued function $q$ satisfies

$$\operatorname*{ess\,sup}_{s \in \Omega} \operatorname{Re} q(s) \le w.$$

This, however, is exactly what is needed for $M_q$ to generate a semigroup (see Propositions I.4.11 and I.4.12). □

**3.28 First-Order Differential Operators and Flows.** We begin by considering a continuously differentiable vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the estimate $\sup_{s \in \mathbb{R}^n} \|DF(s)\| < \infty$ for the derivative $DF(s)$ of $F$ at $s \in \mathbb{R}$. To this vector field we associate the following operator on the space $X := \mathrm{C}_0(\mathbb{R}^n)$.

**Definition 1.** *The first-order differential operator on $\mathrm{C}_0(\mathbb{R}^n)$ corresponding to the vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is*

$$\begin{aligned} Af(s) :&= \langle \operatorname{grad} f(s), F(s) \rangle \\ &= \sum_{i=1}^n F_i(s) \frac{\partial f}{\partial s_i}(s) \end{aligned}$$

*for $f \in \mathrm{C}_c^1(\mathbb{R}^n) := \{ f \in \mathrm{C}^1(\mathbb{R}^n) : f$ has compact support$\}$ and $s \in \mathbb{R}^n$.*

Using Example 3.26.(i) and the fact that ${}^{\partial f(s_0)}/_{\partial s_i} = 0$ if $|f(s_0)| = \|f\|$, it is immediate that $A$ is dissipative. However, in order to show that the closure of $A$ is a generator, there is a natural and explicit choice for what the semigroup generated by $A$ should be. By writing it down, one simply checks that its generator is the closure of $A$.

Since $F$ is globally Lipschitz, it follows from standard results that there exists a *continuous flow* $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\Phi$ is continuous with $\Phi(t + r, s) = \Phi(t, \Phi(r, s))$ and $\Phi(0, s) = s$ for every $r, t \in \mathbb{R}$ and $s \in \mathbb{R}^n$, which solves the differential equation

$$\frac{\partial}{\partial t} \Phi(t, s) = F\big(\Phi(t, s)\big)$$

for all $t \in \mathbb{R}$, $s \in \mathbb{R}^n$ (see [Ama90, Thm. 10.3]). To such a flow we associate a one-parameter group of linear operators on $\mathrm{C}_0(\mathbb{R}^n)$ as follows.

**Definition 2.** *The group defined by the operators*

$$T(t)f(s) := f\big(\Phi(t, s)\big)$$

*for $f \in \mathrm{C}_0(\mathbb{R}^n)$, $s \in \mathbb{R}^n$ and $t \in \mathbb{R}$, is called the group induced by the flow $\Phi$ on the Banach space $\mathrm{C}_0(\mathbb{R}^n)$.*

The group property and the strong continuity follow immediately from the corresponding properties of the flow; we refer to Exercise 3.31.(1) for a closer look at the relations between (nonlinear) semiflows and (linear) semigroups. We now determine the generator of $(T(t))_{t\in\mathbb{R}}$.

**Proposition.** *The generator of the group $(T(t))_{t\in\mathbb{R}}$ on $\mathrm{C}_0(\mathbb{R}^n)$ is the closure of the first-order differential operator*

$$Af(s) := \langle \operatorname{grad} f(s), F(s)\rangle$$

*with domain*

$$D(A) := \mathrm{C}_c^1(\mathbb{R}^n).$$

Proof. Let $(B, D(B))$ denote the generator of $(T(t))_{t\in\mathbb{R}}$. For $f \in \mathrm{C}_c^1(\mathbb{R}^n)$ consider $g := f - Af \in \mathrm{C}_c(\mathbb{R}^n)$ and compute the resolvent using the integral representation (1.13) in Chapter II. This yields

$$\begin{aligned} R(1,B)g(s) &= \int_0^\infty \mathrm{e}^{-t} f\big(\Phi(t,s)\big)\,dt - \int_0^\infty \mathrm{e}^{-t}\left\langle \operatorname{grad} f\big(\Phi(t,s)\big), F\big(\Phi(t,s)\big)\right\rangle dt \\ &= f(s) \end{aligned}$$

after an integration by parts. Accordingly, $\mathrm{C}_c^1(\mathbb{R}^n) \subset D(B)$ and $A \subset B$. On the other hand, $\mathrm{C}_c^1(\mathbb{R}^n)$ is dense in $\mathrm{C}_0(\mathbb{R}^n)$ and invariant under the group $(T(t))_{t\in\mathbb{R}}$ induced by the flow. So, $\mathrm{C}_c^1(\mathbb{R}^n)$ is a core by Proposition 1.7, and the assertion is proved. ☐

For analogous results on first-order differential operators on bounded domains $\Omega \subset \mathbb{R}^n$, we refer to [Ulm92].

**3.29 Delay Differential Operators.** On the space $X := \mathrm{C}[-1,0]$, consider the operator

$$Af := f'$$

with domain

$$D(A) := \big\{ f \in \mathrm{C}^1[-1,0] : f'(0) = Lf \big\},$$

where $L$ is a continuous linear form on $\mathrm{C}[-1,0]$. This can be rewritten as

$$D(A) = \ker \varphi,$$

where $\varphi$ is the linear form on $\mathrm{C}^1[-1,0]$ defined by

$$\mathrm{C}^1[-1,0] \ni f \mapsto f'(0) - Lf \in \mathbb{C}.$$

Since this functional is bounded on the Banach space $\mathrm{C}^1[-1,0]$ but unbounded for the sup-norm, we deduce that $D(A)$ is dense in $\mathrm{C}[-1,0]$ and closed in $\mathrm{C}^1[-1,0]$; cf. Proposition B.5.

Next, we show that the rescaled operator $A - \|L\| \cdot I$ is dissipative. To this end, take $f \in D(A)$. As seen in Example 3.26.(i), the linear form $\overline{f(s_0)}\,\delta_{s_0}$ belongs to $\mathfrak{J}(f)$ if $|f(s_0)| = \|f\|$ for some $s_0 \in [-1,0]$. This means that $A - \|L\|\, I$ is dissipative, provided that

$$\operatorname{Re}\left\langle f' - \|L\|\, f, \overline{f(s_0)}\,\delta_{s_0}\right\rangle \le 0 \qquad \text{or} \qquad \operatorname{Re} \overline{f(s_0)} f'(s_0) \le \|L\| \cdot \|f\|^2. \tag{3.30}$$

In the case $-1 < s_0 < 0$ we have $f'(s_0) = 0$, so that (3.30) certainly holds. The same is true if $s_0 = -1$, since then $2\operatorname{Re} \overline{f(-1)} f'(-1) = (f \cdot \overline{f})'(-1) \le 0$. It remains to consider the case where $s_0 = 0$. Here, we use $f'(0) = Lf$ for $f \in D(A)$ to obtain

$$\operatorname{Re} \overline{f(0)} f'(0) = \operatorname{Re} \overline{f(0)} Lf \le \|f\| \cdot \|L\| \cdot \|f\|.$$

So, we are now well prepared to apply Theorem 3.15 to conclude that $A$ is a generator.

**Proposition.** *Let $L \in \mathrm{C}[-1,0]'$. The delay differential operator*

$$Af := f' \qquad \textit{with} \qquad D(A) := \big\{f \in \mathrm{C}^1[-1,0] : f'(0) = Lf\big\}$$

*on the Banach space $\mathrm{C}[-1,0]$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying*

$$\|T(t)\| \leq \mathrm{e}^{\|L\|t} \qquad \textit{for } t \geq 0.$$

Proof. By the rescaling technique, the assertion follows from Theorem 3.15 and the above consideration, provided that $\lambda - A$ is surjective for some $\lambda > \|L\|$. This means we have to show that for every $g \in \mathrm{C}[-1,0]$ there exists $f \in \mathrm{C}^1[-1,0]$ satisfying both

$$\lambda f - f' = g$$

and

$$f'(0) = Lf, \quad \text{i.e.,} \quad f \in D(A).$$

The first equation has

$$\begin{aligned} f(s) &:= c\,\mathrm{e}^{\lambda s} - \int_0^s \mathrm{e}^{\lambda(s-\tau)} g(\tau)\, d\tau \\ &=: c\,\varepsilon_\lambda(s) - h(s), \quad s \in [-1,0], \end{aligned}$$

as a solution for every constant $c \in \mathbb{C}$. If $\lambda > \|L\|$, then we can choose this constant as

$$c := \frac{g(0) - Lh}{\lambda - L\varepsilon_\lambda}$$

in order to obtain $f \in D(A)$. □

The importance (and name) of this operator stems from the fact that the semigroup it generates solves a *delay differential equation* of the form

$$\begin{cases} \dot{u}(t) = Lu_t & \text{for } t \geq 0, \\ u(s) = f(s) & \text{for } -1 \leq s \leq 0, \end{cases}$$

where $f$ is an initial function from $\mathrm{C}[-1,0]$. Here, $u_t \in \mathrm{C}[-1,0]$ is defined by $u_t(s) := u(t+s)$ for $s \in [-1,0]$. In Section VI.6 we will return to the study of such equations and show how they are related to semigroups. We also refer to Paragraph IV.2.8, where the spectrum of the above delay differential operator is computed.

**3.30 Second-Order Differential Operators.** (i) We first reconsider the operator from Paragraph 2.12, i.e., we take on $X := \mathrm{C}[0,1]$ the operator

$$Af := f'', \quad D(A) := \big\{f \in \mathrm{C}^2[0,1] : f'(0) = f'(1) = 0\big\}.$$

This time, instead of constructing the generated semigroup, we verify the conditions of Theorem 3.15. It is simple to show that $(A, D(A))$ is densely defined

and closed. To show dissipativity, we take $f \in D(A)$ and $s_0 \in [0,1]$ such that $|f(s_0)| = \|f\|$. By Example 3.26.(i) we have

$$\overline{f(s_0)}\,\delta_{s_0} \in \mathfrak{J}(f).$$

Since $t \mapsto \operatorname{Re} \overline{f(s_0)} \cdot f(t)$ takes its maximum at $s_0$, it follows that

$$\operatorname{Re}\left\langle f'', \overline{f(s_0)}\,\delta_0 \right\rangle = \left(\operatorname{Re} \overline{f(s_0)} f\right)''(s_0) \le 0,$$

where we need to use the boundary condition

$$f'(0) = f'(1) = 0$$

if $s_0 = 0$ or $s_0 = 1$. We finally show that $\lambda^2 - A$ is surjective for $\lambda > 0$. Take $g \in \mathrm{C}[0,1]$ and define

$$k(s) := \frac{1}{2\lambda}\left[ \mathrm{e}^{\lambda s} \int_s^1 \mathrm{e}^{-\lambda\tau} g(\tau)\, d\tau - \mathrm{e}^{-\lambda s} \int_s^1 \mathrm{e}^{\lambda\tau} g(\tau)\, d\tau \right] \qquad \text{for } s \in [0,1].$$

Then $k$ is in $\mathrm{C}^2[0,1]$ and satisfies

$$\lambda^2 k - k'' = g.$$

On the other hand, for each $a, b \in \mathbb{C}$, the function

$$h_{a,b}(s) := a\,\mathrm{e}^{\lambda s} + b\,\mathrm{e}^{-\lambda s}, \quad s \in [0,1],$$

satisfies

$$\lambda^2 h_{a,b} - h''_{a,b} = 0.$$

It is now an exercise in linear algebra to determine $\widetilde{a}, \widetilde{b} \in \mathbb{C}$ such that the function

$$f := k + h_{\tilde{a},\tilde{b}}$$

satisfies $f'(0) = f'(1) = 0$. Then $f \in D(A)$ and $\lambda^2 f - f'' = g$, i.e., $\lambda^2 - A$ is surjective. It follows from Theorem 3.15 that $(A, D(A))$ generates a contraction semigroup on $\mathrm{C}[0,1]$.

(ii) The above method is now applied to the same differential operator on a different space and with different boundary conditions. Let $X := \mathrm{L}^2[0,1]$ and

$$Af := f'', \quad D(A) := \{f \in \mathrm{C}^2[0,1] : f(0) = f(1) = 0\}.$$

Then $D(A)$ is dense in $X$, and for $f \in D(A)$ one has

$$(Af \,|\, f) = \int_0^1 f''\overline{f}\, ds = f'\overline{f}\,\Big|_0^1 - \int_0^1 f'\overline{f'}\, ds \le 0. \tag{3.31}$$

By Example 3.26.(iii), this means that $A$ is dissipative on the Hilbert space $\mathrm{L}^2[0,1]$. As in the previous case, for every $g \in \mathrm{C}^2[0,1]$ there exists a function $f \in \mathrm{C}^2[0,1]$ satisfying $f(0) = f(1) = 0$ and

$$\lambda^2 f - f'' = g,$$

i.e., $\operatorname{rg}(\lambda^2 - A)$ is dense. Again by Theorem 3.15 we conclude that $(\overline{A}, D(\overline{A}))$ generates a contraction semigroup on $\mathrm{L}^2[0,1]$.

(iii) As a somewhat less canonical second-order differential operator on $X := \mathrm{C}[0,1]$, consider $(A, D(A))$ defined by

$$Af(s) := s(1-s)f''(s), \qquad s \in [0,1],$$

for $f \in D(A) := \left\{ f \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : \lim_{s\to 0,1} s(1-s)f''(s) = 0 \right\}$. We show that it generates a strongly continuous contraction semigroup by verifying the conditions of Theorem 3.15.

As above, it is easy to show that $(A, D(A))$ is closed, densely defined, and dissipative. Therefore, it suffices to prove that $\lambda - A$ is surjective for some $\lambda > 0$. Observe first that the functions $h_0 : s \mapsto 1$ and $h_1 : s \mapsto s$ belong to $D(A)$ and satisfy

$$(3.32) \qquad (\lambda - A)h_i = \lambda h_i, \qquad i = 0, 1 \text{ and } \lambda > 0.$$

Hence, it suffices to consider the part $A_0$ of $A$ in the closed subspace $X_0 := \{f \in X : f(0) = f(1) = 0\}$ with domain $D(A_0) := \big\{f \in X_0 \cap \mathrm{C}^2(0,1) : \lim_{s\to 0,1} s(1-s)f''(s) = 0\big\}$. Then $(A_0, D(A_0))$ is still dissipative, but is now injective. Its inverse $R$ can be computed as

$$Rf(s) = \int_0^1 \sigma(s,t)\, \frac{f(t)}{t(1-t)}\, dt,$$

where

$$\sigma(s,t) := \begin{cases} s(t-1) & \text{for } 0 \le s \le t \le 1, \\ t(s-1) & \text{for } 0 \le t \le s \le 1, \end{cases}$$

and $f \in X_0$. This shows that $0 \in \rho(A_0)$ and hence $[0,\infty) \subset \rho(A_0)$. From (3.32) we conclude that $(0,\infty) \subset \rho(A)$. Accordingly, $A$ is a generator.

**3.31 Exercises.** (1) Let $\Omega$ be a compact space and take $X := \mathrm{C}(\Omega)$. A *semiflow* $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ is defined by the properties

$$(3.33) \qquad \begin{aligned} \Phi(t+r, s) &= \Phi\big(t, \Phi(r,s)\big), \\ \Phi(0,s) &= s \end{aligned}$$

for every $s \in \Omega$ and $r, t \in \mathbb{R}_+$. Establish the following facts.

(i) The semiflow $\Phi$ is continuous if and only if it induces a strongly continuous semigroup $(T(t))_{t\ge 0}$ on $X$ by the formula

$$(3.34) \qquad \big(T(t)f\big)(s) := f\big(\Phi(t,s)\big) \qquad \text{for } s \in \Omega,\ t \ge 0,\ f \in X.$$

(ii) The generator $(A, D(A))$ of $(T(t))_{t\ge 0}$ is a derivation (cf. Exercise 1.15.(5)).

(iii*) Every strongly continuous semigroup $(T(t))_{t\ge 0}$ on $X$ that consists of algebra homomorphisms originates, via (3.34), from a continuous semiflow on $\Omega$. (Hint: See [Nag86, B-II, Thm. 3.4].)

(2) Show that the semigroup $(T(t))_{t\ge 0}$ on $X := \mathrm{C}[-1,0]$ generated by the delay differential operator from Paragraph 3.29 satisfies the *translation property*, i.e.,

$$(\mathrm{TP}) \qquad \big(T(t)f\big)(s) = \begin{cases} f(t+s) & \text{if } t+s \le 0, \\ [T(t+s)f](0) & \text{if } t+s > 0, \end{cases}$$

for all $f \in X$ (cf. also Lemma VI.6.2).

# 4. Special Classes of Semigroups

Up to now, we have classified semigroups only as being *strongly continuous* in the general case or being *uniformly continuous* as a somewhat uninteresting case. Between these two extreme cases there is room for a wide range of continuity properties. We now introduce several classes of semigroups identified by these continuity, or *regularity*, properties.

We begin with the most important class.

## a. Analytic Semigroups

We return to the exponential formula (3.2), but now impose conditions on the operator $A$ (and its resolvent $R(\lambda, A)$) that make the contour integrals converge even if $A$ and $\sigma(A)$ are unbounded.

**4.1 Definition.** *A closed linear operator $\big(A, D(A)\big)$ with dense domain $D(A)$ in a Banach space $X$ is called sectorial (of angle $\delta$) if there exists $0 < \delta \le {}^{\pi}/_{2}$ such that the sector*

$$\Sigma_{\pi/2+\delta} := \Big\{\lambda \in \mathbb{C} : |\arg \lambda| < \frac{\pi}{2} + \delta\Big\} \setminus \{0\}$$

*is contained in the resolvent set $\rho(A)$, and if for each $\varepsilon \in (0, \delta)$ there exists $M_\varepsilon \ge 1$ such that*

$$\|R(\lambda, A)\| \le \frac{M_\varepsilon}{|\lambda|} \qquad \text{for all } 0 \ne \lambda \in \overline{\Sigma}_{\pi/2+\delta-\varepsilon}. \tag{4.1}$$

For sectorial operators and appropriate paths $\gamma$, the exponential function "$\mathrm{e}^{tA}$" can now be defined via the Cauchy integral formula.

**4.2 Definition.** *Let $\big(A, D(A)\big)$ be a sectorial operator of angle $\delta$. Define $T(0) := I$ and operators $T(z)$, for $z \in \Sigma_\delta$, by*

$$T(z) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\mu z} R(\mu, A)\, d\mu, \tag{4.2}$$

*where $\gamma$ is any piecewise smooth curve in $\Sigma_{\pi/2+\delta}$ going from $\infty\, \mathrm{e}^{-\mathrm{i}(\pi/2+\delta')}$ to $\infty\, \mathrm{e}^{\mathrm{i}(\pi/2+\delta')}$ for some $\delta' \in (|\arg z|, \delta)$.*[1]

As a first step, we need to justify this definition. In particular, we show that the essential properties of the analytic functional calculus for bounded operators (cf. Definition I.3.4) prevail in this situation.

[1] See Figure 1.

**4.3 Proposition.** *Let $\big(A, D(A)\big)$ be a sectorial operator of angle $\delta$. Then, for all $z \in \Sigma_\delta$, the maps $T(z)$ are bounded linear operators on $X$ satisfying the following properties.*

(i) *$\|T(z)\|$ is uniformly bounded for $z \in \Sigma_{\delta'}$ if $0 < \delta' < \delta$.*
(ii) *The map $z \mapsto T(z)$ is analytic in $\Sigma_\delta$.*
(iii) *$T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.*
(iv) *The map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'} \cup \{0\}$ if $0 < \delta' < \delta$.*

Proof. We first verify that for $z \in \Sigma_{\delta'}$, with $\delta' \in (0, \delta)$ fixed, the integral in (4.2) defining $T(z)$ converges uniformly in $\mathcal{L}(X)$ with respect to the operator norm. Since the integrand is analytic in $\mu \in \Sigma_{\pi/2+\delta}$, this integral, if it exists, is by Cauchy's integral theorem independent of the particular choice of $\gamma$. Hence, we may choose $\gamma = \gamma_r$ as in Figure 1, i.e., $\gamma$ consists of the three parts

$$
\begin{aligned}
&\gamma_{r,1} : \Big\{-\rho \mathrm{e}^{-\mathrm{i}(\pi/2+\delta-\varepsilon)} : -\infty \le \rho \le -r\Big\},\\
&\gamma_{r,2} : \big\{r\mathrm{e}^{\mathrm{i}\alpha} : -(\pi/2+\delta-\varepsilon) \le \alpha \le (\pi/2+\delta-\varepsilon)\big\},\\
&\gamma_{r,3} : \Big\{\rho \mathrm{e}^{\mathrm{i}(\pi/2+\delta-\varepsilon)} : r \le \rho \le \infty\Big\},
\end{aligned}
\tag{4.3}
$$

where $\varepsilon := (\delta-\delta')/2 > 0$ and $r := 1/|z|$.

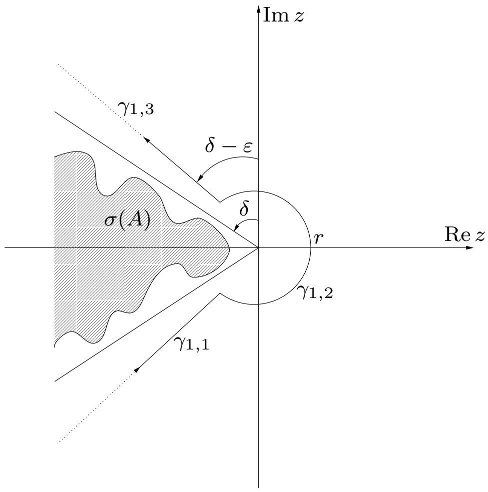



**Figure 1**

Then, for $\mu \in \gamma_{r,3}$, $z \in \Sigma_{\delta'}$, we can write

$$\mu z = |\mu z|\, \mathrm{e}^{\mathrm{i}(\arg \mu + \arg z)},$$

where ${}^{\pi}/_{2} + \varepsilon \le \arg\mu + \arg z \le \frac{3\pi}{2} - \varepsilon$. Hence, we have

$$\frac{1}{|\mu z|}\operatorname{Re}(\mu z) = \cos(\arg\mu + \arg z) \le \cos\left({}^{\pi}/_{2} + \varepsilon\right) = -\sin\varepsilon,$$

and therefore

$$|\mathrm{e}^{\mu z}| \le \mathrm{e}^{-|\mu z|\sin\varepsilon} \tag{4.4}$$

for all $z \in \Sigma_{\delta'}$ and $\mu \in \gamma_{r,3}$. Similarly, one shows that (4.4) is true for $z \in \Sigma_{\delta'}$ and $\mu \in \gamma_{r,1}$, from which we conclude

$$\|\mathrm{e}^{\mu z} R(\mu, A)\| \le \mathrm{e}^{-|\mu z|\sin\varepsilon}\,\frac{M_\varepsilon}{|\mu|} \tag{4.5}$$

for all $z \in \Sigma_{\delta'}$ and $\mu \in \gamma_{r,1} \cup \gamma_{r,3}$. On the other hand, the estimate

$$\|\mathrm{e}^{\mu z} R(\mu, A)\| \le \mathrm{e}\,\frac{M_\varepsilon}{|\mu|} = \mathrm{e}M_\varepsilon|z| \tag{4.6}$$

holds for all $z \in \Sigma_{\delta'}$ and $\mu \in \gamma_{r,2}$. Using the estimates (4.5) and (4.6), we then conclude

$$\begin{aligned}\left\|\int_{\gamma_r} \mathrm{e}^{\mu z} R(\mu, A)\,d\mu\right\| &\le \sum_{k=1}^{3}\left\|\int_{\gamma_{r,k}} \mathrm{e}^{\mu z} R(\mu, A)\,d\mu\right\| \\ &\le 2M_\varepsilon \int_{1/|z|}^{\infty} \frac{1}{\rho}\,\mathrm{e}^{-\rho|z|\sin\varepsilon}\,d\rho + \mathrm{e}M_\varepsilon|z|\cdot\frac{2\pi}{|z|} \\ &= 2M_\varepsilon \int_{1}^{\infty} \frac{1}{\rho}\,\mathrm{e}^{-\rho\sin\varepsilon}\,d\rho + 2\pi\mathrm{e}M_\varepsilon\end{aligned}$$

for all $z \in \Sigma_{\delta'}$. This shows that the integral defining $T(z)$ converges in $\mathcal{L}(X)$ absolutely and uniformly for $z \in \Sigma_{\delta'}$, i.e., the operators $T(z)$ are well-defined and satisfy (i).

Moreover, from the above considerations, it follows that the map $z \mapsto T(z)$ is analytic for $z \in \Sigma_\delta = \cup_{0<\delta'<\delta}\Sigma_{\delta'}$, which proves (ii).

Next, we verify the semigroup property (iii). To this end, we choose some constant $c > 0$ such that $\gamma \cap \gamma' := \gamma_1 \cap (\gamma_1 + c) = \emptyset$, where $\gamma_1$ is as in (4.3) with $r = 1$. Then, for $z_1, z_2 \in \Sigma_{\delta'}$, we obtain using the resolvent equation in Paragraph IV.1.2 and Fubini's theorem that

$$\begin{aligned}T(z_1)T(z_2) &= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\gamma'} \mathrm{e}^{\mu z_1}\mathrm{e}^{\lambda z_2} R(\mu, A)R(\lambda, A)\,d\lambda\,d\mu \\ &= \frac{1}{(2\pi\mathrm{i})^2}\int_\gamma\int_{\gamma'} \frac{\mathrm{e}^{\mu z_1}\mathrm{e}^{\lambda z_2}}{\lambda - \mu}\big(R(\mu, A) - R(\lambda, A)\big)\,d\lambda\,d\mu \\ &= \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{e}^{\mu z_1} R(\mu, A)\left(\frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \frac{\mathrm{e}^{\lambda z_2}}{\lambda - \mu}\,d\lambda\right)d\mu \\ &\quad - \frac{1}{2\pi\mathrm{i}}\int_{\gamma'} \mathrm{e}^{\lambda z_2} R(\lambda, A)\left(\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{e}^{\mu z_1}}{\lambda - \mu}\,d\mu\right)d\lambda.\end{aligned}$$

By closing the curves $\gamma$ and $\gamma'$ by circles with increasing diameter on the left and using the fact that $\gamma$ lies to the left of $\gamma'$, Cauchy's integral theorem implies

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{e}^{\mu z_1}}{\lambda - \mu}\, d\mu = 0 \quad \text{and} \quad \frac{1}{2\pi \mathrm{i}} \int_{\gamma'} \frac{\mathrm{e}^{\lambda z_2}}{\lambda - \mu}\, d\lambda = \mathrm{e}^{\mu z_2}.$$

Thus, we conclude

$$\begin{aligned} T(z_1)T(z_2) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\mu z_1} \mathrm{e}^{\mu z_2} R(\mu, A)\, d\mu \\ &= T(z_1 + z_2) \end{aligned}$$

for all $z_1, z_2 \in \Sigma_{\delta'}$, which proves (iii).

It remains only to show (iv), i.e., that the map $z \mapsto T(z)$ is strongly continuous in $\Sigma_{\delta'} \cup \{0\}$ for every $0 < \delta' < \delta$. By (i) and (ii), it suffices, as usual, to verify that

$$\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x - x = 0 \quad \text{for all } x \in D(A). \tag{4.7}$$

We start from estimate (4.4) and Cauchy's integral formula and obtain for $\gamma = \gamma_1$ that

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{e}^{\mu z}}{\mu}\, d\mu = 1$$

for all $z \in \Sigma_{\delta'}$. Hence, the identity $R(\mu, A)Ax = \mu R(\mu, A)x - x$ for $x \in D(A)$ yields

$$\begin{aligned} T(z)x - x &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{\mu z} \left( R(\mu, A) - \frac{1}{\mu} \right) x\, d\mu \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{e}^{\mu z}}{\mu} R(\mu, A)Ax\, d\mu \end{aligned}$$

for all $z \in \Sigma_{\delta'}$. Now, by (4.1) and (4.5), we have

$$\left\| \frac{\mathrm{e}^{\mu z}}{\mu} R(\mu, A)Ax \right\| \le \frac{M_\varepsilon}{|\mu|^2} \left( 1 + \mathrm{e}^{|z|} \right) \|Ax\|$$

for all $\mu \in \gamma$ and $z \in \Sigma_{\delta'}$. Using this estimate and since $\lim_{z \to 0} \mathrm{e}^{\mu z} = 1$, Lebesgue's dominated convergence theorem implies

$$\lim_{\Sigma_{\delta'} \ni z \to 0} T(z)x - x = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\mu} R(\mu, A)Ax\, d\mu = 0,$$

where the second equality follows from Cauchy's integral theorem by closing the path $\gamma$ by circles with increasing diameter on the right. This proves (4.7), and the proof is complete. ☐

If in Definition 4.2 we only consider values $z \in \mathbb{R}_+$, we obtain, by the previous proposition, a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$. Having in mind Proposition I.3.5, the following result determining its generator is not very surprising.

**4.4 Proposition.** *The generator of the strongly continuous semigroup defined by* (4.2) *is the sectorial operator* $\big(A, D(A)\big)$.

PROOF. Denoting by $\big(B, D(B)\big)$ the generator of $\big(T(t)\big)_{t\geq 0}$, it suffices to show that

$$R(\lambda, A) = R(\lambda, B) \tag{4.8}$$

for $\lambda = |\,\omega_0\,| + 2$, where $\omega_0$ denotes the growth bound of $\big(T(t)\big)_{t\geq 0}$, cf. Definition I.5.6. However, from Theorem 1.10 we know that the resolvent of $B$ in $\lambda$ is given as the integral

$$R(\lambda, B)x = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x\, dt \qquad \text{for all } x \in X.$$

Take now $t_0 > 0$ and choose $\gamma = \gamma_1$ as in (4.3). Then, by Fubini's theorem, we obtain

$$\begin{aligned}\int_0^{t_0} \mathrm{e}^{-\lambda t} T(t)x\, dt &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{t_0(\mu-\lambda)} - 1}{\mu - \lambda} R(\mu, A)x\, d\mu \\ &= R(\lambda, A)x + \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{e}^{t_0(\mu-\lambda)}}{\mu - \lambda} R(\mu, A)x\, d\mu.\end{aligned}$$

Here, we used the formula $\int_\gamma \frac{R(\mu,A)}{\mu-\lambda}\, x\, d\mu = -2\pi\mathrm{i} R(\lambda, A)x$, which can be verified using Cauchy's integral formula and by closing $\gamma$ on the right by circles of diameter converging to $\infty$. Since $\operatorname{Re}(\mu - \lambda) \leq -1$, for $\varepsilon = {}^{(\delta-\delta')}\!/_2$ we can estimate

$$\left\| \int_\gamma \frac{\mathrm{e}^{t_0(\mu-\lambda)}}{\mu - \lambda} R(\mu, A)x\, d\mu \right\| \leq \mathrm{e}^{-t_0} \cdot \|x\| \int_\gamma \frac{M_\varepsilon}{|\mu - \lambda| \cdot |\mu|}\, |d\mu|$$

and obtain (4.8) by taking the limit as $t_0 \to \infty$. □

Combining the two previous results, we see that a sectorial operator is always the generator of a strongly continuous semigroup that can be extended analytically to some sector $\Sigma_\delta$ containing $\mathbb{R}_+$. At this point, we remark that sectorial operators are characterized by the single resolvent estimate (4.1), while the Hille–Yosida Generation Theorem 3.8 requires estimates on all powers of the resolvent.

As we will see later (e.g. in Theorem III.2.10, Corollary IV.3.12, Corollary VI.3.6, and Corollary VI.7.17), semigroups that can be extended analytically enjoy many nice properties. Therefore, we will give various characterizations of these *analytic* semigroups. First, we introduce the appropriate terminology.

**4.5 Definition.** *A family of operators $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}} \subset \mathcal{L}(X)$ is called an analytic semigroup (of angle $\delta \in (0, \pi/2]$) if*

(i) *$T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.*

(ii) *The map $z \mapsto T(z)$ is analytic in $\Sigma_\delta$.*

(iii) *$\lim_{\Sigma_{\delta'}\ni z\to 0} T(z)x = x$ for all $x \in X$ and $0 < \delta' < \delta$.*

*If, in addition,*

(iv) *$\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$,*

*we call $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}}$ a bounded analytic semigroup.*

In our next result, we give various equivalences characterizing generators of bounded analytic semigroups.

**4.6 Theorem.** *For an operator $\big(A, D(A)\big)$ on a Banach space $X$, the following statements are equivalent.*

(a) *$A$ generates a bounded analytic semigroup $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}}$ on $X$.*

(b) *There exists $\vartheta \in (0, \pi/2)$ such that the operators $e^{\pm i\vartheta}A$ generate bounded strongly continuous semigroups on $X$.*

(c) *$A$ generates a bounded strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$ such that $\operatorname{rg}\big(T(t)\big) \subset D(A)$ for all $t > 0$, and*

$$M := \sup_{t>0} \|tAT(t)\| < \infty. \tag{4.9}$$

(d) *$A$ generates a bounded strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$, and there exists a constant $C > 0$ such that*

$$\|R(r + is, A)\| \leq \frac{C}{|s|} \tag{4.10}$$

*for all $r > 0$ and $0 \neq s \in \mathbb{R}$.*

(e) *$A$ is sectorial.*

Proof. We will show that (a) ⇒ (b) ⇒ (d) ⇒ (e) ⇒ (c) ⇒ (a).

(a) ⇒ (b). For $\vartheta \in (0, \delta)$, we define $T_\vartheta(t) := T(e^{i\vartheta}t)$. Then, by Definition 4.5, the operator family $\big(T_\vartheta(t)\big)_{t\geq 0} \subset \mathcal{L}(X)$ is a bounded strongly continuous semigroup on $X$. In order to determine its generator, we define

$\gamma : [0,\infty) \to \mathbb{C}$ by $\gamma(r) := \mathrm{e}^{\mathrm{i}\vartheta} r$. Then, by analyticity and Cauchy's integral theorem, we obtain

$$\begin{aligned} R(1,A)x &= \int_0^\infty \mathrm{e}^{-t} T(t)x\,dt = \int_\gamma \mathrm{e}^{-r} T(r)x\,dr \\ &= \mathrm{e}^{\mathrm{i}\vartheta} \int_0^\infty \mathrm{e}^{-\mathrm{e}^{\mathrm{i}\vartheta} r} T_\vartheta(r)x\,dr = \mathrm{e}^{\mathrm{i}\vartheta} R\big(\mathrm{e}^{\mathrm{i}\vartheta}, A_\vartheta\big)x \end{aligned}$$

for all $x \in X$, hence $A_\vartheta = \mathrm{e}^{\mathrm{i}\vartheta} A$. Similarly, it follows that $\big(T(\mathrm{e}^{-\mathrm{i}\vartheta} t)\big)_{t\geq 0}$ is a bounded strongly continuous semigroup with generator $\mathrm{e}^{-\mathrm{i}\vartheta} A$, i.e., (b) is proved.

(b) $\Rightarrow$ (d). Let $\mathrm{e}^{-\mathrm{i}\vartheta} = a - ib$ for $a, b > 0$. Then, applying the Hille–Yosida Generation Theorem 3.8 to the generator $\mathrm{e}^{-\mathrm{i}\vartheta} A$, we obtain a constant $\widetilde{C} \geq 1$ such that

$$\begin{aligned} \|R(r+is, A)\| &= \big\|\mathrm{e}^{-\mathrm{i}\vartheta} R\big(\mathrm{e}^{-\mathrm{i}\vartheta}(r+is), \mathrm{e}^{-\mathrm{i}\vartheta} A\big)\big\| \\ &\leq \frac{\widetilde{C}}{ar+bs} \leq \frac{C}{s} \end{aligned}$$

for all $r, s > 0$ and $C := \tilde{C}/b$. For $s < 0$, we obtain a similar estimate using the fact that $\mathrm{e}^{\mathrm{i}\vartheta} A$ is a generator on $X$.

(d) $\Rightarrow$ (e). By assumption, $A$ generates a bounded strongly continuous semigroup, and therefore we have $\Sigma_{\pi/2} \subset \rho(A)$ by Theorem 1.10. From Corollary IV.1.14, we know that

$$\|R(\lambda, A)\| \geq \frac{1}{\operatorname{dist}(\lambda, \sigma(A))} \qquad \text{for all } \lambda \in \rho(A).$$

Therefore, the estimate (4.10) implies $\mathrm{i}\mathbb{R} \setminus \{0\} \subset \rho(A)$ and, by continuity of the resolvent map,

$$\|R(\mu, A)\| \leq \frac{C}{|\mu|} \qquad \text{for all } 0 \neq \mu \in \mathrm{i}\mathbb{R}. \tag{4.11}$$

We now develop the resolvent of $A$ in $0 \neq \mu \in \mathrm{i}\mathbb{R}$ in its Taylor series (see Proposition IV.1.3)

$$R(\lambda, A) = \sum_{n=0}^\infty (\mu - \lambda)^n R(\mu, A)^{n+1}. \tag{4.12}$$

This series converges uniformly in $\mathcal{L}(X)$, provided that $|\mu - \lambda| \cdot \|R(\mu, A)\| \leq q < 1$ for some fixed $q \in (0,1)$. In particular, for $\mu = \mathrm{i}\operatorname{Im}\lambda$, we see from (4.11) that this is the case if $|\operatorname{Re}\lambda| \leq q/C\,|\operatorname{Im}\lambda|$. Since this is true for arbitrary $0 < q < 1$, we conclude that

$$\left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0 \text{ and } \left|\frac{\operatorname{Re}\lambda}{\operatorname{Im}\lambda}\right| < \frac{1}{C}\right\} \subset \rho(A),$$

and hence $\Sigma_{\pi/2+\delta} \subseteq \rho(A)$ for $\delta := \arctan 1/C$.

It remains to estimate $\|R(\lambda, A)\|$ for $\lambda \in \Sigma_{\pi/2+\delta-\varepsilon}$ and $\varepsilon \in (0,\delta)$. We assume first that $\operatorname{Re}\lambda > 0$. Then, by the Hille–Yosida Generation Theorem 3.8 for the bounded semigroup $(T(t))_{t\geq 0}$, there exists a constant $\widetilde{M} \geq 1$ such that $\|R(\lambda, A)\| \leq \tilde{M}/\operatorname{Re}\lambda$. Moreover, by (4.10), we have $\|R(\lambda, A)\| \leq C/|\operatorname{Im}\lambda|$; hence there exists $M \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|} \qquad \text{if } \operatorname{Re}\lambda > 0.$$

In the case $\operatorname{Re}\lambda \leq 0$, we choose $q \in (0,1)$ such that $\delta - \varepsilon = \arctan(q/C)$. Then $|\operatorname{Re}\lambda/\operatorname{Im}\lambda| \leq q/C$, and from estimate (4.11) combined with the Taylor expansion (4.12) for $\mu = \mathrm{i}\operatorname{Im}\lambda$ we obtain

$$\begin{aligned}\|R(\lambda, A)\| &\leq \sum_{n=0}^{\infty} |\operatorname{Re}\lambda|^n \frac{C^{n+1}}{|\operatorname{Im}\lambda|^{n+1}} \\ &\leq \frac{1}{1-q} \cdot \frac{C}{|\operatorname{Im}\lambda|} \leq \frac{\sqrt{C^2+1}}{1-q} \cdot \frac{1}{|\lambda|}.\end{aligned}$$

(e) $\Rightarrow$ (c). By Propositions 4.3 and 4.4, $A$ generates a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$, and the map

$$(0,\infty) \ni t \mapsto T(t)x \in X$$

is differentiable for all $x \in X$. In particular, the limit

$$\lim_{h\downarrow 0} \frac{T(t+h) - T(t)}{h} x = \lim_{h\downarrow 0} \frac{T(h) - I}{h} T(t)x$$

exists for all $x \in X$ and $t > 0$; hence $\operatorname{rg}(T(t)) \subset D(A)$ for $t > 0$.

Since for $t > 0$ the operator $AT(t)$ is closed with domain $D(AT(t)) = X$, it is bounded by the closed graph theorem.

To estimate its norm, we use the integral representation (4.2) of $T(t)$ and obtain, using the closedness of $A$, the resolvent equation, and Cauchy's integral theorem that

$$\begin{aligned}AT(t) &= A\,\frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\mu t} R(\mu, A)\, d\mu \\ &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{e}^{\mu t} \big(\mu R(\mu, A) - I\big)\, d\mu \\ &= \frac{1}{2\pi\mathrm{i}} \int_\gamma \mu \mathrm{e}^{\mu t} R(\mu, A)\, d\mu.\end{aligned}$$

Since by analyticity we may choose $\gamma = \gamma_r$ for $r := 1/t$ as in the proof of Proposition 4.3, we conclude, using (4.5) and (4.6), that

$$\begin{aligned}\left\| \int_\gamma \mu \mathrm{e}^{\mu t} R(\mu, A)\, d\mu \right\| &\leq 2M_\varepsilon \int_{1/t}^{\infty} \mathrm{e}^{-\rho t \sin\varepsilon} d\rho + \frac{2\pi \mathrm{e} M_\varepsilon}{t} \\ &\leq 2M_\varepsilon \left( \frac{1}{\sin\varepsilon} + \pi\mathrm{e} \right) \cdot \frac{1}{t},\end{aligned}$$

where $\varepsilon := (\delta-\delta')/2$ for some $\delta' \in (0,\delta)$. This proves (c).

(c) $\Rightarrow$ (a). We claim first that the map $t \mapsto T(t)x \in X$ is infinitely many times differentiable for all $t > 0$ and $x \in X$. In fact, using the formula $AT(s)y = T(s)Ay$, valid for $s \geq 0$ and $y \in D(A)$ (see Lemma 1.3), one easily verifies by induction that $\operatorname{rg}\big(T(t)\big) \subset D(A^\infty) = \cap_{n\in\mathbb{N}} D(A^n)$ and

$$A^n T(t) = \big(AT(t/n)\big)^n$$

for all $t > 0$ and $n \in \mathbb{N}$. We now fix some $\varepsilon \in (0, t)$. Then, by Lemma 1.3,

$$\begin{aligned} A^n T(t)x &= AT(t-\varepsilon)A^{n-1}T(\varepsilon)x \\ &= \tfrac{d}{dt} T(t-\varepsilon)A^{n-1}T(\varepsilon)x \\ &\;\;\vdots \\ &= \tfrac{d^n}{dt^n} T(t)x \end{aligned}$$

for all $x \in X$. This establishes our claim. Combining this with (4.9) and the inequality[2] $n!\,\mathrm{e}^n \geq n^n$, we obtain, while writing $T^{(n)}(t) := \frac{d^n}{dt^n} T(t)$,

$$\frac{1}{n!} \left\| T^{(n)}(t) \right\| \leq \left( \frac{\mathrm{e}M}{t} \right)^n \qquad \text{for all } n \in \mathbb{N} \text{ and } t > 0. \tag{4.13}$$

Next, we develop $T(t)$ in its Taylor series. To this end, we choose $t > 0$ and $x \in X$ arbitrary. Then, by Taylor's theorem, we have for $|h| < t$ and all $n \in \mathbb{N}$

$$T(t+h)x = \sum_{k=0}^{n} \frac{h^k}{k!} T^{(k)}(t)x + \frac{1}{n!} \int_t^{t+h} (t+h-s)^n T^{(n+1)}(s)x \, ds. \tag{4.14}$$

Denoting the integral term on the right-hand side of (4.14) by $R_{n+1}(t+h)x$, we see from (4.13) that

$$\lim_{n\to\infty} \|R_{n+1}(t+h)\| = 0$$

uniformly for $|h| \leq q \cdot t/\mathrm{e}M$ for every fixed $q \in (0,1)$. On the other hand, the series

$$T(z) := \sum_{k=0}^{\infty} \frac{(z-t)^k}{k!} T^{(k)}(t)$$

converges uniformly for all $z \in \mathbb{C}$ satisfying $|z - t| \leq q \cdot t/\mathrm{e}M$; hence it extends the given semigroup $\big(T(t)\big)_{t\geq 0}$ analytically to the sector $\Sigma_\delta$ for $\delta := \arctan\left(1/\mathrm{e}M\right)$. This proves (ii) of Definition 4.5.

[2] Taking logarithms, this inequality can be restated as $1/n \sum_{k=1}^{n} \log k/n \geq -1$, which follows from $\int_0^1 \log x \, dx = -1$.

In order to verify the semigroup property for $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}}$, we first take some $t>0$. Then the map $\Sigma_\delta \ni z \mapsto T(t)T(z) \in \mathcal{L}(X)$ is analytic and satisfies $T(t)T(z) = T(t+z)$ for $z \geq 0$. Hence, by the identity theorem for analytic functions, we conclude that $T(t)T(z) = T(t+z)$ for all $z \in \Sigma_\delta$. Now fix some $z_1 \in \Sigma_\delta$ and consider the map $\Sigma_\delta \ni z \mapsto T(z_1)T(z) \in \mathcal{L}(X)$. This map is analytic as well and satisfies $T(z_1)T(z) = T(z_1+z)$ for $z \geq 0$. Using the analyticity again, we obtain the functional equation $T(z_1)T(z_2) = T(z_1+z_2)$ for all $z_1, z_2 \in \Sigma_\delta$.

To verify that $z \mapsto T(z)$ is uniformly bounded on the sector $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$, we choose $q \in (0,1)$ such that $\delta' := \arctan\left({}^{q}\!/_{\mathrm{e}M}\right)$. Then, by equation (4.14),

$$\|T(z)\| = \left\|\sum_{k=0}^{\infty} \frac{(\mathrm{i}\,\mathrm{Im}\,z)^k}{k!}\, T^{(k)}(\mathrm{Re}\,z)\right\|$$

$$\leq \sum_{k=0}^{\infty} |\,\mathrm{Im}\,z|^k \left(\frac{\mathrm{e}M}{\mathrm{Re}\,z}\right)^k \leq \frac{1}{1-q}. \tag{4.15}$$

It remains only to prove that the map

$$\Sigma_{\delta'} \cup \{0\} \ni z \mapsto T(z) \in \mathcal{L}(X)$$

is strongly continuous in $z=0$. To this end, we choose $x \in X$ and $\varepsilon > 0$. Since $\big(T(t)\big)_{t\geq 0}$ is strongly continuous, there exists $h_0 > 0$ such that $\|T(h)x - x\| < \varepsilon(1-q)$ for all $0 < h < h_0$. Then, using (4.15), we obtain

$$\begin{aligned}\|T(z)x - x\| &\leq \big\|T(z)\big(x - T(h)x\big)\big\| + \|T(z+h)x - T(h)x\| + \|T(h)x - x\|\\ &< 2\varepsilon + \|T(z+h) - T(h)\| \cdot \|x\|\end{aligned}$$

for all $h \in (0, h_0)$. Since the map $z \mapsto T(z+h) \in \mathcal{L}(X)$ is analytic in some neighborhood of $z=0$, we have $\lim_{z\to 0} \|T(z+h) - T(h)\| = 0$, which completes the proof of the implication (c) ⇒ (a). □

After this long proof, we pause for a moment and give some abstract and concrete examples of analytic semigroups.

**4.7 Corollary.** *If $A$ is a normal operator on a Hilbert space $H$ satisfying*

$$\sigma(A) \subseteq \{z \in \mathbb{C} : \arg(-z) < \delta\} \tag{4.16}$$

*for some $\delta \in [0, {}^{\pi}\!/_{2})$, then $A$ generates a bounded analytic semigroup.*

Proof. Since $A$ is normal, the same is true for $R(\lambda, A)$ for all $\lambda \in \rho(A)$. Hence, by [TL80, Thm. VI.3.5] or [Wei80, Thm. 5.44], we have

$$\|R(\lambda, A)\| = \mathrm{r}\big(R(\lambda, A)\big),$$

and the assertion follows from Theorem 4.6.(d) combined with the Spectral Mapping Theorem for the Resolvent IV.1.13. □

A different proof of the previous result is indicated in Exercise 4.12.(9).

In particular, Corollary 4.7 shows that the semigroup generated by a self-adjoint operator that is bounded above (see Example 3.27) is analytic of angle $\pi/2$.

**4.8 Example.** In Paragraph 3.30.(ii) we showed that the closure $\overline{A}$ of the operator

$$Af := f'', \quad D(A) := \{f \in \mathrm{C}^2[0,1] : f(0) = f(1) = 0\}$$

generates a strongly continuous contraction semigroup $\big(T(t)\big)_{t\ge0}$ on the Hilbert space $H = \mathrm{L}^2[0,1]$. Since it is not difficult to show that

$$\overline{A}f := f'', \quad D(\overline{A}) := \{f \in \mathrm{H}^2[0,1] : f(0) = f(1) = 0\}$$

is self-adjoint, the semigroup $\big(T(t)\big)_{t\ge0}$ is analytic. See Exercise 4.12.(10) and, for more general operators, Section VI.4.d.

It is, however, even simpler to verify the inequality in (3.31) with $A$ replaced by $\mathrm{e}^{\pm i\vartheta}\overline{A}$ for some $\vartheta \in (0, \pi/2)$ in order to conclude that $\mathrm{e}^{\pm i\vartheta}\overline{A}$ are dissipative. Since $\rho(\mathrm{e}^{\pm i\vartheta}\overline{A}) = \mathrm{e}^{\pm i\vartheta}\rho(\overline{A})$, we then conclude by the Lumer–Phillips Theorem 3.15 that $\mathrm{e}^{\pm i\vartheta}\overline{A}$ are generators of contraction semigroups. Hence, Theorem 4.6.(b) implies that the operator $\overline{A}$ generates a bounded analytic semigroup on $H$.

Another important class of generators of analytic semigroups is provided by squares of group generators.

**4.9 Corollary.** *Let $A$ be the generator of a strongly continuous group. Then $A^2$ generates an analytic semigroup of angle $\pi/2$.*

Proof. Here we consider only the case where $A$ generates a bounded group and refer to [Nag86, A-II, Thm. 1.15] for the general case.

Take some $0 < \delta' < \pi/2$ and $\lambda \in \Sigma_{\pi/2+\delta'}$. Then there exists a square root $r\mathrm{e}^{i\alpha}$ of $\lambda$ with $0 < r$ and $|\alpha| < (\pi/2+\delta')/2 < \pi/2$, and we obtain

$$(\lambda - A^2) = (r\mathrm{e}^{i\alpha} - A)(r\mathrm{e}^{i\alpha} + A).$$

This implies $\lambda \in \rho(A^2)$ and $R(\lambda, A^2) = R(r\mathrm{e}^{i\alpha}, A)R(r\mathrm{e}^{i\alpha}, -A)$. Since $A$ generates a bounded group, there exists a constant $\widetilde{M} \ge 1$ such that

$$\|R(\mu, \pm A)\| \le \frac{\widetilde{M}}{\operatorname{Re}\mu} \qquad \text{for all } \mu \in \Sigma_{\pi/2}.$$

Consequently, one has

$$\begin{aligned} \big\|R(\lambda, A^2)\big\| &\le \frac{\widetilde{M}^2}{(r\cos\alpha)^2} \le \frac{1}{r^2}\left(\frac{\widetilde{M}}{\cos\left(\frac{\pi/2+\delta'}{2}\right)}\right)^2 \\ &= \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\pi/2+\delta'}, \end{aligned}$$

and the assertion follows from Propositions 4.3 and 4.4. □

**4.10 Example.** It is immediately clear from the discussion of the translation groups in Paragraph 2.10 that starting from $Af := f'$ (and appropriate domain) on $\mathrm{C}_0(\mathbb{R})$ or $\mathrm{L}^p(\mathbb{R})$, $1 \le p < \infty$, the operator

$$A^2 f = f''$$

generates a bounded analytic semigroup.

We now consider the slightly more involved case of several space dimensions, i.e., we consider the spaces $\mathrm{C}_0(\mathbb{R}^n)$ or $\mathrm{L}^p(\mathbb{R}^n)$, $1 \le p < \infty$. Denote by $\big(U_i(t)\big)_{t\in\mathbb{R}}$ the strongly continuous group given by

$$\big(U_i(t)f\big)(x) := f(x_1, \dots, x_{i-1}, x_i + t, \dots, x_n),$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $1 \le i \le n$, and let $A_i$ be its generator. Obviously, these semigroups commute as do the resolvents of $A_i$ and hence of $A_i^2$. Denote by $\big(T_i(t)\big)_{t\ge 0}$ the semigroup generated by $A_i^2$, which by Corollary 4.9 has an analytic extension $\big(T_i(z)\big)_{z\in\Sigma_{\pi/2}}$. These extensions also commute, and therefore

$$T(z) := T_1(z) \cdots T_n(z), \quad z \in \Sigma_{\pi/2},$$

defines a bounded analytic semigroup of angle $\frac{\pi}{2}$. The domain $D(A)$ of the generator $A$ of $\big(T(z)\big)_{z\in\Sigma_{\pi/2}\cup\{0\}}$ contains $D(A_1^2) \cap \cdots \cap D(A_n^2)$ by Paragraph 2.7. In particular, it contains

$$D_0 := \big\{ f \in X \cap \mathrm{C}^2(\mathbb{R}^n) : D^\alpha f \in X \text{ for every multi-index } \alpha \text{ with } |\alpha| \le 2 \big\},$$

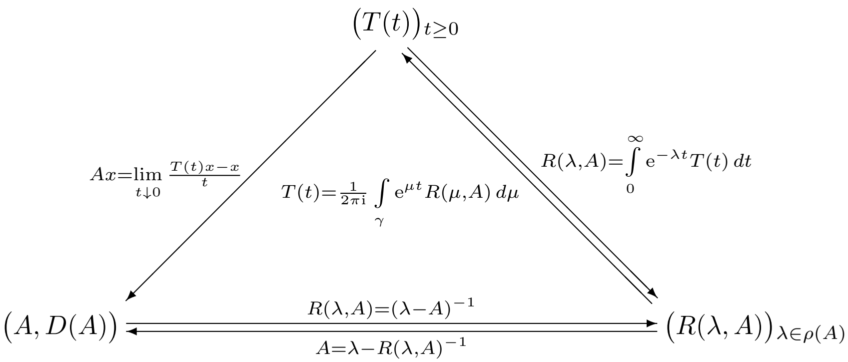
and for every $f \in D_0$ the generator is given by

$$Af = \big(A_1^2 + \cdots + A_n^2\big) f = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f = \Delta f.$$

For the characterization of multiplication operators generating analytic semigroups, we refer to Paragraph 4.32.

**4.11 Comment.** We point out that in most of the above results the density of the domain of $A$ is not needed. In fact, the integral (4.2) exists even for non-densely defined sectorial operators and yields an *analytic semigroup* without, however, the strong continuity in Proposition 4.3.(iv). This is treated in detail in [Lun95], but we indicate how this can be obtained within our semigroup framework (see Exercise 4.12.(8).)

We close this subsection by adding an arrow to Diagram 1.14 in the case of analytic semigroups.




**4.12 Exercises.** (1) Let $X$ be a Banach space and consider a function $F:\Omega\to\mathcal{L}(X)$ defined on an open set $\Omega\subseteq\mathbb{C}$. Show that the following assertions are equivalent.

(a) $F:\Omega\to\mathcal{L}(X)$ is analytic.

(b) $F(\cdot)x:\Omega\to X$ is analytic for all $x\in X$.

(c) $\langle F(\cdot)x,x'\rangle:\Omega\to\mathbb{C}$ is analytic for all $x\in X$ and $x'\in X'$.

(Hint: Use Cauchy's integral formula and the uniform boundedness principle.)

(2) Show that an analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ is exponentially bounded on $\Sigma_{\delta'}$ for every $0<\delta'<\delta$.

(3) Show that the generator $A$ of an analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ coincides with the "complex" generator, i.e.,

$$Ax=\lim_{\Sigma_{\delta'}\ni z\to 0}\frac{T(z)x-x}{z},\qquad D(A)=\left\{x\in X:\lim_{\Sigma_{\delta'}\ni z\to 0}\frac{T(z)x-x}{z}\text{ exists}\right\}$$

for every $0<\delta'<\delta$

(4) Show that for an analytic semigroup $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ on a Banach space $X$ one always has $T(t)X\subset D(A^\infty)$ for all $t>0$.

(5*) Give a proof of Corollary 4.9 in case the group $(T(t))_{t\in\mathbb{R}}$ is not necessarily bounded. (Hint: See [Nag86, A-II, Thm. 1.15].)

(6) Let $(A,D(A))$ be a closed, densely defined linear operator on a Banach space $X$. If there exist constants $\delta>0$, $r>0$, and $M\geq 1$ such that $\Sigma:=\{\lambda\in\mathbb{C}:|\lambda|>r\text{ and }|\arg(\lambda)|<\pi/2+\delta\}\subseteq\rho(A)$ and $\|R(\lambda,A)\|\leq M/|\lambda|$ for all $\lambda\in\Sigma$, then $A-w$ is sectorial for $w$ sufficiently large. In particular, $A$ generates an analytic semigroup.

(7) For an operator $(A,D(A))$ on a Banach space $X$ define on $\mathcal{X}:=X\times X$ the operator matrix

$$\mathcal{A}:=\begin{pmatrix}A & A\\ 0 & A\end{pmatrix}\quad\text{with domain}\quad D(\mathcal{A}):=D(A)\times D(A).$$

Show that the following assertions are equivalent.

(i) $A$ generates an analytic semigroup on $X$.

(ii) $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{X}$.

(iii) $\mathcal{A}$ generates an analytic semigroup on $\mathcal{X}$.

(Hint: If $A$ generates the semigroup $(T(t))_{t\geq 0}$, then the candidate for the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is given by $\mathcal{T}(t) = \begin{pmatrix} T(t) & tAT(t) \\ 0 & T(t) \end{pmatrix}$. Now use Theorem 4.6.(c).)

(8) Let $(A, D(A))$ be a sectorial operator, i.e., satisfying (4.1), with not necessarily dense domain in the Banach space $X$. Prove the following assertions in order to obtain an analytic semigroup generated by $A$.

(i) The part $(A_0, D(A_0))$ of $A$ in $X_0 := \overline{D(A)}$ is densely defined and sectorial, hence generates a bounded analytic semigroup $(T_0(z))_{z\in\Sigma_\delta\cup\{0\}}$ on $X_0$.

(ii) Construct the associated Sobolev spaces $X_n$, $n \in \mathbb{Z}$, and observe that each of the extended/restricted semigroups $(T_n(z))_{z\in\Sigma_\delta\cup\{0\}}$ is also analytic.

(iii) For $0 \neq z \in \Sigma_\delta$, the operators $T_n(z)$ are bounded from $X_n$ to $X_m$ whenever $m \geq n$, and the integral (4.2) converges in the corresponding operator norm.

(iv) For the given Banach space $X$, one has

$$X_0 \subset X \hookrightarrow X_{-1},$$

and hence $(T_{-1}(z))_{z\in\Sigma_\delta\cup\{0\}}$ induces an analytic semigroup on $X$ that is not strongly continuous in the case $X_0 \neq X$. (Hint: See also Exercise 5.23.(3).)

(9) Give an alternative proof of Corollary 4.7 based on the Spectral Theorem I.4.9 and multiplication semigroups from Section I.4.b. (Hint: Observe the theorem in Paragraph 4.32.)

(10) Show that the operator $A$ in Example 4.8 is self-adjoint.

(11*) Show that for every closed and densely defined operator $T$ on a Hilbert space $H$ the operator $T^*T$ is self-adjoint and positive semidefinite. (Hint: See [Ped89, Thm. 5.1.9].)

(12) Consider the first derivative $D := {}^{d}/_{dx}$ on $\mathrm{L}^2[a, b]$ with the domains

$$D\big(D_0\big) := \mathrm{H}^1_0[a, b] := \big\{f \in \mathrm{H}^1[a, b] : f(a) = 0 = f(b)\big\} \quad \text{and} \quad D\big(D_m\big) := \mathrm{H}^1[a, b].$$

(i) Show that $(D_0)^* = -D_m$ and $(D_m)^* = -D_0$.

(ii) Show that $\Delta_D := D_m D_0$ and $\Delta_N := D_0 D_m$ generate analytic semigroups. Write down these operators explicitly. Compare this with Example 4.8. (Hint: Use Exercises (11).)

## *b. Differentiable Semigroups*

To motivate the class of semigroups to be introduced now, we recall that for a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$ on a Banach space $X$ the orbit maps

$$\xi_x : t \mapsto T(t)x$$

are differentiable for $t \geq 0$ if (and only if) $x \in D(A)$ (see (1.3)). Hence, these orbits are differentiable for all $x \in X$ only if $A$ is bounded (and $D(A) = X$). In the following definition, we now require $\xi_x$ to be differentiable for *all* $x \in X$, but not for all $t \geq 0$.

**4.13 Definition.** *A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ is called eventually differentiable if there exists $t_0 \geq 0$ such that the orbit maps $\xi_x : t \mapsto T(t)x$ are differentiable on $(t_0, \infty)$ for every $x \in X$. The semigroup is called immediately differentiable if $t_0$ can be chosen as $t_0 = 0$.*

Using the definition of $D(A)$ and the semigroup law (FE), eventual differentiability for $(T(t))_{t\geq 0}$ means that the operators $T(t)$ map $X$ into $D(A)$ as soon as $t > t_0$. Since $A$ is closed and $T(t)$ is bounded, it follows from the closed graph theorem that $AT(t)$ is bounded on $X$ for $t > t_0$, or, in other words, $T(t) \in \mathcal{L}(X_0, X_1)$, where $X_1$ denotes the Sobolev space $(D(A), \|\cdot\|_A)$ as in Definition 5.1 below. Moreover, it follows again from the semigroup law (FE) that the $n$th derivative

$$\frac{d^n}{dt^n}\xi_x(t) = A^n T(t)x = \big(AT(t/n)\big)^n x$$

exists on $(nt_0, \infty)$ for all $x \in X$ and that $T(t) \in \mathcal{L}(X_0, X_n)$ for $t > nt_0$. In particular, for an immediately differentiable semigroup, the orbit maps $\xi_x$ are infinitely differentiable on $(0, \infty)$ for every $x \in X$. This property already appeared in the proof of (c) $\Rightarrow$ (a) in Theorem 4.6 and clearly holds for analytic semigroups.

We now look for a characterization of differentiable semigroups in terms of the spectrum and the growth of the resolvent of its generator. We obtain that the spectrum has to be limited by a function of (at most) exponential growth. This should be compared with the fact, already shown in Sections 1 and 4.a, that the spectrum is included in a left half-plane (for strongly continuous semigroups) or in a sector (for analytic semigroups).

**4.14 Theorem.** *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying the norm estimate $\|T(t)\| \leq M\mathrm{e}^{wt}$ and having generator $(A, D(A))$, the following properties are equivalent.*

(a) *$(T(t))_{t\geq 0}$ is eventually differentiable.*

(b) *There exist constants $a > 0$, $b > 0$, and $C > 0$ such that*

$$\Theta := \Big\{\lambda \in \mathbb{C} : a\,\mathrm{e}^{-b\operatorname{Re}\lambda} \leq |\operatorname{Im}\lambda|\Big\} \subset \rho(A) \tag{4.17}$$

*and*

$$\|R(\lambda, A)\| \leq C|\operatorname{Im}\lambda| \tag{4.18}$$

*for all $\lambda \in \Theta$ with $\operatorname{Re}\lambda \leq w$.*

Proof. Here we show only that (a) $\Rightarrow$ (b) and refer to [Paz83, Chap. 2, Thm. 4.7] for the other implication.

Assume $(T(t))_{t\geq 0}$ to be differentiable for $t > t_0$. Then from Lemma 1.9, we obtain

$$\lambda\mathrm{e}^{t\lambda}x - AT(t)x = (\lambda - A)\Big(T(t) + \lambda\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)\,ds\Big)x \quad \text{for } x \in X \tag{4.19}$$

$$= \Big(T(t) + \lambda\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)\,ds\Big)(\lambda - A)x \quad \text{for } x \in D(A) \tag{4.20}$$

for all $\lambda \in \mathbb{C}$. This implies that $\lambda\mathrm{e}^{t\lambda} - AT(t)$ fails to be bijective for all $\lambda \in \sigma(A)$, and hence we conclude that

$$\lambda\mathrm{e}^{t\lambda} \in \sigma\big(AT(t)\big) \qquad \text{for all } \lambda \in \sigma(A). \tag{4.21}$$

Therefore, if we set $a(t) := \|AT(t)\|$, we obtain

$$\sigma(A) \subseteq \big\{\lambda \in \mathbb{C} : \lambda\mathrm{e}^{t\lambda} \in \sigma\big(AT(t)\big)\big\} \subseteq \big\{\lambda \in \mathbb{C} : \big|\lambda\mathrm{e}^{t\lambda}\big| \leq a(t)\big\}.$$

Consequently, we have

$$\rho(A) \supset \left\{ \lambda \in \mathbb{C} : (1+\delta)a(t)\mathrm{e}^{-\operatorname{Re}\lambda t} \le |\operatorname{Im}\lambda| \right\}$$

for all $\delta > 0$. This shows (4.17) for

$$a := (1+\delta)a(t) \quad \text{and} \quad b := t.$$

In order to prove the resolvent estimate (4.18), we multiply (4.20) by $R(\lambda, A)$ from the right and obtain

$$\lambda \mathrm{e}^{\lambda t} R(\lambda, A) = AT(t)R(\lambda, A) + T(t) + \lambda \mathrm{e}^{\lambda t} \int_0^t \mathrm{e}^{-\lambda s} T(s)\, ds. \tag{4.22}$$

If $\lambda \in \Theta$, we have

$$\frac{a(t)\mathrm{e}^{-t \operatorname{Re}\lambda}}{|\operatorname{Im}\lambda|} \le \frac{1}{1+\delta}, \tag{4.23}$$

and together with (4.22) this implies

$$\begin{aligned} \|R(\lambda, A)\| &\le \frac{\mathrm{e}^{-t \operatorname{Re}\lambda}}{|\operatorname{Im}\lambda|} \left( a(t)\|R(\lambda, A)\| + M\mathrm{e}^{wt} \right) + M \int_0^t \mathrm{e}^{(w-\operatorname{Re}\lambda)s}\, ds \\ &\le \frac{1}{1+\delta}\|R(\lambda, A)\| + \frac{M\mathrm{e}^{wt}}{(1+\delta)a(t)} + Mt\mathrm{e}^{(w-\operatorname{Re}\lambda)t} \end{aligned}$$

if $\operatorname{Re}\lambda \le w$. Using (4.23) again, this yields

$$\begin{aligned} \|R(\lambda, A)\| &\le \frac{1+\delta}{\delta} M\mathrm{e}^{wt} \left( \frac{1}{(1+\delta)a(t)} + t\mathrm{e}^{-\operatorname{Re}\lambda t} \right) \\ &\le \frac{M\mathrm{e}^{wt}}{\delta a(t)} \left( \frac{\mathrm{e}^{wt}}{(1+\delta)a(t)} + t \right) \cdot |\operatorname{Im}\lambda| =: C \cdot |\operatorname{Im}\lambda| \end{aligned}$$

for all $\lambda \in \Theta$ satisfying $\operatorname{Re}\lambda \le w$. This shows (4.18), and the proof of (a) $\Rightarrow$ (b) is complete. □

The above proof shows that for a strongly continuous semigroup that is differentiable for $t > t_0$ the following holds: For every $b > t_0$, we can find positive constants $a := a_b$ and $C := C_b$ such that condition (b) in Theorem 4.14 is satisfied. This remark proves one-half of the following characterization for immediately differentiable semigroups. For the proof of the converse implication we refer to [Paz83, Chap. 2, Thm. 4.8].

**4.15 Corollary.** *For a strongly continuous semigroup $(T(t))_{t\ge 0}$ with generator $(A, D(A))$ and satisfying $\|T(t)\| \le M\mathrm{e}^{wt}$, the following properties are equivalent.*

(a) *$(T(t))_{t\ge 0}$ is immediately differentiable.*

(b) *For all $b > 0$, there exist constants $a_b > 0$ and $C_b > 0$ such that*

$$\Theta_b := \left\{ \lambda \in \mathbb{C} : a_b\, \mathrm{e}^{-b \operatorname{Re}\lambda} \le |\operatorname{Im}\lambda| \right\} \subset \rho(A)$$

*and*

$$\|R(\lambda, A)\| \le C_b |\operatorname{Im}\lambda|$$

*for all $\lambda \in \Theta_b$ with $\operatorname{Re}\lambda \le w$.*

The simplest example of a strongly continuous semigroup being eventually differentiable but not immediately differentiable is provided by a nilpotent semigroup. More examples, e.g., semigroups that are immediately differentiable but not analytic, can be obtained from multiplication semigroups (see Counterexample 4.33).

**4.16 Exercises.** (1) Show that for a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ the following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ is eventually differentiable for $t > t_0$.

(ii) The map $t_0 < t \mapsto T(t) \in (\mathcal{L}(X), \|\cdot\|)$ is differentiable.

(2*) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$ with generator $A$ and growth bound $\omega_0$. If for some $r > \omega_0$

$$\overline{\lim_{|s|\to\infty}} \log|s| \cdot \|R(r+is, A)\| = 0,$$

then $(T(t))_{t\geq 0}$ is immediately differentiable. (Hint: Use a Taylor series expansion of $R(\lambda, A)$ in order to verify the condition (b) of Corollary 4.15.)

## c. Eventually Norm-Continuous Semigroups

Continuity of $t \mapsto T(t)$ on $[0,\infty)$ for the operator norm makes the semigroup $\big(T(t)\big)_{t\geq 0}$ "trivial" in the sense of Theorem I.3.7. Norm continuity at some later time $s_0 > 0$, however, is an interesting property. Before stating the adequate definition, we observe that due to the semigroup law, the condition

$$\lim_{t\downarrow s_0} \|T(t) - T(s_0)\| = 0 \qquad \text{for some } s_0 \geq 0$$

already implies that the function $t \mapsto T(t)$ is norm continuous on the half-line $[s_0,\infty)$. This leads to the following concept.

**4.17 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ is called eventually norm continuous if there exists $t_0 \geq 0$ such that the function*

$$t \mapsto T(t)$$

*is norm continuous from $(t_0,\infty)$ into $\mathcal{L}(X)$. The semigroup is called immediately norm continuous if $t_0$ can be chosen to be $t_0 = 0$.*

This is a very important class of semigroups. In particular, it is nontrivial in the sense that the generator of a semigroup in this class can be unbounded, but it still enjoys special properties like the spectral mapping theorem stated in Lemma I.3.13 (see Section IV.3). In addition, it includes analytic and differentiable semigroups; see Exercise 4.21.(1).

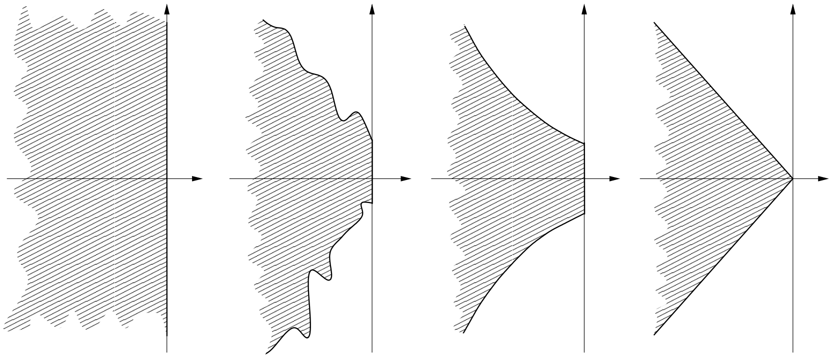



**Figure 2**

As before, we now strive for a characterization of eventually norm-continuous semigroups through the spectrum of its generator. From the previous results on analytic and differentiable semigroups, we expect some kind of "imaginary boundedness." Before stating the result precisely, we visualize in Figure 2 the location of the spectrum for the various classes of semigroups introduced so far.

We now prove the statement expressed by this figure.

**4.18 Theorem.** *Let $\big(A, D(A)\big)$ be the generator of an eventually norm-continuous semigroup $\big(T(t)\big)_{t\geq 0}$. Then, for every $b \in \mathbb{R}$, the set*

$$\big\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq b\big\}$$

*is bounded.*

Proof. Fix $a \in \mathbb{R}$ larger than the growth bound $\omega_0$ of $\big(T(t)\big)_{t\geq 0}$. If we show that for every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $r_0 \geq 0$ such that

$$\|R(a + \mathrm{i}r, A)^n\|^{1/n} < \varepsilon \qquad \text{for all } r \in \mathbb{R} \text{ with } |r| \geq r_0,$$

then the assertion follows from the inequality

$$\begin{aligned}\operatorname{dist}\big(a + \mathrm{i}r, \sigma(A)\big) &= \frac{1}{\mathrm{r}\big(R(a + \mathrm{i}r, A)\big)}\\ &\geq \|R(a + \mathrm{i}r, A)^n\|^{-1/n} > \frac{1}{\varepsilon}\end{aligned}$$

(see Corollary IV.1.14).

First, we obtain from the integral representation of the resolvent (see Corollary 1.11) that

$$R(\lambda, A)^{n+1}x = \frac{1}{n!}\int_0^\infty \mathrm{e}^{-\lambda t} t^n T(t)x\, dt$$

for all $x \in X$, $n \in \mathbb{N}$, and $\operatorname{Re}\lambda > \omega_0$. Now choose $t_1 > 0$ such that $t \mapsto T(t)$ is norm continuous on $[t_1, \infty)$ and choose $w \in (\omega_0, a)$, $M \geq 1$ such that $\|T(t)\| \leq M\mathrm{e}^{wt}$ for $t \geq 0$. Finally, set $N := M \cdot \int_0^{t_1} \mathrm{e}^{-at}\mathrm{e}^{wt}\, dt$ and take $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$ and $t_2 > t_1$ such that

$$\frac{N \cdot t_1^n}{n!} < \frac{\varepsilon^{n+1}}{3} \qquad \text{and} \qquad \frac{1}{n!}\int_{t_2}^\infty t^n \mathrm{e}^{-at}\, \|T(t)\|\, dt < \frac{\varepsilon^{n+1}}{3}.$$

Now apply the Riemann–Lebesgue lemma (see Theorem C.8) to the norm-continuous function $t \mapsto t^n\mathrm{e}^{-at}T(t)$ on $[t_1, t_2]$ to obtain $r_0 \geq 0$ such that

$$\left\| \frac{1}{n!}\int_{t_1}^{t_2} t^n \mathrm{e}^{-\mathrm{i}rt}\mathrm{e}^{-at} T(t)\, dt \right\| < \frac{\varepsilon^{n+1}}{3}$$

whenever $|r| \geq r_0$. The combination of these three estimates yields

$$\begin{aligned}
\big\|R(a+\mathrm{i}r, A)^{n+1}x\big\| &= \frac{1}{n!}\left\| \int_0^\infty \mathrm{e}^{-(a+\mathrm{i}r)t} t^n T(t)x\, dt \right\| \\
&\leq \frac{1}{n!}\int_0^{t_1} \mathrm{e}^{-at}t^n\, \|T(t)x\|\, dt + \frac{1}{n!}\left\| \int_{t_1}^{t_2} t^n \mathrm{e}^{-\mathrm{i}rt}\mathrm{e}^{-at} T(t)x\, dt \right\| \\
&\quad + \frac{1}{n!}\int_{t_2}^\infty \mathrm{e}^{-at}t^n\, \|T(t)x\|\, dt \\
&\leq \left( \frac{1}{n!}\, t_1^n \int_0^{t_1} \mathrm{e}^{-at} M\mathrm{e}^{wt}\, dt + \frac{2}{3}\,\varepsilon^{n+1} \right) \cdot \|x\| \\
&\leq \left( \frac{1}{n!}\, t_1^n N + \frac{2}{3}\,\varepsilon^{n+1} \right) \cdot \|x\| \leq \varepsilon^{n+1} \cdot \|x\|
\end{aligned}$$

for all $x \in X$. □

By analyzing the previous proof, one sees that in the case where $\big(T(t)\big)_{t\geq 0}$ is immediately norm continuous, one can choose $t_1 = 0$ and $n = 0$. This observation yields the following result.

**4.19 Corollary.** *If $\big(A, D(A)\big)$ is the generator of an immediately norm-continuous semigroup $\big(T(t)\big)_{t\geq 0}$, then*

$$\lim_{r \to \pm\infty} \|R(a+\mathrm{i}r, A)\| = 0 \tag{4.24}$$

*for all $a > \omega_0$.*

**4.20 Immediately Norm-Continuous Semigroups on Hilbert Spaces.** A satisfactory characterization of eventually norm-continuous semigroups is still lacking. On Hilbert spaces, however, the converse of Corollary 4.19 is true, i.e., (4.24) characterizes generators of immediately norm-continuous semigroups. More precisely, we have the following result.

**Theorem.** *Let $A$ be the generator of a uniformly exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $H$. Then the following conditions are equivalent.*

(a) $(T(t))_{t\geq 0}$ *is immediately norm-continuous.*

(b) $\lim_{\mathbb{R}\ni r\to\pm\infty} \|R(\mathrm{i}r, A)\| = 0.$

Proof. The assertion that (a) ⇒ (b) is just Corollary 4.19.

In order to prove that (b) ⇒ (a), it suffices to show that the operator family $\big(t^2T(t)\big)_{t\geq 0}$ is norm continuous for $t > 0$. To this end, we choose $k = 3$ in Corollary III.5.16 below and obtain

$$\begin{aligned}\big\|t^2T(t)x - s^2T(s)x\big\| &= \frac{1}{\pi}\left\|\int_{-\infty}^{\infty}\big(\mathrm{e}^{\mathrm{i}\lambda t}-\mathrm{e}^{\mathrm{i}\lambda s}\big)R(\mathrm{i}\lambda, A)^3x\,d\lambda\right\| \\ &\leq \frac{1}{\pi}\left\|\int_{|\lambda|\geq N}\big(\mathrm{e}^{\mathrm{i}\lambda t}-\mathrm{e}^{\mathrm{i}\lambda s}\big)R(\mathrm{i}\lambda, A)^3x\,d\lambda\right\| \\ &\quad+\frac{1}{\pi}\int_{|\lambda|\leq N}\big|\mathrm{e}^{\mathrm{i}\lambda t}-\mathrm{e}^{\mathrm{i}\lambda s}\big|\cdot\big\|R(\mathrm{i}\lambda, A)^3x\big\|\,d\lambda \\ &=: I_1(N) + I_2(N).\end{aligned}$$

Let $\varepsilon > 0$. Then $I_1(N) < \varepsilon\,\|x\|$ for $N$ sufficiently large. To prove this claim take $x^* \in H$. Then, by the Cauchy–Schwarz and Hölder inequalities, we conclude that

(4.25)
$$\begin{aligned}&\left|\left(\int_{|\lambda|\geq N}\big(\mathrm{e}^{\mathrm{i}\lambda t}-\mathrm{e}^{\mathrm{i}\lambda s}\big)R(\mathrm{i}\lambda, A)^3x\,d\lambda\,\middle|\,x^*\right)\right| \\ &\leq 2\int_{|\lambda|\geq N}\big|\big(R(\mathrm{i}\lambda, A)^2x\,|\,R(\mathrm{i}\lambda, A)^*x^*\big)\big|\,d\lambda \\ &\leq 2\left(\int_{|\lambda|\geq N}\big\|R(\mathrm{i}\lambda, A)^2x\big\|^2\,d\lambda\right)^{1/2}\cdot\left(\int_{|\lambda|\geq N}\|R(\mathrm{i}\lambda, A)^*x^*\|^2\,d\lambda\right)^{1/2} \\ &\leq 2\sup_{|r|\geq N}\|R(\mathrm{i}r, A)\|\cdot\left(\int_{-\infty}^{\infty}\|R(\mathrm{i}\lambda, A)x\|^2\,d\lambda\right)^{1/2}\cdot\left(\int_{-\infty}^{\infty}\|R(\mathrm{i}\lambda, A)^*x^*\|^2\,d\lambda\right)^{1/2} \\ &= 4\pi\sup_{|r|\geq N}\|R(\mathrm{i}r, A)\|\cdot\|T(\cdot)x\|_2\cdot\|T(\cdot)^*x^*\|_2\,,\end{aligned}$$

where in the last step we used the equality

$$\begin{aligned}\|T(\cdot)x\|_2 := \left(\int_0^{\infty}\|T(t)x\|^2\,dt\right)^{1/2} &= \left(\int_{-\infty}^{\infty}\left\|\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\mathrm{e}^{-\lambda\mathrm{i}t}T(t)x\,dt\right\|^2\,d\lambda\right)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}\|R(\mathrm{i}\lambda, A)x\|^2\,d\lambda\right)^{1/2},\end{aligned}$$

which follows from Plancherel's theorem for the Hilbert space-valued Fourier transform (see Theorem C.14). Since $(T(t))_{t\geq 0}$ and its adjoint $(T(t)^*)_{t\geq 0}$ are both uniformly exponentially stable, there exists $M > 0$ such that

$$\|T(\cdot)x\|_2 \leq M\,\|x\| \qquad \text{and} \qquad \|T(\cdot)^*x^*\|_2 \leq M\,\|x^*\|.$$

Combining this with estimate (4.25), we obtain

$$\begin{aligned}
&\left\|\int_{|\lambda|\geq N} \left(e^{i\lambda t} - e^{i\lambda s}\right) R(i\lambda, A)^3 x\, d\lambda\right\| \\
&\qquad = \sup_{\|x^*\|\leq 1} \left| \left( \int_{|\lambda|\geq N} \left(e^{i\lambda t} - e^{i\lambda s}\right) R(i\lambda, A)^3 x\, d\lambda \,\middle|\, x^* \right) \right| \\
&\qquad \leq 4\pi M^2 \cdot \sup_{|r|\geq N} \|R(ir, A)\| \cdot \|x\|,
\end{aligned}$$

and hence

$$I_1(N) \leq 4M^2 \cdot \sup_{|r|\geq N} \|R(ir, A)\| \cdot \|x\|.$$

Since $\lim_{\mathbb{R}\ni r\to\pm\infty} \|R(ir, A)\| = 0$, there exists $N > 0$ such that

$$4M^2 \cdot \sup_{|r|\geq N} \|R(ir, A)\| < \varepsilon,$$

which yields the desired estimate $I_1(N) < \varepsilon\,\|x\|$ for every $x \in D(A^2)$.

In order to estimate $I_2(N)$, choose $\delta \in (0, t)$ such that

$$\left|e^{i\lambda t} - e^{i\lambda s}\right| \cdot \left\|R(i\lambda, A)^3\right\| < \frac{\varepsilon}{2N}$$

for all $s \in (t - \delta, t + \delta)$, $\lambda \in [-N, N]$. Then $I_2(N) < \varepsilon\|x\|$ for all $x \in H$.

Using these estimates and the fact that $D(A^2)$ is dense in $H$ we obtain

$$\left\|t^2 T(t) - s^2 T(s)\right\| < 2\varepsilon$$

for all $s \in (t - \delta, t + \delta)$. This completes the proof. □

Clearly, the assumption of exponential stability can be achieved by a simple rescaling of the original semigroup; cf. Paragraph I.5.11.

On general Banach spaces it seems to be unknown whether the condition (4.24) forces a semigroup to be immediately norm continuous, cf. [EME96] or [BM96].

**4.21 Exercises.** (1) If a strongly continuous semigroup is eventually differentiable for some $t_0 \geq 0$, then it is eventually norm continuous for the same $t_0$.
(2) Let $A$ be the generator of an eventually norm-continuous semigroup $(T(t))_{t\geq 0}$ of positive operators on a Banach lattice $X$. If $(T(t))_{t\geq 0}$ is bounded and $s(A) = 0$, or if $A$ has compact resolvent, then the *boundary spectrum*

$$\sigma_+(A) := \sigma(A) \cap \left(s(A) + i\mathbb{R}\right)$$

of $A$ satisfies $\sigma_+(A) \subseteq \{s(A)\}$. (Hint: Use Theorem VI.1.12.(i) or [Nag86, C-III, Sec. 2].)

## d. Eventually Compact Semigroups

Up to now we have classified the semigroups according to the smoothness (or regularity) properties of the maps $t \mapsto T(t)$. In this subsection we introduce a property of the semigroup based on the "regularity" of a single operator. We prepare for the definition with the following lemma.

**4.22 Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$. If $T(t_0)$ is compact for some $t_0 > 0$, then $T(t)$ is compact for all $t \geq t_0$, and the map $t \mapsto T(t)$ is norm continuous on $[t_0, \infty)$.*

PROOF. The first assertion follows immediately from the semigroup law (FE). By Lemma I.5.2, we know that $\lim_{h\to 0} T(s+h)x = T(s)x$ for all $s \geq 0$ uniformly for $x$ in any compact subset $K$ of $X$. Let $U$ be the unit ball in $X$. Since $T(t_0)$ is compact, we have that $K := \overline{T(t_0)U}$ is compact, and hence

$$\lim_{s\to t}\big(T(t)x - T(s)x\big) = \lim_{s\to t}\big(T(t-t_0) - T(s-t_0)\big)T(t_0)x = 0$$

for arbitrary $t \geq t_0$ and uniformly for $x \in U$. □

**4.23 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ is called immediately compact if $T(t)$ is compact for all $t > 0$ and eventually compact if there exists $t_0 > 0$ such that $T(t_0)$ is compact.*

From Lemma 4.22 we obtain that an immediately (eventually) compact semigroup is immediately (eventually) norm continuous. In addition, one might expect some relation between the compactness of the semigroup and compactness of the resolvent of its generator. Before introducing the appropriate terminology, we observe that due to the resolvent equation, a resolvent operator is compact for one $\lambda \in \rho(A)$ if and only if it is compact for all $\lambda \in \rho(A)$.

**4.24 Definition.** *A linear operator $A$ with $\rho(A) \neq \emptyset$ has compact resolvent if its resolvent $R(\lambda, A)$ is compact for one (and hence all) $\lambda \in \rho(A)$.*

Operators with compact resolvent on infinite-dimensional Banach spaces are necessarily unbounded (see Exercise 4.30.(1)). For concrete operators, the following characterization is quite useful.

**4.25 Proposition.** *Let $\big(A, D(A)\big)$ be an operator on $X$ with $\rho(A) \neq \emptyset$ and take $X_1 := \big(D(A), \|\cdot\|_A\big)$ (see Section 5.a and Exercise 5.9.(1)). Then the following assertions are equivalent.*

(a) *The operator $A$ has compact resolvent.*

(b) *The canonical injection $i : X_1 \hookrightarrow X$ is compact.*

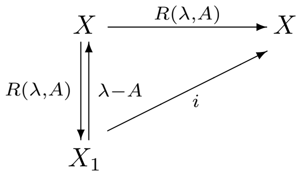
PROOF. Observe that for every $\lambda \in \rho(A)$, the graph norm $\|\cdot\|_A$ is equivalent to the norm

$$|||x|||_\lambda := \|(\lambda - A)x\|$$

(see the proof of Proposition 5.2.(i)). Therefore, the operator

$$R(\lambda, A) : X \to X_1$$

is an isomorphism with continuous inverse $\lambda - A$. The assertion then follows from the following factorization.




□

This proposition allows us to prove that differential operators on certain function spaces have compact resolvent. It suffices to apply appropriate *Sobolev embedding theorems*; see, e.g., [RR93, Sec. 6.4]. Here is a very simple example.

**4.26 Example.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and take $X = \mathrm{C}_0(\Omega)$. Assume that $\big(A, D(A)\big)$ is an operator on $X$ such that $D(A)$ is a continuously embedded subspace of the Banach space

$$\mathrm{C}_0^1(\Omega) := \big\{f \in \mathrm{C}_0(\Omega) : f \text{ is differentiable and } f' \in \mathrm{C}_0(\Omega)\big\}.$$

By the Arzelà–Ascoli theorem, the injection $i : \mathrm{C}_0^1(\Omega) \hookrightarrow \mathrm{C}_0(\Omega)$ is compact, whence $A$ has compact resolvent whenever $\rho(A) \neq \emptyset$. See Exercise 4.30.(4) for the analogous $\mathrm{L}^p$-result.

The relation between compactness of the semigroup and the resolvent is not simple. We show first what is not true.

**4.27 Examples.** (i) Consider the translation semigroup on the Banach space $\mathrm{L}^1([0,1] \times [0,1])$ defined by

$$T(t)f(r,s) := \begin{cases} f(r+t,s) & \text{for } r+t \le 1; \\ 0 & \text{for } r+t > 1. \end{cases}$$

This semigroup is nilpotent, hence eventually compact. However, its generator does not have compact resolvent. (See Exercise 4.30.(5).)

(ii) The generator of the periodic translation group (or rotation group, see Paragraph I.4.18 and Example IV.2.29) has compact resolvent. The group, however, does not have any of the smoothness properties defined above.

**4.28 Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup with generator $A$. Moreover, assume that the map $t \mapsto T(t)$ is norm continuous at some point $t_0 \geq 0$ and that $R(\lambda, A)T(t_0)$ is compact for some (and hence all) $\lambda \in \rho(A)$. Then the operators $T(t)$ are compact for all $t \geq t_0$.*

PROOF. As usual, we may assume that $0 \in \rho(A)$. For the operators $V(t)$ defined by $V(t)x := \int_0^t T(s)x\,ds$ for $x \in X$ and $t \geq 0$ one has

$$AV(t)x = T(t)x - x \qquad \text{for all } x \in X;$$

hence

$$V(t) = R(0, A)\big(I - T(t)\big).$$

The norm continuity for $t \geq t_0$ implies

$$T(t_0) = \lim_{h\downarrow 0} \frac{1}{h}\big(V(t_0 + h) - V(t_0)\big)$$

in operator norm. Since it follows from the assumptions that $V(t_0 + h) - V(t_0)$ is compact for all $h > 0$, this implies that $T(t_0)$ as the norm limit of compact operators is compact as well. ☐

**4.29 Theorem.** *For a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ the following properties are equivalent.*

(a) *$\big(T(t)\big)_{t\geq 0}$ is immediately compact.*

(b) *$\big(T(t)\big)_{t\geq 0}$ is immediately norm continuous, and its generator has compact resolvent.*

PROOF. If $\big(T(t)\big)_{t\geq 0}$ is immediately compact, it is immediately norm continuous by Lemma 4.22. Therefore, the integral representation for the resolvent in Theorem 1.10.(i) exists in the norm topology; hence $R(\lambda, A)$ is compact. The converse implication follows from Lemma 4.28. ☐

We close these considerations by visualizing the implications between the various classes of semigroups in the following diagram:

$$\begin{array}{ccccc}
\text{analytic} \Longrightarrow & \text{immediately differentiable} & \Longrightarrow & \text{eventually differentiable} \\
 & \Downarrow & & \Downarrow \\
 & \text{immediately norm continuous} & \Longrightarrow & \text{eventually norm continuous} \\
 & \Uparrow & & \Uparrow \\
 & \text{immediately compact} & \Longrightarrow & \text{eventually compact}
\end{array} \tag{4.26}$$

That all these classes are different will be shown in the following subsection.

**4.30 Exercises.** (1) A bounded operator $A \in \mathcal{L}(X)$ has compact resolvent if and only if $X$ is finite-dimensional.

(2) Let $(A, D(A))$ be an operator on a Banach space $X$ having compact resolvent and let $B \in \mathcal{L}(X)$ be such that $\rho(A+B) \neq \emptyset$. Then $A+B$ has compact resolvent. (Hint: Use the formula $U^{-1} - V^{-1} = U^{-1}(V-U) - V^{-1}$ valid for each pair of invertible operators having the same domain.)

(3) Let $(A, D(A))$ be an operator on a Banach space $X$ having finite-dimensional kernel $\ker(A)$. If $A_1, A_2 \subset A$ are two invertible restrictions of $A$, then $A_1$ has compact resolvent if and only if $A_2$ has.

(4) Let $X := \mathrm{L}^p(\Omega)$ for $1 \leq p < \infty$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. If $(A, D(A))$ is an operator on $X$ satisfying $\rho(A) \neq \emptyset$ and $D(A) \subset \mathrm{W}^{1,p}(\Omega)$, then $A$ has compact resolvent. (Hint: Use Corollary B.7 and Sobolev's embedding theorem.)

(5) Show that the generator of the semigroup in Example 4.27 does not have compact resolvent. (Hint: Compute the resolvent, using the integral representation (1.14), on functions of the form $f(r,s) := h(r)g(s)$ for $0 \leq r, s \leq 1$ and $h, g \in \mathrm{L}^1[0,1]$.)

(6) Show that an analytic, eventually compact semigroup $(T(t))_{t\geq 0}$ is immediately compact. (Hint: Use the power series expansion of $T(t)$ near the point $a := \inf\{s > 0 : T(s) \text{ compact}\}$.)

## e. Examples

First, we show that the "eventual" do not imply the "immediate" properties.

**4.31 Nilpotent Semigroups.** The nilpotent semigroup

$$T(t)f(s) := \begin{cases} f(s+t) & \text{for } s+t \leq 1, \\ 0 & \text{for } s+t > 1, \end{cases}$$

on $\mathrm{C}_0[0,1)$ (or $\mathrm{L}^p[0,1]$) is the standard example for a semigroup with "good" properties for $t > 1$, but no smoothing effect for $t < 1$. In particular, it is eventually differentiable, eventually compact, and eventually norm continuous, but is not immediately norm continuous and consequently not immediately differentiable nor immediately compact. For a characterization of nilpotent semigroups see Exercise 4.34.(4).

Next, we study regularity properties of multiplication semigroups and characterize them in terms of the function defining its generator.

**4.32 Multiplication Semigroups.** As in Definition I.4.3, we consider a multiplication operator

$$M_q : f \mapsto q \cdot f$$

on $X := \mathrm{C}_0(\Omega)$ (or, if one prefers, on $\mathrm{L}^p(\Omega, \mu)$) for some continuous function $q : \Omega \to \mathbb{C}$. If $\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty$, then

$$T_q(t)f := \mathrm{e}^{tq} \cdot f$$

defines a strongly continuous semigroup (see Proposition I.4.5) for which the following holds.

**Theorem.** *Let $\big(T_q(t)\big)_{t\geq 0}$ be the strongly continuous multiplication semigroup on $X$ generated by the multiplication operator $M_q$. Then $\big(T_q(t)\big)_{t\geq 0}$ has one of the above regularity properties if and only if the spectrum $\sigma(M_q) = \overline{q(\Omega)}$ satisfies the conditions stated in Theorem 4.6, Theorem 4.14, or Theorem 4.18. More precisely, the following holds.*

(i) *$\big(T_q(t)\big)_{t\geq 0}$ is bounded analytic of angle $\delta$ if and only if*

$$\Sigma_{\delta+\pi/2} \subset \mathbb{C} \setminus \overline{q(\Omega)} = \rho(M_q).$$

(ii) *$\big(T_q(t)\big)_{t\geq 0}$ is eventually (and immediately) differentiable for $t > t_0$ if and only if there exists $c > 0$ such that*

$$\big\{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| > c \cdot \mathrm{e}^{-t_0 \operatorname{Re}\lambda}\big\} \subset \mathbb{C} \setminus \overline{q(\Omega)} = \rho(M_q). \tag{4.27}$$

(iii) *$\big(T_q(t)\big)_{t\geq 0}$ is eventually (and immediately) norm continuous if and only if*

$$\overline{q(\Omega)} \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq b\}$$

*is bounded for every $b \in \mathbb{R}$.*

We point out that for multiplication semigroups the "eventual" and "immediate" properties coincide. Moreover, the location of the spectrum of the generator already characterizes the properties of the semigroup, and no estimates on the resolvent are needed. Since the spectrum of the generator is just the (closure of the) range of the function, we again confirm the statement made before Proposition I.4.2:

*Properties of the function $q$ directly correspond to properties of the operator $M_q$ and then to properties of the semigroup $\big(T_q(t)\big)_{t\geq 0}$.*

Proof. (i) The condition is necessary by Theorem 4.6. Conversely, if $\Sigma_{\delta+\pi/2}$ is contained in $\mathbb{C} \setminus \overline{q(\Omega)}$, it follows that the functions $q_\pm := \mathrm{e}^{\pm \mathrm{i}\delta} \cdot q$ still have nonpositive real part. By Proposition I.4.5, this implies that

$$\mathrm{e}^{\pm \mathrm{i}\delta} \cdot M_q$$

are both generators of bounded strongly continuous semigroups. By Theorem 4.6.(b), this proves that $M_q$ generates a bounded analytic semigroup.

(ii) (Right) differentiability in $t_0$ for $\big(T_q(t)\big)_{t\geq 0}$ means that $T_q(t_0)X \subset D(M_q)$, i.e., $\mathrm{e}^{t_0 q} \cdot f \cdot q \in X$ for all $f \in X$. This is true (by Proposition I.4.2.(ii)) if and only if $\mathrm{e}^{t_0 q} \cdot q$ is bounded. Since $\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty$, we know that $\mathrm{e}^{t_0 q} \cdot \operatorname{Re} q$ is bounded. Hence $\mathrm{e}^{t_0 q} \cdot q$ is bounded if and only if $\mathrm{e}^{t_0 \operatorname{Re} q} \cdot \operatorname{Im} q$ is bounded, which again is equivalent to the existence of $c > 0$ such that

$$q(\Omega) \subset \big\{\lambda \in \mathbb{C} : \mathrm{e}^{t_0 \operatorname{Re}\lambda} |\operatorname{Im}\lambda| \leq c\big\},$$

hence to condition (4.27).

(iii) Take $t_0 > 0$, $\varepsilon > 0$ and choose $b \in \mathbb{R}$ such that $2\mathrm{e}^{(t_0+1)b} < \varepsilon$. If $\operatorname{Re} q(s) \le b$, then

$$\left|\mathrm{e}^{tq(s)} - \mathrm{e}^{t_0 q(s)}\right| \le \mathrm{e}^{t \operatorname{Re} q(s)} + \mathrm{e}^{t_0 \operatorname{Re} q(s)} \le 2\mathrm{e}^{(t_0+1)b} < \varepsilon,$$

whenever $|t - t_0| < 1$. By hypothesis, $H := \{q(s) : s \in \Omega,\ \operatorname{Re} q(s) \ge b\}$ is bounded in $\mathbb{C}$. Thus, $\lim_{t \downarrow t_0} |\mathrm{e}^{tz} - \mathrm{e}^{t_0 z}| = 0$ exists uniformly for $z \in H$. Now choose $1 > \delta > 0$ such that

$$\sup\left\{\left|\mathrm{e}^{tq(s)} - \mathrm{e}^{t_0 q(s)}\right| : s \in \Omega,\ \operatorname{Re} q(s) > b\right\} < \varepsilon,$$

whenever $|t - t_0| < \delta$. This and the inequality above imply

$$\|T_q(t) - T_q(t_0)\| = \sup\left\{\left|\mathrm{e}^{tq(s)} - \mathrm{e}^{t_0 q(s)}\right| : s \in \Omega\right\} < \varepsilon,$$

whenever $|t - t_0| < \delta$. Since $t_0 > 0$ was arbitrary, we obtain immediate norm continuity for the semigroup $\big(T_q(t)\big)_{t \ge 0}$. ☐

In order to proceed, we characterize compact multiplication operators.

**Lemma.** *For a (bounded) multiplication operator $M_q$ on $X := \mathrm{C}_0(\Omega)$, $\Omega$ locally compact, the following assertions are equivalent.*

(a) *$M_q$ is compact.*

(b) *The range $q(\Omega) = \{\lambda_n : n \in \mathbb{N}\}$ is finite or a null sequence, such that $q^{-1}(\lambda_n)$ is a finite set for each $0 \ne \lambda_n$.*

Proof. If (a) holds, then $\overline{q(\Omega)} = \sigma(M_q)$ is countable with only zero as an accumulation point. Moreover, each $0 \ne \lambda \in q(\Omega)$ is an eigenvalue with finite-dimensional eigenspace $F_\lambda$. Since $\Omega_\lambda := q^{-1}(\lambda)$ is open and closed in $\Omega$ and the eigenspace $F_\lambda$ coincides with $\mathrm{C}_0(\Omega_\lambda)$, this forces $q^{-1}(\lambda)$ to be finite. If (b) holds, one can approximate $M_q$ by finite-dimensional, hence compact, (multiplication) operators. ☐

As a consequence of this lemma, we observe that on spaces like $\mathrm{C}_0(\mathbb{R})$ there are no (nonzero) compact multiplication operators. On the other hand, on $X := \mathrm{c}_0$ (or $X := \ell^p$) a multiplication operator defined by a sequence $(q_n)_{n\in\mathbb{N}}$ is compact if and only if $(q_n)_{n\in\mathbb{N}}$ is a null sequence. This yields directly to the following characterizations.

**Proposition.** *Let $X := \mathrm{c}_0$ and $M_q$ be the multiplication operator defined by the sequence $q := (q_n)_{n\in\mathbb{N}}$ satisfying $\sup_n \operatorname{Re} q_n < \infty$.*

(i) *The operator $M_q$ has compact resolvent if and only if $\lim_{n\to\infty} |q_n| = \infty$.*

(ii) *The semigroup $\big(T_q(t)\big)_{t\ge 0}$ is eventually (and immediately) compact if and only if $\lim\limits_{n\to\infty} \operatorname{Re} q_n = -\infty$.*

As a consequence of the above characterizations, we can now construct semigroups with prescribed behavior showing that none of the arrows in diagram (4.26) can be reversed.

**4.33 Counterexamples.** On the Banach space $X := \mathrm{c}_0$ take the multiplication operator $M_q$ corresponding to one of the sequences $q = (q_n)_{n\in\mathbb{N}}$ below. Then the following holds.
(i) If $q_n := -n + \mathrm{i}e^{n^2}$, then the semigroup generated by $M_q$ is immediately compact and (consequently) immediately norm continuous, but is not eventually differentiable.

(ii) If $q_n := -n+\mathrm{i}e^{t_0 n}$, then the semigroup generated by $M_q$ is differentiable for $t > t_0$, but not differentiable in $[0, t_0)$.

(iii) If $q_n := -n+\mathrm{i}n^2$, then the semigroup generated by $M_q$ is immediately differentiable, but is not analytic.

**4.34 Exercises.** (1) Show that the operator $A := {}^{d^2}\!/_{dx^2}$ with domain $D(A) := \{f \in \mathrm{C}^2[0,1] : f'(0) = 0 = f'(1)\}$ generates an immediately compact, analytic contraction semigroup $(T(t))_{t\geq 0}$ on $X := \mathrm{C}[0,1]$. In addition, show that $T(t)f \geq 0$ for every $f \geq 0$, i.e., $(T(t))_{t\geq 0}$ is positive (see Section VI.1.b). (Hint: Observe Paragraphs 2.12 and 3.30. Moreover, use Example 4.26 and Theorem 4.29 to show that $(T(t))_{t\geq 0}$ is immediately compact.)

(2) Take $p(s) := -s^2$ for $s \in \mathbb{R}$ and $q(n) := \mathrm{i}n$ for $n \in \mathbb{N}$. Show that the semigroups generated by the multiplication operators $M_p$ on $\mathrm{C}_0(\mathbb{R})$ and $M_q$ on $\mathrm{c}_0$, resp., have the following properties.

(i) $(T_p(t))_{t\geq 0}$ is analytic, but not eventually compact.

(ii) $(T_q(t))_{t\geq 0}$ is not eventually compact, but its generator has compact resolvent.

(3) Show that the multiplication semigroup on $X := \mathrm{C}_0(\mathbb{R})$ associated to the function $q(s) := \mathrm{i}s^2 - \log(1 + s^2)$ is immediately norm continuous and eventually differentiable for $t > 1$.

(4*) Let $(T(t))_{t\geq 0}$ be a strongly continuous contraction semigroup on a Banach space $X$ with generator $A$. Show that $(T(t))_{t\geq 0}$ is eventually nilpotent with $T(r) = 0$ if and only if

$$\bigl(n!\,\|R(1,A)^n\|\bigr)^{1/n} \leq r \qquad \text{for all } n \in \mathbb{N}.$$

(Hint: See [Sin82, Thm. 6.11].)

# 5. Interpolation and Extrapolation Spaces for Semigroups
**(by Simon Brendle)**

In the spirit of Section 2.a, we continue to associate new semigroups on new spaces to a given strongly continuous semigroup. The constructions here are inspired by the classical Sobolev and distribution spaces and yield an important tool for the abstract theory as well as for concrete applications. We start by defining semigroups on a discrete scale of spaces.

## a. Sobolev Towers

Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup with generator $\big(A, D(A)\big)$ on a Banach space $X$. After applying the rescaling procedure, and hence without loss of generality (see Paragraph 2.2 and Exercise 5.9.(1)), we can assume that its growth bound $\omega_0$ is negative. Therefore, its generator $A$ is invertible with $A^{-1} \in \mathcal{L}(X)$. On the domains $D(A^n)$ of its powers $A^n$, we now introduce new norms $\|\cdot\|_n$.

**5.1 Definition.** *For each $n \in \mathbb{N}$ and $x \in D(A^n)$, we define the $n$-norm*

$$\|x\|_n := \|A^n x\|$$

*and call*

$$X_n := (D(A^n), \|\cdot\|_n)$$

*the Sobolev space of order $n$ associated to $\big(T(t)\big)_{t\geq 0}$. The operators $T(t)$ restricted to $X_n$ will be denoted by*

$$T_n(t) := T(t)_{|X_n}.$$

It turns out that the restrictions $T_n(t)$ behave surprisingly well on $X_n$.

**5.2 Proposition.** *With the above definitions, the following holds.*

(i) *Each $X_n$ is a Banach space.*

(ii) *The operators $T_n(t)$ form a strongly continuous semigroup $\big(T_n(t)\big)_{t\geq 0}$ on $X_n$.*

(iii) *The generator $A_n$ of $\big(T_n(t)\big)_{t\geq 0}$ is given by the part of $A$ in $X_n$, i.e.,*

$$A_n x = Ax \qquad \text{for } x \in D(A_n) \text{ with}$$
$$D(A_n) := \{x \in X_n : Ax \in X_n\} = D(A^{n+1}) = X_{n+1}.$$

Proof. The assertion follows by induction if we prove the case $n = 1$. Assertion (i) follows, since $A$ is a closed operator and $\|\cdot\|_1$ is equivalent to the graph norm, as can be seen from the estimate

$$\|x\|_A = \big\|A^{-1}Ax\big\| + \|Ax\| \leq \big(\big\|A^{-1}\big\| + 1\big) \cdot \|x\|_1 \leq \big(\big\|A^{-1}\big\| + 1\big) \cdot \|x\|_A$$

for $x \in X_1$. From Lemma 1.3.(ii), we know that $T(t)$ maps $X_1$ into $X_1$. Each $T_1(t)$ is bounded, since

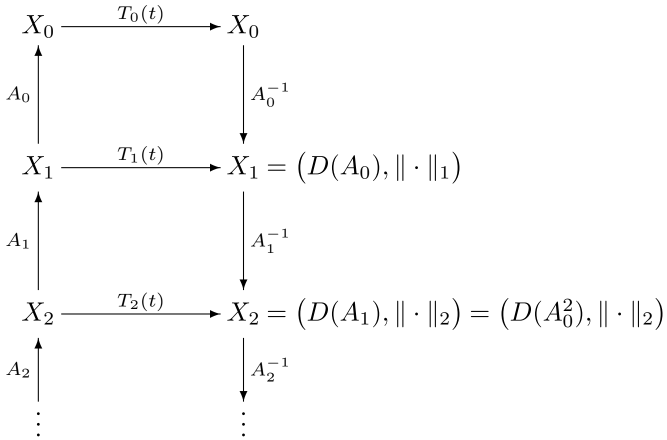
$$\|T_1(t)x\|_1 = \|T(t)Ax\| \leq \|T(t)\| \cdot \|x\|_1 \qquad \text{for } x \in X_1,$$

so $\big(T_1(t)\big)_{t\geq 0}$ is a semigroup on $X_1$. The strong continuity follows from

$$\|T_1(t)x - x\|_1 = \|T(t)Ax - Ax\| \to 0 \qquad \text{for } t \downarrow 0 \qquad \text{and } x \in X_1.$$

Finally, (iii) follows from the proposition in Paragraph 2.3 on subspace semigroups. □

We suggest visualizing the above spaces and semigroups by a diagram. Before doing so, we point out that by definition, $A_n$ is an isometry (with inverse $A_n^{-1}$) from $X_{n+1}$ onto $X_n$. Moreover, we write $X_0 := X$, $T_0(t) := T(t)$ and $A_0 := A$.




Observe that each $X_{n+1}$ is densely embedded in $X_n$ but also, via $A_n$, isometrically isomorphic to $X_n$. In addition, the semigroup $\big(T_{n+1}(t)\big)$ is the restriction of $\big(T_n(t)\big)_{t\geq 0}$, but also similar to $\big(T_n(t)\big)_{t\geq 0}$. We state this important property explicitly.

**5.3 Corollary.** *All the strongly continuous semigroups $\big(T_n(t)\big)_{t\geq 0}$ on the spaces $X_n$ are similar. More precisely, one has*

$$T_{n+1}(t) = A_n^{-1} T_n(t) A_n = T_n(t)_{|X_{n+1}} \quad \textit{for all } n \geq 0.$$

This similarity implies that spectrum, spectral bound, growth bound, etc. coincide for all the semigroups $\big(T_n(t)\big)_{t\geq 0}$.

In our construction, we obtained the $(n+1)$st Sobolev space from the $n$th Sobolev space. However, since $X_{n+1}$ is a dense subspace of $X_n$ (by Theorem 1.4), it is possible to invert this procedure and obtain $X_n$ from $X_{n+1}$ as the completion for the norm

$$\|x\|_n := \left\|A_{n+1}^{-1} x\right\|_{n+1}.$$

This observation permits us to extend the above diagram to the negative integers and to define *extrapolation spaces* or Sobolev spaces of negative order.

**5.4 Definition.** *For each $n \in \mathbb{N}$ and $x \in X_{-n+1}$, we define (recursively) the norm*

$$\|x\|_{-n} := \left\|A_{-n+1}^{-1}x\right\|_{-n+1}$$

*and call the completion*

$$X_{-n} := \left(X_{-n+1}, \|\cdot\|_{-n}\right)^{\sim}$$

*the Sobolev space of order $-n$ associated to $\left(T_0(t)\right)_{t\geq 0}$. Moreover, we denote the continuous extensions of the operators $T_{-n+1}(t)$ to the extrapolated space $X_{-n}$ by $T_{-n}(t)$.*

Note that these extended operators $T_{-n}(t)$ have properties analogous to the ones stated in Proposition 5.2; hence our previous results hold for all $n \in \mathbb{Z}$.

**5.5 Theorem.** *With the above definitions, the following hold for all $m \geq n \in \mathbb{Z}$.*

(i) *Each $X_n$ is a Banach space containing $X_m$ as a dense subspace.*
(ii) *The operators $T_n(t)$ form a strongly continuous semigroup $\left(T_n(t)\right)_{t\geq 0}$ on $X_n$.*
(iii) *The generator $A_n$ of $\left(T_n(t)\right)_{t\geq 0}$ has domain $D(A_n) = X_{n+1}$ and is the unique continuous extension of $A_m : X_{m+1} \to X_m$ to an isometry from $X_{n+1}$ onto $X_n$.*

Proof. It suffices to prove the assertions for $m = 0$ and $n = -1$ only. In this case, (i) holds true by definition. From

$$\|T_0(t)x\|_{-1} = \left\|T_0(t)A_0^{-1}x\right\|_0 \leq \|T_0(t)\| \cdot \|x\|_{-1},$$

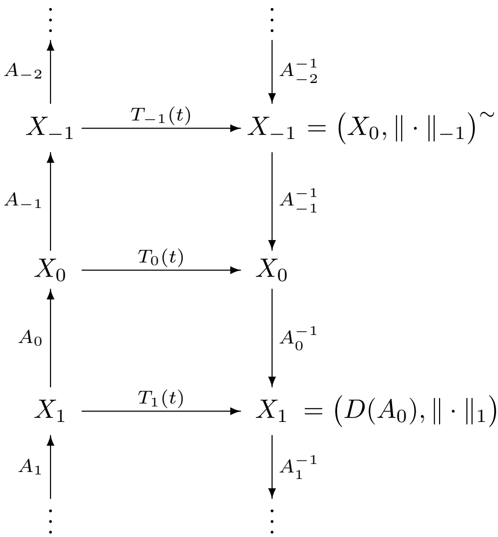
we see that $T_0(t)$ extends continuously to $X_{-1}$. The semigroup property holds for $\left(T_0(t)\right)_{t\geq 0}$ on $X_0$, hence for $\left(T_{-1}(t)\right)_{t\geq 0}$ on $X_{-1}$. Similarly, the strong continuity follows, since it holds on the dense subset $X_0$ (even for the stronger norm $\|\cdot\|_0$).

To prove (iii), we observe first that $A_{-1}$ extends $A_0$, since $T_{-1}(t)$ extends $T_0(t)$. The closedness of $A_{-1}$ then implies $X_0 \subseteq D(A_{-1})$. Since $X_0$ is dense in $X_{-1}$ and $\left(T_{-1}(t)\right)_{t\geq 0}$-invariant, it is a core for $A_{-1}$ by Proposition 1.7. Now, on $X_0$ the graph norm $\|\cdot\|_{A_{-1}}$ is equivalent to $\|\cdot\|$; hence $X_0$ is a Banach space for $\|\cdot\|_{A_{-1}}$, and therefore $X_0 = D(A_{-1})$.

The remaining assertions follow from the fact that $A_0 : D(A_0) \subset X_0 \to X_{-1}$, by definition of the norms, is an isometry. □

So, we have constructed a two-sided infinite sequence of Banach spaces and strongly continuous semigroups thereon. Again we visualize this *Sobolev tower* associated to the semigroup $\left(T_0(t)\right)_{t\geq 0}$ by a diagram. Note that Corollary 5.3 now holds for all $n \in \mathbb{Z}$. In addition, if we start this construction from any level, i.e., from the semigroup $\left(T_k(t)\right)_{t\geq 0}$ on the space $X_k$ for some $k \in \mathbb{Z}$, we will obtain the same scale of spaces and semigroups.

**5.6 Diagram.**




We point out again that each space $X_n$ is the completion (unique up to isomorphism) of any of its subspaces $X_m$ whenever $m \geq n \in \mathbb{Z}$.

For multiplication semigroups it is easy to identify all Sobolev spaces with concrete function spaces.

**5.7 Example.** We take $X_0 := \mathrm{C}_0(\Omega)$ and $q : \Omega \to \mathbb{C}$ continuous assuming, for simplicity, that $\sup_{s\in\Omega} \operatorname{Re} q(s) < 0$. As in Section I.4.a, we define $M_q f := q \cdot f$ and the corresponding multiplication semigroup by

$$T_q(t)f := \mathrm{e}^{tq} \cdot f$$

for $t \geq 0$, $f \in X$. The Sobolev spaces $X_n$ are then given by

$$X_n = \big\{q^{-n} \cdot f : f \in X\big\} = \big\{g \in \mathrm{C}(\Omega) : q^n \cdot g \in X_0\big\} \tag{5.1}$$

for all $n \in \mathbb{Z}$.

Note that the analogous statement holds if we start from

$$X_0 := \mathrm{L}^p(\Omega, \mu) \qquad \text{for } 1 \leq p < \infty,$$

a measurable function $q : \Omega \to \mathbb{C}$ satisfying $\operatorname{ess\,sup}_{s\in\Omega} \operatorname{Re} q(s) < 0$, and the corresponding multiplication semigroup $\big(T_q(t)\big)_{t\geq 0}$ (cf. Section I.4.b). In particular, (5.1) becomes

$$X_n = \mathrm{L}^p\big(\Omega, |q|^{np} \cdot \mu\big) \tag{5.2}$$

for all $n \in \mathbb{Z}$.

Our abstract Sobolev spaces look quite familiar if we consider the translation semigroups and their generators from Paragraph 2.10.

**5.8 Examples.** (i) First, we look at the (left) translation group $\big(T_l(t)\big)_{t\in\mathbb{R}}$ on $X := \mathrm{L}^2(\mathbb{R})$ as discussed in Paragraph 2.10. If by $\mathcal{F}$ we denote the Fourier transform, then $(2\pi)^{-1/2}\,\mathcal{F}$ maps $\mathrm{L}^2(\mathbb{R})$ isometrically onto $\mathrm{L}^2(\mathbb{R})$ (see Theorem VI.5.6 and the following remark) and transforms $\big(T_l(t)\big)_{t\in\mathbb{R}}$ into the multiplication group $\big(\widehat{T}(t)\big)_{t\in\mathbb{R}}$ given by

$$\widehat{T}(t)f(\xi) = \mathrm{e}^{\mathrm{i}t\xi}\cdot f(\xi) \qquad \text{for } f\in\mathrm{L}^2(\mathbb{R}),\ \xi\in\mathbb{R}.$$

(Note that this is a concrete version of the Spectral Theorem I.4.9.) The generator of $\big(\widehat{T}(t)\big)_{t\in\mathbb{R}}$ is the multiplication operator given by the function $\widehat{q}:\xi\mapsto \mathrm{i}\xi$; hence the associated Sobolev spaces have been determined in Example 5.7 as

$$\widehat{X}_n = \big\{\xi\mapsto(1-\mathrm{i}\xi)^{-n}\cdot f(\xi) : f\in\mathrm{L}^2(\mathbb{R})\big\}$$

for all $n\in\mathbb{Z}$. If we now apply the inverse Fourier transform (and its extension to the space of distributions), we obtain the Sobolev spaces associated to the translation group as

$$X_n = \big\{(1-D)^{-n}f : f\in\mathrm{L}^2(\mathbb{R})\big\},$$

where $D$ denotes the distributional derivative. Hence, $X_n$ coincides with the usual Sobolev space $\mathrm{W}^{2,n}(\mathbb{R})$ for all $n\in\mathbb{Z}$.

(ii) In the case of the translation group $\big(T_l(t)\big)_{t\in\mathbb{R}}$ on $X := \mathrm{C}_0(\mathbb{R})$, we can avoid the use of the Fourier transform and work in the space of test functions $\mathscr{D}(\mathbb{R})$ and its dual $\mathscr{D}(\mathbb{R})'$ (see [Rud73, Chap. 6]) to obtain an analogous characterization of $X_n$. For $n\ge 1$, the spaces $X_n$ are easy to identify as

$$X_n = \left\{ f\in\mathrm{C}_0(\mathbb{R}) : \begin{array}{l} f \text{ is } n\text{-times differentiable and} \\ f^{(k)}\in\mathrm{C}_0(\mathbb{R}) \text{ for } k=1,\dots,n \end{array}\right\}.$$

To find the Sobolev spaces of negative order, we only consider the case $n=-1$ and recall that $X_{-1}$ is the set of (equivalence classes of) Cauchy sequences in $X$ for the norm $\|f\|_{-1} := \|R(1,A)f\|$ for $f\in X$ and $A$ the generator of $\big(T_l(t)\big)_{t\in\mathbb{R}}$. Then each such $\|\cdot\|_{-1}$-Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ defines a distribution $F\in\mathscr{D}(\mathbb{R})'$ by

$$\langle F,\varphi\rangle := \Big\langle \lim_{n\to\infty} R(1,A)f_n, \varphi+\varphi'\Big\rangle$$

for $\varphi\in\mathscr{D}(\mathbb{R})$. This shows that $X_{-1}$ is continuously embedded in the space $\big(\mathscr{D}'(\mathbb{R}),\sigma(\mathscr{D}',\mathscr{D})\big)$. Since $A_{-1}$ is the continuous extension of the classical derivative defined on $X_1$, it coincides with the distributional derivative $D$, and hence

$$X_{-1} = \big\{F\in\mathscr{D}' : F = f - Df \text{ for some } f\in\mathrm{C}_0(\mathbb{R})\big\}.$$

**5.9 Exercises.** (1) Let $(A, D(A))$ be a densely defined operator on $X$ such that $\rho(A) \neq \emptyset$. Prove the following.

(i) For each fixed $n \in \mathbb{N}$, all the norms

$$\|x\|_{n,\lambda} := \left\|(\lambda - A)^n x\right\|, \quad x \in D(A^n),$$

are equivalent for $\lambda \in \rho(A)$.

(ii) For each fixed $n \in \mathbb{N}$, all the norms

$$\|x\|_{-n,\lambda} := \left\|R(\lambda - A)^n x\right\|, \quad x \in X,$$

are equivalent for $\lambda \in \rho(A)$.

(iii) Now take $\lambda = 0 \in \rho(A)$ and define the *Sobolev spaces* $X_n$, $n \in \mathbb{Z}$, as in Definition 5.1 and Definition 5.4. Then the operator $A$ can be restricted/extended to an isometry from $X_{n+1}$ onto $X_n$ for each $n \in \mathbb{Z}$.

(2) Identify the abstract Sobolev spaces $X_n$ in Example 5.7 assuming only that $\sup_{s\in\mathbb{R}} \operatorname{Re} q(s) < \infty$.

(3) Show that an operator $(A, D(A))$ on $X$ with $\rho(A) \neq \emptyset$ is bounded if and only if $X_n = X$ for all $n \in \mathbb{Z}$.

(4) Take an operator $(A, D(A))$ with $\rho(A) \neq \emptyset$ on the Banach space $X$. Show that the dual of the extrapolated Sobolev space $X_{-1}$ is canonically isomorphic to the domain $D(A')$ with the graph norm of the adjoint $A'$ in $X'$.

(5) Show that for two densely defined operators $(A, D(A))$ with $\rho(A) \neq \emptyset$ and $(B, D(B))$ on the Banach space $X$ the following assertions are equivalent.

(i) $D(A') \subseteq D(B')$.

(ii) $\overline{R(\lambda, A)B} \in \mathcal{L}(X)$ for one (hence, all) $\lambda \in \rho(A)$.

(iii) $B : D(B) \subseteq X \to X_{-1}^A$ is bounded, i.e., $B$ can be extended to a bounded operator from $X$ to $X_{-1}^A$.

## *b. Favard and Abstract Hölder Spaces*

Into a given Sobolev tower, constructed from a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$, we will now insert a continuous scale of spaces, called the Favard and abstract Hölder spaces, and describe the behavior of the semigroup thereon. This is not only a theoretical exercise in Banach space construction, but will lead to important applications, e.g., to perturbation problems (see Corollary III.2.14, Corollary III.3.6) or to inhomogeneous Cauchy problems (see Corollary VI.7.17).

For simplicity, we will first consider the case $\alpha \in (0, 1]$.

**5.10 Definition.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and assume that $\omega_0 < 0$. For each $\alpha \in (0, 1]$, the space*

$$F_\alpha := \left\{ x \in X : \sup_{t>0} \left\| \frac{1}{t^\alpha} \big(T(t)x - x\big) \right\| < \infty \right\}$$

*with norm*

$$\|x\|_{F_\alpha} := \sup_{t>0} \left\| \frac{1}{t^\alpha} \big(T(t)x - x\big) \right\|$$

*is called the Favard space of order $\alpha$. For $\alpha \in (0,1)$, the space*

$$X_\alpha := \left\{ x \in X : \lim_{t \downarrow 0} \left\| \frac{1}{t^\alpha} \big(T(t)x - x\big) \right\| = 0 \right\}$$

*equipped with the norm $\|\cdot\|_{X_\alpha}$ induced by $\|\cdot\|_{F_\alpha}$ is called the abstract Hölder space of order $\alpha$.*

Using the Sobolev tower from Section 5.a, we extend this definition to all $\alpha \in \mathbb{R}$.

**5.11 Definition.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup, let $\alpha \in \mathbb{R}$, and choose $w > \omega_0$. Write $\alpha = k + \gamma$ with $k \in \mathbb{Z}$ and $\gamma \in (0,1]$. Then the Favard space of order $\alpha$ associated to the semigroup $(T(t))_{t\geq 0}$ is defined as the $\gamma$th Favard space associated to the rescaled semigroup $(\mathrm{e}^{-wt}T_k(t))_{t\geq 0}$. Analogously, the abstract Hölder space of order $\alpha$ associated to $(T(t))_{t\geq 0}$ is defined as the $\gamma$th abstract Hölder space associated to $(\mathrm{e}^{-wt}T_k(t))_{t\geq 0}$.*

It is not difficult to show that this definition is independent of the choice of $w > \omega_0$; see Exercise 5.23.(1). In a first step, we characterize the Favard and abstract Hölder spaces in terms of the generator.

**5.12 Proposition.** *Assume that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup with growth bound $\omega_0 < 0$. If $\alpha \in (0,1]$, then one has for the Favard space*

$$F_\alpha = \left\{ x \in X : \sup_{\lambda > 0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty \right\},$$

*and the Favard norm $\|\cdot\|_{F_\alpha}$ is equivalent to the norm*

$$|||x|||_{F_\alpha} := \sup_{\lambda > 0} \|\lambda^\alpha AR(\lambda, A)x\|.$$

*Moreover, if $\alpha \in (0,1)$, then*

$$X_\alpha = \left\{ x \in X : \lim_{\lambda \to \infty} \|\lambda^\alpha AR(\lambda, A)x\| = 0 \right\}.$$

PROOF. Let $x \in F_\alpha$. Then $\sup_{t>0} \|{}^1\!/\!_{t^\alpha}(T(t)x - x)\| =: J < \infty$. Using the integral representation of the resolvent, we obtain

$$\lambda^\alpha AR(\lambda, A)x = \lambda^{\alpha+1}R(\lambda, A)x - \lambda^\alpha x = \lambda^{\alpha+1} \int_0^\infty \mathrm{e}^{-\lambda s} \big(T(s)x - x\big)\, ds$$

for $\lambda > 0$, and so

$$\|\lambda^\alpha AR(\lambda, A)x\| \leq \lambda^{\alpha+1} \int_0^\infty \mathrm{e}^{-\lambda s} J s^\alpha \, ds = J \int_0^\infty \mathrm{e}^{-r} r^\alpha \, dr.$$

Therefore, $\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| < \infty$.

Conversely, assume that $\sup_{\lambda>0} \|\lambda^\alpha AR(\lambda, A)x\| =: K < \infty$. We write

$$x = \lambda R(\lambda, A)x - AR(\lambda, A)x =: x_\lambda - y_\lambda.$$

Then

$$\begin{aligned}\|T(t)x_\lambda - x_\lambda\| &= \left\| \int_0^t T(s)Ax_\lambda \, ds \right\| \le M\|Ax_\lambda\|t \\ &= M \left\|\lambda^\alpha AR(\lambda, A)x\right\| t\lambda^{1-\alpha} \le MKt\lambda^{1-\alpha}\end{aligned}$$

and

$$\begin{aligned}\|T(t)y_\lambda - y_\lambda\| &\le 2M\|y_\lambda\| = 2M \left\|\lambda^\alpha AR(\lambda, A)x\right\| \lambda^{-\alpha} \\ &\le 2MK\lambda^{-\alpha}.\end{aligned}$$

This implies

$$\begin{aligned}\left\| \frac{1}{t^\alpha}\big(T(t)x - x\big) \right\| &\le \left\| \frac{1}{t^\alpha}\big(T(t)x_\lambda - x_\lambda\big) \right\| + \left\| \frac{1}{t^\alpha}\big(T(t)y_\lambda - y_\lambda\big) \right\| \\ &\le MK(t\lambda)^{1-\alpha} + 2MK(t\lambda)^{-\alpha}.\end{aligned}$$

Taking $\lambda = 1/t$, we obtain

$$\left\| \frac{1}{t^\alpha}\big(T(t)x - x\big) \right\| \le 3MK;$$

hence $\sup_{t>0} \left\| 1/t^\alpha (T(t)x - x) \right\| < \infty$.

The assertion concerning $X_\alpha$ is proved similarly. ☐

For analytic semigroups one can characterize these spaces as follows.

**5.13 Proposition.** *Assume that $(T(t))_{t\ge0}$ is an analytic semigroup with growth bound $\omega_0 < 0$. If $\alpha \in (0, 1]$, then one has for the Favard space*

$$F_\alpha = \left\{ x \in X : \sup_{t>0} \left\| t^{1-\alpha} AT(t)x \right\| < \infty \right\},$$

*and the Favard norm $\|\cdot\|_{F_\alpha}$ is equivalent to the norm*

$$[\![x]\!]_{F_\alpha} := \sup_{t>0} \|t^{1-\alpha} AT(t)x\|.$$

*Moreover, if $\alpha \in (0, 1)$, then*

$$X_\alpha = \left\{ x \in X : \lim_{t\downarrow0} \|t^{1-\alpha} AT(t)x\| = 0 \right\}.$$

Proof. Let $x \in F_\alpha$. Then

$$\sup_{t>0} \left\| \frac{1}{t^\alpha}\big(T(t)x - x\big) \right\| =: J < \infty.$$

Now we write

$$t^{1-\alpha} AT(t)x = \frac{1}{t^\alpha} T(t) \cdot \big(T(t)x - x\big) - \frac{1}{t^\alpha} AT(t) \int_0^t \big(T(s)x - x\big)\, ds.$$

Using Theorem 4.6.(c) this implies

$$\|t^{1-\alpha} AT(t)x\| \le MJ + \|tAT(t)\| \frac{J}{\alpha + 1};$$

hence $\sup_{t>0} \|t^{1-\alpha} AT(t)x\| < \infty$.

Conversely, suppose that $\sup_{t>0} \|t^{1-\alpha} AT(t)x\| =: L < \infty$. Then

$$\left\|\frac{1}{t^\alpha}\big(T(t)x - x\big)\right\| = \left\|\frac{1}{t^\alpha}\int_0^t AT(s)x\,ds\right\| \le \frac{1}{t^\alpha}\int_0^t \|AT(s)x\|\,ds$$
$$\le \frac{1}{t^\alpha}\int_0^t Ls^{\alpha-1}\,ds = \frac{L}{\alpha}$$

for every $t > 0$, and therefore $x \in F_\alpha$.

The assertion for $X_\alpha$ follows in the same way. □

Returning to arbitrary semigroups we obtain natural inclusions between the Favard and abstract Hölder spaces for different indices.

**5.14 Proposition.** *For $\alpha > \beta$ we have $X_\alpha \subset F_\alpha \hookrightarrow X_\beta \subset F_\beta$.*

PROOF. We write $\alpha = k + \gamma$, $\beta = l + \delta$ with $k, l \in \mathbb{Z}$, $\gamma, \delta \in (0, 1]$. The assertion follows directly from the definition if $k = l$. On the other hand, if $k \ge l+1$, then $F_\alpha \hookrightarrow X_k \hookrightarrow X_{l+1} \hookrightarrow X_\beta$. □

We now study the behavior of the induced semigroups on these spaces.

**5.15 Theorem.** *Let $(T(t))_{t\ge0}$ be a strongly continuous semigroup with growth bound $\omega_0 < 0$. Then, for $\alpha \in (0, 1]$, the following statements hold.*

(i) *The Favard space $F_\alpha$ is a Banach space.*

(ii) *The restrictions $(T(t)_{|F_\alpha})_{t\ge0}$ form a semigroup of bounded operators on $F_\alpha$ for which $X_\alpha$ is its space of strong continuity; more precisely,*

$$X_\alpha = \left\{ x \in F_\alpha : \|\cdot\|_{F_\alpha}\text{-}\lim_{t\downarrow0} T(t)x = x \right\} = \overline{D(A)}^{\|\cdot\|_{F_\alpha}}.$$

(iii) *The generator of the strongly continuous semigroup $(T(t)_{|X_\alpha})_{t\ge0}$ is given by the part $A_{|X_\alpha}$ of $A$ in $X_\alpha$ with domain $D(A_{|X_\alpha}) = X_{\alpha+1}$.*

(iv) *For the spectra of the parts of $A$ in $F_\alpha$ and in $X_\alpha$ one has*

$$\sigma(A_{|F_\alpha}) = \sigma(A_{|X_\alpha}) = \sigma(A).$$

PROOF. (i) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $F_\alpha$. Since $F_\alpha$ is continuously embedded in $X$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$ as well, and by the completeness of $X$ the limit $\|\cdot\|\text{-}\lim_{n\to\infty} x_n =: x$ exists. For each $s > 0$ we obtain

$$\left\|\frac{1}{s^\alpha}\big(T(s)x - x\big)\right\| = \lim_{m\to\infty}\left\|\frac{1}{s^\alpha}\big(T(s)x_m - x_m\big)\right\| \le \overline{\lim_{m\to\infty}}\,\|x_m\|_{F_\alpha}.$$

Therefore, $x \in F_\alpha$ and $\|x\|_{F_\alpha} \le \overline{\lim}_{m\to\infty} \|x_m\|_{F_\alpha}$. An analogous argument yields that $\|x - x_n\|_{F_\alpha} \le \overline{\lim}_{m\to\infty} \|x_m - x_n\|_{F_\alpha}$. Hence we have $\overline{\lim}_{n\to\infty} \|x - x_n\|_{F_\alpha} \le \overline{\lim}_{n,m\to\infty} \|x_m - x_n\|_{F_\alpha} = 0$, and so we obtain $\|\cdot\|_{F_\alpha}\text{-}\lim_{n\to\infty} x_n = x$. This shows that $F_\alpha$ is complete.

(ii) We first prove the assertion for $\alpha \in (0,1)$. Take $x \in X_\alpha$, and let $\varepsilon > 0$ be given. Then we can find some $\delta > 0$ such that $\left\| \frac{1}{s^\alpha}(T(s) - I)x \right\| \leq \varepsilon/_{2M}$ for $0 < s < \delta$. Furthermore, if $t$ is sufficiently small, we also have $\|(T(t) - I)x\| \leq \delta^\alpha \varepsilon/_{2M}$. Therefore,

$$\begin{aligned} \left\| \frac{1}{s^\alpha} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| &\leq 2M \left\| \frac{1}{s^\alpha} \big(T(s) - I\big)x \right\| \\ &\leq 2M \frac{\varepsilon}{2M} = \varepsilon \qquad \text{for } 0 < s < \delta, \end{aligned}$$

whereas

$$\begin{aligned} \left\| \frac{1}{s^\alpha} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| &\leq 2M \frac{1}{s^\alpha} \left\| \big(T(t) - I\big)x \right\| \\ &\leq 2M \frac{1}{\delta^\alpha} \frac{\delta^\alpha \varepsilon}{2M} = \varepsilon \qquad \text{for } s \geq \delta. \end{aligned}$$

This implies

$$\left\| \big(T(t) - I\big)x \right\|_{F_\alpha} = \sup_{s>0} \left\| \frac{1}{s^\alpha} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| \leq \varepsilon$$

for $t$ sufficiently small. Hence $\| \cdot \|_{F_\alpha}\text{-}\lim_{t\downarrow 0} T(t)x = x$.

Suppose now that $\| \cdot \|_{F_\alpha}\text{-}\lim_{t\downarrow 0} T(t)x = x$. This implies

$$\| \cdot \|_{F_\alpha}\text{-}\lim_{r\downarrow 0} \frac{1}{r} \int_0^r T(s)x\, ds = x,$$

and therefore $x \in \overline{D(A)}^{\|\cdot\|_{F_\alpha}}$.

For $x \in \overline{D(A)}^{\|\cdot\|_{F_\alpha}}$ and $\varepsilon > 0$ we can find some $y \in D(A)$ such that $\|x - y\|_{F_\alpha} \leq \varepsilon/_2$. It follows that

$$\begin{aligned} \left\| \frac{1}{t^\alpha} \big(T(t)x - x\big) \right\| &\leq \left\| \frac{1}{t^\alpha} \big(T(t)y - y\big) \right\| + \left\| \frac{1}{t^\alpha} \big(T(t)(x - y) - (x - y)\big) \right\| \\ &\leq Mt^{1-\alpha} \|Ay\| + \|x - y\|_{F_\alpha} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $t$ sufficiently small, and hence $x \in X_\alpha$. This proves the assertion for $\alpha \in (0,1)$.

We now consider the case $\alpha = 1$. To this end we observe that the norms $\| \cdot \|_1$ and $\| \cdot \|_{F_1}$ are equivalent on $D(A)$. If $x \in X_1$, then $\| \cdot \|_1\text{-}\lim_{t\downarrow 0} T(t)x = x$ and so $\| \cdot \|_{F_1}\text{-}\lim_{t\downarrow 0} T(t)x = x$. If $\| \cdot \|_{F_1}\text{-}\lim_{t\downarrow 0} T(t)x = x$, then $x \in \overline{D(A)}^{\|\cdot\|_{F_1}}$ as above. But $D(A)$ is a Banach space for $\| \cdot \|_1$, hence also for $\| \cdot \|_{F_1}$. Therefore, $x \in \overline{D(A)}^{\|\cdot\|_{F_1}}$ implies $x \in D(A) = X_1$.

(iii) By the proposition in Paragraph 2.3, the generator of the restricted semigroup $(T(t)_{|X_\alpha})_{t\geq 0}$ is the part $A_{|X_\alpha}$ of $A$ in $X_\alpha$ with the domain $D(A_{|X_\alpha}) = \{x \in D(A) : Ax \in X_\alpha\} = X_{\alpha+1}$.

(iv) This follows immediately from Proposition IV.2.17. □

In general, $A_{|F_\alpha}$ is not a generator. However, the following corollary shows that we are not far from semigroup generators.

**5.16 Corollary.** *For $\alpha \in (0,1]$ consider the abstract Hölder spaces $X_\alpha$ and the Favard spaces $F_\alpha$ associated to a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq M\mathrm{e}^{wt}$ for $t \geq 0$. Then the part $A_{|F_\alpha}$ of $A$ in $F_\alpha$ satisfies*

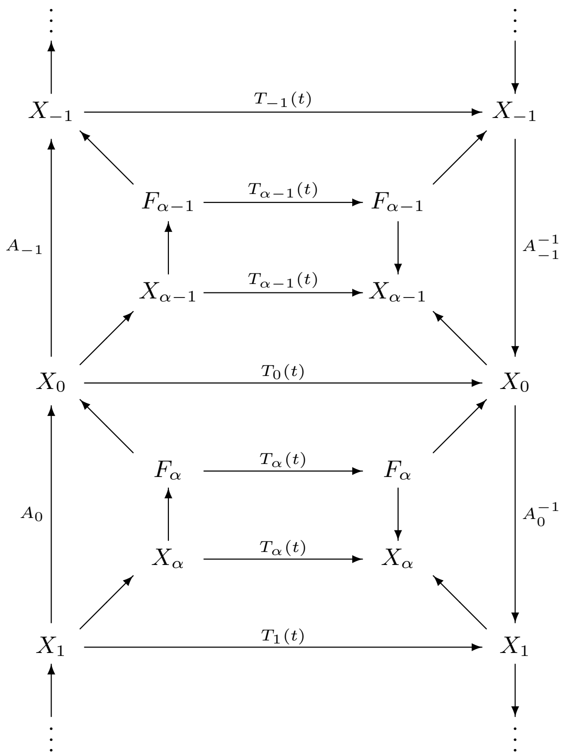
(i) $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > w\} \subset \rho(A_{|F_\alpha})$, *and*

(ii) $\|R(\lambda, A_{|F_\alpha})^n\| \leq \frac{M}{(\operatorname{Re}\lambda - w)^n}$ *for all $n \in \mathbb{N}$ and $\operatorname{Re}\lambda > w$,*

*and hence, in the terminology of Definition 3.22, $A_{|F_\alpha}$ is a Hille–Yosida operator.*

Proof. The assertions follow from the corresponding statements for the resolvent of the operator $A$. □

**5.17 Diagram.** The results of the previous theorem can now be illustrated by completing Diagram 5.6.




In the final part of this subsection we are concerned with the dual situation and look at the Favard spaces ${F^{\odot}}_1$, ${F^{\odot}}_0$, and ${F^{\odot}}_{-1}$ associated to the sun dual semigroup $(T(t)^{\odot})_{t\geq 0}$ of $(T(t))_{t\geq 0}$. As it turns out, these Favard spaces can be identified with well-known spaces. First, we need a preliminary lemma.[3]

**5.18 Lemma.** *The spaces ${X^{\odot}}_{-1}$ and ${X_1}^{\odot}$ are canonically isomorphic, and after this identification, we have $({T^{\odot}}_{-1}(t))_{t\geq 0} = ({T_1}^{\odot}(t))_{t\geq 0}$.*

[3] Here it will be more convenient to use the notation $T^{\odot}(t)$ instead of $T(t)^{\odot}$.

PROOF. The operator ${A_1}'$ maps $X'$, which can be canonically identified with the domain $D({A_1}')$, isomorphically onto ${X_1}'$. Under this isomorphism, the semigroup $(T(t)')_{t\geq 0}$ becomes $(T_1(t)')_{t\geq 0}$. In particular, the space of strong continuity of $(T(t)')_{t\geq 0}$ is mapped onto the space of strong continuity of $(T_1(t)')_{t\geq 0}$. Hence, ${A_1}'$ maps $X^{\odot}$ onto ${X_1}^{\odot}$. Therefore, ${X^{\odot}}_{-1}$ is canonically isomorphic to ${X_1}^{\odot}$. □

**5.19 Proposition.** *The Favard space for the sun dual semigroup is given by* ${F^{\odot}}_1 = D(A')$.

PROOF. For $x' \in D(A')$, we have for every $t > 0$ and all $x \in X$

$$\Big\langle x, \frac{1}{t}\big(T(t)'x' - x'\big)\Big\rangle = \Big\langle \frac{1}{t}\big(T(t)x - x\big), x'\Big\rangle = \Big\langle A\,\frac{1}{t}\int_0^t T(s)x\,ds, x'\Big\rangle$$
$$= \Big\langle \frac{1}{t}\int_0^t T(s)x\,ds, A'x'\Big\rangle .$$

Assuming $\|T(t)\| \leq M$ for all $t \geq 0$, this implies $\sup_{t>0} \|{}^1\!/\!_t(T(t)'x' - x')\| \leq M\|A'x'\|$, and therefore $x' \in {F^{\odot}}_1$.

Conversely, if $x' \in {F^{\odot}}_1$, then $\sup_{t>0} \|{}^1\!/\!_t(T(t)'x' - x')\| < \infty$. Hence, for every $x \in D(A)$ one has

$$\big|\langle Ax, x'\rangle\big| = \lim_{t\downarrow 0}\Big|\Big\langle \frac{1}{t}\big(T(t)x - x\big), x'\Big\rangle\Big| = \lim_{t\downarrow 0}\Big|\Big\langle x, \frac{1}{t}\big(T(t)'x' - x'\big)\Big\rangle\Big|$$
$$\leq \|x\| \sup_{t>0}\Big\|\frac{1}{t}\big(T(t)'x' - x'\big)\Big\|.$$

By the definition of the dual operator, this implies $x' \in D(A')$ and therefore ${F^{\odot}}_1 = D(A')$. □

We now determine the extrapolated Favard spaces of the sun dual semigroup.

**5.20 Corollary.** *The following assertions are true.*

(i) ${F^{\odot}}_0$ *and* $X'$ *are canonically isomorphic.*

(ii) ${F^{\odot}}_{-1}$ *and* ${X_1}'$ *are canonically isomorphic.*

PROOF. The space ${F^{\odot}}_0$ is the Favard space of order 1 associated to $({T^{\odot}}_{-1}(t))_{t\geq 0} \cong ({T_1}^{\odot}(t))_{t\geq 0}$; cf. Lemma 5.18. However, by Proposition 5.19, ${F^{\odot}}_0 \cong D({A_1}') \cong X'$, which proves (i).

Observe next that the space ${F^{\odot}}_{-1}$ is the Favard space of order 0 associated to $({T^{\odot}}_{-1}(t))_{t\geq 0} \cong ({T_1}^{\odot}(t))_{t\geq 0}$. Hence, by part (i), we obtain that ${F^{\odot}}_{-1} \cong {X_1}'$, which is just (ii). □

As a final consequence we obtain that in reflexive spaces the Favard and abstract Hölder spaces of order 1 coincide.

**5.21 Corollary.** *If* $X$ *is reflexive, then* $F_1 = D(A)$.

PROOF. Since $X$ is reflexive and $A$ is densely defined, the dual operator $A'$ is also densely defined. Hence $X^{\odot} = X'$ and $T(t)^{\odot} = T(t)'$. Analogously, $X^{\odot\odot} = X'' = X$ and $T(t)^{\odot\odot} = T(t)'' = T(t)$. Proposition 5.19 implies that $F_1 = F_1'' = D(A'') = D(A)$. □

As usual, we visualize the various duals and sun duals in the form of a diagram.

$$
\begin{array}{ccccl}
X_{-1} & \cdots\cdots\cdots & F^{\odot}{}_{-1} & = & (X_1)' = D(A)' \\
\uparrow & & \cup & & \\
| & & X^{\odot}{}_{-1} & & \\
| & & \uparrow & & \\
X_0 & \cdots\cdots\cdots & F^{\odot}{}_{0} & = & X_0' \\
\uparrow & & \cup & & \\
| & & X^{\odot} & & \\
| & & \uparrow & & \\
X_1 & \cdots\cdots\cdots & F^{\odot}{}_{1} & = & (X_{-1})' = D(A') \\
 & & \cup & & \\
 & & X^{\odot}{}_{1} & = & D(A^{\odot})
\end{array}
$$

**5.22 Examples.** We continue the discussion of the examples in Paragraph 2.6.
(i) We first consider the left translation semigroup on $\mathrm{L}^1(\mathbb{R})$ and its adjoint semigroup given by the right translations on $\mathrm{L}^\infty(\mathbb{R})$. Since the sun dual semigroup is the right translation semigroup on $\mathrm{C_{ub}}(\mathbb{R})$, the Favard spaces for this semigroup are given by

$$F_1 = \mathrm{Lip_u}(\mathbb{R}) \qquad \text{and} \qquad F_0 = \mathrm{L}^\infty(\mathbb{R}).$$

(ii) We now start with the left translation semigroup on $\mathrm{C_0}(\mathbb{R})$ and its dual semigroup given by the right translation semigroup on $\mathrm{M_b}(\mathbb{R})$. Since the sun dual semigroup is the right translation semigroup on $\mathrm{L}^1(\mathbb{R})$, by Proposition 5.19, the Favard spaces for this semigroup are given by

$$F_1 = \mathrm{UBV}(\mathbb{R}) \qquad \text{and} \qquad F_0 = \mathrm{M_b}(\mathbb{R}).$$

The details are left as Exercise 5.23.(4).

**5.23 Exercises.** (1) Prove that the definitions of the Favard and abstract Hölder spaces given in 5.11 are independent of the choice of $w$.

(2) Show that $x \in X$ belongs to $F_1$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\sup_{n\in\mathbb{N}} \|Ax_n\| < \infty$.

(3) Let $(A, D(A))$ be a Hille–Yosida operator on a Banach space $X$ and consider the semigroup $(T_0(t))_{t\geq 0}$ generated by the part of $A$ in $X_0 := \overline{D(A)}$ (cf. Corollary 3.21 and Definition 3.22). Show that $X$ can be canonically identified with a closed subspace of the extrapolated Favard space $F_0$ associated to $(T_0(t))_{t\geq 0}$.

(4) Work out the details of Example 5.22. (For the definitions of the function spaces involved see Appendix A.)

(5) Let $(T(t))_{t\geq 0}$ be the left translation semigroup on $X := \mathrm{C_b}(\mathbb{R}_+)$. Show that the abstract Favard and Hölder spaces corresponding to $(T(t))_{t\geq 0}$ coincide with the classical Hölder spaces, i.e.,

$$F_\alpha = \mathrm{C}^\alpha(\mathbb{R}_+) := \left\{ f \in \mathrm{C_b}(\mathbb{R}_+) : \begin{array}{l} \text{the functions } t \mapsto {}^1\!/_{t^\alpha}|f(t+s) - f(s)| \\ \text{are uniformly bounded for } s > 0 \end{array} \right\}$$

and

$$X_\alpha = \mathrm{h}^\alpha(\mathbb{R}_+) := \left\{ f \in \mathrm{C_b}(\mathbb{R}_+) : \begin{array}{l} \lim_{t\downarrow 0} {}^1\!/_{t^\alpha} |f(s+t) - f(s)| = 0 \\ \text{uniformly for } s > 0 \end{array} \right\}$$

for each $\alpha \in (0,1)$.

(6) Let $\Omega$ be a locally compact space, $q : \Omega \to \mathbb{C}$ a continuous function satisfying $\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty$, and let $(T_q(t))_{t\geq 0}$ be the multiplication semigroup generated by the multiplication operator $M_q$ on $X := \mathrm{C_0}(\Omega)$ (see Definition I.4.3). Show that for each $\alpha \in (0,1)$ we have

$$F_\alpha = \left\{ f \in \mathrm{C_0}(\Omega) : |q|^\alpha f \in \mathrm{C_b}(\Omega) \right\}$$

and

$$X_\alpha = \left\{ f \in \mathrm{C_0}(\Omega) : |q|^\alpha f \in \mathrm{C_0}(\Omega) \right\}.$$

## c. Fractional Powers

It is the aim of this subsection to introduce fractional powers of linear operators. We prove that they form a semigroup and that their domains fit nicely into the Favard and abstract Hölder spaces. Finally, we derive the important moment inequality, which can be viewed as a continuous analogue of the Landau–Kolmogorov inequality from Exercise 1.15.(6).

In the sequel, we assume $A$ to be a closed operator such that $(0,\infty) \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{M}{1+\lambda}$ for all $\lambda \in (0,\infty)$ and some constant $M$.

**5.24 Proposition.** *There exists an open sector $\Sigma$ in $\mathbb{C}$ such that $\mathbb{R}_+ \subset \Sigma \subset \rho(A)$ and*

$$\|R(\lambda, A)\| \leq \frac{2M}{1+|\lambda|}$$

*for all $\lambda \in \Sigma$.*

The proof is left as Exercise 5.36.(1).

**5.25 Definition.** *For $\alpha > 0$, the bounded linear operator $A^{-\alpha}$ is defined by*

$$A^{-\alpha} := \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-\alpha} R(\lambda, A)\, d\lambda,$$

*where $\gamma$ is a piecewise smooth path in $\Sigma \setminus \mathbb{R}_+$ going from $\infty \mathrm{e}^{-i\delta}$ to $\infty \mathrm{e}^{\mathrm{i}\delta}$ for some $\delta > 0$, cf. Figure 3.*

In this definition the function $\lambda \mapsto \lambda^{-\alpha}$ is a branch of the fractional power function on $\mathbb{C} \setminus \mathbb{R}_+$, i.e., $\lambda^{-\alpha} = \mathrm{e}^{-\alpha \log \lambda}$, where log denotes a branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_+$ (cf. [Con73, Def. 2.18]). The estimate in Proposition 5.24 assures that the integral exists. Moreover, by Cauchy's integral theorem the expression is independent of the particular choice of the path $\gamma$.

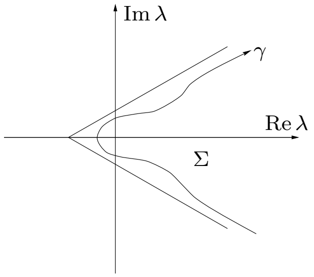



**Figure 3**

**5.26 Proposition.** *For $\alpha \in \mathbb{N}_0$ the above definition yields the inverse power $A^{-\alpha}$ of $A$.*

PROOF. Since the integrand has an isolated singularity at the origin, the residue theorem applies, and the assertion follows. □

**5.27 Theorem.** *Let $\alpha \in (0, n+1)$. In the case $\alpha \notin \mathbb{N}$ we have the formula*

$$A^{-\alpha} = \frac{1}{2\pi \mathrm{i}} \frac{n!}{(1-\alpha)\cdots(n-\alpha)} (1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{n-\alpha} R(s,A)^{n+1}\, ds.$$

*In the case $\alpha \in \mathbb{N}$, the formula is true if both sides are extended continuously in $\alpha$.*

PROOF. First, we have to specify our path of integration. Let $a$ and $\delta$ be positive real numbers. Let $\gamma$ be the path consisting of the half-lines going from $\infty \mathrm{e}^{-i\delta}$ to $-a$ and from $-a$ to $\infty \mathrm{e}^{\mathrm{i}\delta}$. If $a$ and $\delta$ are sufficiently small, then $\gamma$ is contained in $\Sigma \setminus \mathbb{R}_+$. Therefore, one has

$$A^{-\alpha} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-\alpha} R(\lambda, A)\, d\lambda.$$

Integrating $n$ times by parts gives

$$A^{-\alpha} = \frac{1}{2\pi \mathrm{i}} \frac{n!}{(1-\alpha)\cdots(n-\alpha)} \int_\gamma \lambda^{n-\alpha} R(\lambda, A)^{n+1}\, d\lambda,$$

we hence

$$\begin{aligned} A^{-\alpha} = \frac{1}{2\pi \mathrm{i}} \frac{n!}{(1-\alpha)\cdots(n-\alpha)} \Bigg[ & \int_0^\infty (s\mathrm{e}^{\mathrm{i}\delta} - a)^{n-\alpha} R(s\mathrm{e}^{\mathrm{i}\delta} - a, A)^{n+1}\, \mathrm{e}^{\mathrm{i}\delta}\, ds \\ & - \int_0^\infty (s\mathrm{e}^{-i\delta} - a)^{n-\alpha} R(s\mathrm{e}^{-i\delta} - a, A)^{n+1}\, \mathrm{e}^{-i\delta}\, ds \Bigg]. \end{aligned}$$

We now take successively the limits as $a \downarrow 0$ and $\delta \downarrow 0$. With the dominating function

$$s \mapsto \begin{cases} K\left(1 + s^{n-\alpha}\right) & \text{if } s \le 1, \\ K s^{-\alpha-1} & \text{if } s > 1 \end{cases}$$

with $K > 0$ sufficiently large, Lebesgue's dominated convergence theorem implies

$$A^{-\alpha} = \frac{1}{2\pi \mathrm{i}} \frac{n!}{(1-\alpha)\cdots(n-\alpha)} (1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{n-\alpha} R(s,A)^{n+1}\, ds.$$

This proves the assertion. □

If $n = 0$, i.e., $\alpha \in (0,1)$, we obtain the following representation for $A^{-\alpha}$.

**5.28 Corollary.** *For $\alpha \in (0,1)$, we have*

$$A^{-\alpha} = \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{-\alpha} R(s,A)\, ds.$$

Next, we study the properties of the operator family $(A^{-\alpha})_{\alpha \geq 0}$.

**5.29 Theorem.** *The operators $(A^{-\alpha})_{\alpha \geq 0}$ form a semigroup. If $A$ is densely defined, this semigroup is strongly continuous.*

PROOF. The first step consists in verifying the semigroup law. To this end, we choose curves $\gamma$ and $\gamma'$ in $\Sigma \setminus \mathbb{R}_+$ such that $\gamma$ lies to the left of $\gamma'$. Using the resolvent identity, we obtain

$$\begin{aligned}
A^{-\alpha} A^{-\beta} &= \frac{1}{(2\pi \mathrm{i})^2} \int_\gamma \int_{\gamma'} \lambda^{-\alpha} \mu^{-\beta} R(\lambda, A) R(\mu, A)\, d\mu\, d\lambda \\
&= \frac{1}{(2\pi \mathrm{i})^2} \int_\gamma \int_{\gamma'} \lambda^{-\alpha} \mu^{-\beta} \left[ \frac{R(\lambda, A)}{\mu - \lambda} + \frac{R(\mu, A)}{\lambda - \mu} \right] d\mu\, d\lambda \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-\alpha} \left[ \frac{1}{2\pi \mathrm{i}} \int_{\gamma'} \frac{\mu^{-\beta}}{\mu - \lambda}\, d\mu \right] R(\lambda, A)\, d\lambda \\
&\quad + \frac{1}{2\pi \mathrm{i}} \int_{\gamma'} \mu^{-\beta} \left[ \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\lambda^{-\alpha}}{\lambda - \mu}\, d\lambda \right] R(\mu, A)\, d\mu.
\end{aligned}$$

By Cauchy's integral theorem, one has

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_{\gamma'} \frac{\mu^{-\beta}}{\mu - \lambda}\, d\mu &= \lambda^{-\beta} \qquad \text{for } \lambda \in \gamma, \\
\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\lambda^{-\alpha}}{\lambda - \mu}\, d\lambda &= 0 \qquad \text{for } \mu \in \gamma',
\end{aligned}$$

and hence

$$A^{-\alpha} A^{-\beta} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \lambda^{-\alpha} \lambda^{-\beta} R(\lambda, A)\, d\lambda = A^{-(\alpha+\beta)}.$$

The uniform boundedness of the operators $A^{-\alpha}$ for $\alpha \in (0,1)$ follows from

$$\begin{aligned}
\|A^{-\alpha}\| &= \left\| \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{-\alpha} R(s,A)\, ds \right\| \\
&\leq \left| \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{-\alpha} \frac{M}{1+s}\, ds \right| \\
&= |M(-1)^{-\alpha}| = M.
\end{aligned}$$

Finally, we show that the semigroup $(A^{-\alpha})_{\alpha \geq 0}$ is strongly continuous when $D(A)$ is dense in $X$. To this end it suffices to verify that $\lim_{\alpha \downarrow 0} A^{-\alpha} x = x$ for every $x \in D(A)$. For $x \in D(A)$ and $\alpha \in (0,1)$ we have

$$\begin{aligned}
A^{-\alpha} x - (-I)^{-\alpha} x &= \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{-\alpha} [R(s,A) - R(s,-I)] x\, ds \\
&= \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{-2\pi \mathrm{i}\alpha}) \int_0^\infty s^{-\alpha} R(s,-I) R(s,A)(I + A) x\, ds.
\end{aligned}$$

From this it follows that

$$\lim_{\alpha\downarrow 0} A^{-\alpha}x - x = \lim_{\alpha\downarrow 0}[A^{-\alpha}x - (-I)^{-\alpha}x] + [(-I)^{-\alpha}x - x] = 0.$$

This completes the proof. □

In order to define $A^\alpha$ for positive values of $\alpha$ we need the following result.

**5.30 Proposition.** *The operator $A^{-\alpha}$ is injective for every $\alpha > 0$.*

PROOF. Choose $\beta > 0$ such that $\alpha + \beta =: n \in \mathbb{N}$. Then $A^{-\beta}A^{-\alpha} = A^{-n}$, and hence $A^n A^{-\beta}A^{-\alpha} = I$. Therefore, $A^{-\alpha}$ is injective. □

We are now prepared to define the fractional powers of $A$.

**5.31 Definition.** *Let $\alpha > 0$. Then the operator $A^\alpha$ defined as the inverse of $A^{-\alpha}$ with domain $D(A^\alpha) = \operatorname{rg}(A^{-\alpha})$ is called the $\alpha$-power of $A$.*

This terminology is justified by the following result.

**5.32 Theorem.** *Let $\alpha, \beta \in \mathbb{R}$. Then the operators $A^\alpha A^\beta$ and $A^{\alpha+\beta}$ agree on $D(A^\gamma)$ for $\gamma := \max\{\alpha, \beta, \alpha+\beta\}$.*

PROOF. The assertion is a consequence of Theorem 5.29. For example, if $\alpha, \beta \geq 0$, then

$$A^\alpha A^\beta x = A^\alpha A^\beta \bigl(A^{-\beta}A^{-\alpha}A^{\alpha+\beta}\bigr)x = \bigl(A^\alpha A^\beta A^{-\beta}A^{-\alpha}\bigr)A^{\alpha+\beta}x = A^{\alpha+\beta}x$$

for each $x \in D(A^{\alpha+\beta})$. The other cases follow similarly. □

Our next aim is to relate the domains of the fractional powers to the Favard and abstract Hölder spaces from Section 5.b.

**5.33 Proposition.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ with growth bound $\omega_0 < 0$, and let $\alpha, \beta \in (0,1)$ such that $\alpha > \beta$. Then $X_\alpha \hookrightarrow D(A^\beta) \hookrightarrow X_\beta$.*

PROOF. Let $x \in D(A)$. Then, by Corollary 5.28,

$$\begin{aligned} A^\beta x = AA^{\beta-1}x &= \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{2\pi \mathrm{i}\beta}) \int_0^\infty s^{\beta-1}AR(s,A)x\,ds \\ &= \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{2\pi \mathrm{i}\beta}) \int_0^1 s^{\beta-1}AR(s,A)x\,ds \\ &\quad + \frac{1}{2\pi \mathrm{i}}(1 - \mathrm{e}^{2\pi \mathrm{i}\beta}) \int_1^\infty s^{\beta-\alpha-1}s^\alpha AR(s,A)x\,ds. \end{aligned}$$

For each $s > 0$, one has $\|AR(s,A)x\| \leq (M+1)\|x\|$ and $\|s^\alpha AR(s,A)x\| \leq |\!|\!| x |\!|\!|_{F_\alpha}$ (see Proposition 5.12). From this it follows that

$$\|A^\beta x\| \leq K \; |\!|\!| x |\!|\!|_{F_\alpha}$$

for a suitable constant $K > 0$. The closedness of the fractional power $A^\beta$ then implies $X_\alpha = \overline{D(A)}^{|\!|\!|\cdot|\!|\!|_{F_\alpha}} \hookrightarrow D(A^\beta)$.

To verify the second embedding, we first show that $\|A^{-\beta}\lambda^\beta AR(\lambda,A)\| \le L$ for every $\lambda > 0$ and a suitable constant $L$. By Corollary 5.28, we have

$$A^{-\beta} = \frac{1}{2\pi\mathrm{i}}(1-\mathrm{e}^{-2\pi\mathrm{i}\beta}) \int_0^\infty (\lambda s)^{-\beta} R(\lambda s, A)\, d(\lambda s).$$

This implies

$$\begin{aligned} A^{-\beta}\lambda^\beta AR(\lambda,A) &= \frac{1}{2\pi\mathrm{i}}(1-\mathrm{e}^{-2\pi\mathrm{i}\beta}) \int_0^1 s^{-\beta}\, AR(\lambda s,A)\,\lambda R(\lambda,A)\, ds \\ &\quad + \frac{1}{2\pi\mathrm{i}}(1-\mathrm{e}^{-2\pi\mathrm{i}\beta}) \int_1^\infty s^{-\beta-1}\, \lambda s R(\lambda s,A)\, AR(\lambda,A)\, ds. \end{aligned}$$

For each $\lambda > 0$, we have $\|AR(\lambda,A)\| \le M+1$ and $\|\lambda R(\lambda,A)\| \le M$. From this it follows that

$$\|A^{-\beta}\lambda^\beta AR(\lambda,A)\| \le L$$

for all $\lambda > 0$ and a suitable constant $L$. Hence, we obtain

$$\|\lambda^\beta AR(\lambda,A)x\| \le L\,\|A^\beta x\|$$

for each $x \in D(A)$ and $\lambda > 0$ as claimed. This yields

$$\|x\|_{F_\beta} \le L\,\|A^\beta x\|,$$

and therefore $D(A^\beta) \hookrightarrow \overline{D(A)}^{\|\cdot\|_{F_\beta}} = X_\beta$. □

Next, we use the above results to prove the *moment inequality*, cf. [Kre71, Thm. 5.2]. It allows us to estimate $\|A^\beta x\|$ in terms of $\|A^\alpha x\|$ and $\|A^\gamma x\|$ if $\alpha < \beta < \gamma$ and therefore can be considered as a generalization of the Landau–Kolmogorov inequality in Exercise 1.15.(6).

**5.34 Theorem.** *Let $\alpha < \beta < \gamma$. Then there exists a constant $L = L(\alpha,\beta,\gamma)$ such that*

$$\|A^\beta x\| \le L\,\|A^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \cdot \|A^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}$$

*for every $x \in D(A^\gamma)$.*

Proof. We first consider a slightly different situation. Suppose that $\alpha_0 > \beta_0 > 0$, and let $n$ be an integer satisfying $\alpha_0 \in (n, n+1]$. Then $\beta_0 \in (0, n+1)$, and from the proof of Proposition 5.33 we know that

$$\begin{aligned} &\left\|s^{n-\beta_0} R(s,A)^{n+1} x_0\right\| \\ &\qquad \le s^{\alpha_0-\beta_0-1} \left\|A^{-n-1+\alpha_0} s^{n+1-\alpha_0} AR(s,A)\right\| \cdot \|A^n R(s,A)^n\| \cdot \left\|A^{-\alpha_0} x_0\right\| \\ &\qquad \le K s^{\alpha_0-\beta_0-1} \left\|A^{-\alpha_0} x_0\right\| \end{aligned}$$

and

$$\left\|s^{n-\beta_0} R(s,A)^{n+1} x_0\right\| \le K s^{n-\beta_0} s^{-n-1} \|x_0\| = K s^{-\beta_0-1} \|x_0\|$$

for a suitable constant $K$. Using these inequalities we obtain

$$\begin{aligned}
\|A^{-\beta_0}x_0\| &\le K'\Big\|\int_0^\infty s^{n-\beta_0}R(s,A)^{n+1}x_0\,ds\Big\| \\
&= K'\Big\|\int_0^\tau s^{n-\beta_0}R(s,A)^{n+1}x_0\,ds + \int_\tau^\infty s^{n-\beta_0}R(s,A)^{n+1}x_0\,ds\Big\| \\
&\le K''\int_0^\tau s^{\alpha_0-\beta_0-1}\,\|A^{-\alpha_0}x_0\|\,ds + K''\int_\tau^\infty s^{-\beta_0-1}\|x_0\|\,ds \\
&= \frac{K''}{\alpha_0-\beta_0}\tau^{\alpha_0-\beta_0}\|A^{-\alpha_0}x_0\| + \frac{K''}{\beta_0}\tau^{-\beta_0}\|x_0\|
\end{aligned}$$

for all $\tau>0$. Taking $\tau := \|A^{-\alpha_0}x_0\|^{-1/\alpha_0}\cdot\|x_0\|^{1/\alpha_0}$ yields

$$\|A^{-\beta_0}x_0\| \le L\,\|A^{-\alpha_0}x_0\|^{\frac{\beta_0}{\alpha_0}}\cdot\|x_0\|^{\frac{\alpha_0-\beta_0}{\alpha_0}}$$

for a suitable constant $L$.

We now turn to the proof of the theorem. If we apply the previous result to $\alpha_0 := \gamma-\alpha$, $\beta_0 := \gamma-\beta$, and $x_0 := A^\gamma x$, we obtain

$$\|A^\beta x\| \le L\,\|A^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}}\cdot\|A^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}}.$$

This proves the assertion. □

We close this subsection by looking at iterated abstract Hölder spaces, i.e., spaces of the form $(X_\alpha)_\beta$ where $\alpha$ and $\beta$ are positive real numbers.

**5.35 Proposition.** *Let $\alpha,\beta\in(0,1)$ satisfy $\alpha+\beta\ne 1$. Then the iterated abstract Hölder space $(X_\alpha)_\beta$ coincides with the abstract Hölder space $X_{\alpha+\beta}$.*

PROOF. We divide the proof into two cases and first assume that $\alpha+\beta<1$. Let $x\in(X_\alpha)_\beta$. Using the identity

$$2\big(T(t)-I\big) = \big(T(2t)-I\big) - \big(T(t)-I\big)^2,$$

we obtain

$$\begin{aligned}
2\Big\|\frac{1}{t^{\alpha+\beta}}\big(T(t)-I\big)x\Big\| &\le \Big\|\frac{1}{t^{\alpha+\beta}}\big(T(2t)-I\big)x\Big\| + \Big\|\frac{1}{t^{\alpha+\beta}}\big(T(t)-I\big)^2x\Big\| \\
&\le 2^{\alpha+\beta}\Big\|\frac{1}{(2t)^{\alpha+\beta}}\big(T(2t)-I\big)x\Big\| + \Big\|\frac{1}{t^{\alpha}}\big(T(t)-I\big)\cdot\frac{1}{t^\beta}\big(T(t)-I\big)x\Big\| \\
&\le 2^{\alpha+\beta}\Big\|\frac{1}{(2t)^{\alpha+\beta}}\big(T(2t)-I\big)x\Big\| + \Big\|\frac{1}{t^\beta}\big(T(t)-I\big)x\Big\|_{X_\alpha}
\end{aligned}$$

for all $t>0$. For a given $\varepsilon>0$ we choose $\delta>0$ such that

$$\sup_{0<t\le\delta}\Big\|\frac{1}{t^\beta}\big(T(t)-I\big)x\Big\|_{X_\alpha} \le 2\left(1-2^{\alpha+\beta-1}\right)\varepsilon \qquad \text{for } 0<t\le\delta.$$

Then we obtain

$$\begin{aligned}
&\sup_{2^{-n-1}\delta\le t\le 2^{-n}\delta}\Big\|\frac{1}{t^{\alpha+\beta}}\big(T(t)-I\big)x\Big\| \\
&\qquad\le 2^{\alpha+\beta-1}\sup_{2^{-n}\delta\le t\le 2^{-n+1}\delta}\Big\|\frac{1}{t^{\alpha+\beta}}\big(T(t)-I\big)x\Big\| + \left(1-2^{\alpha+\beta-1}\right)\varepsilon.
\end{aligned}$$

By induction on $n \in \mathbb{N}$, it follows that

$$\sup_{2^{-n-1}\delta \le t \le 2^{-n}\delta} \left\| \frac{1}{t^{\alpha+\beta}} \big(T(t) - I\big)x \right\| \le 2^{n(\alpha+\beta-1)} \sup_{\delta/2 \le t \le \delta} \left\| \frac{1}{t^{\alpha+\beta}} \big(T(t) - I\big)x \right\| + \varepsilon.$$

Taking successively the limit for $n \to \infty$ and then for $\varepsilon \downarrow 0$, we obtain

$$\lim_{t \downarrow 0} \left\| \frac{1}{t^{\alpha+\beta}} \big(T(t) - I\big)x \right\| = 0,$$

and so $x \in X_{\alpha+\beta}$.

Conversely, let $x \in X_{\alpha+\beta}$. Then we have

$$\left\| \frac{1}{t^{\alpha+\beta}} \big(T(t) - I\big)x \right\| \le \frac{\varepsilon}{2M} \qquad \text{for } 0 < t \le \delta.$$

Now let $t \in (0, \delta]$. Then we have

$$\begin{aligned} \left\| \frac{1}{s^\alpha} \frac{1}{t^\beta} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| &\le \left(\frac{s}{t}\right)^\beta \|T(t) - I\| \cdot \left\| \frac{1}{s^{\alpha+\beta}} \big(T(s) - I\big)x \right\| \\ &\le \varepsilon \qquad \text{for } s \le t \end{aligned}$$

and, similarly,

$$\begin{aligned} \left\| \frac{1}{s^\alpha} \frac{1}{t^\beta} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| &\le \left(\frac{t}{s}\right)^\alpha \|T(s) - I\| \cdot \left\| \frac{1}{t^{\alpha+\beta}} \big(T(t) - I\big)x \right\| \\ &\le \varepsilon \qquad \text{for } t \le s. \end{aligned}$$

Therefore,

$$\left\| \frac{1}{t^\beta} \big(T(t) - I\big)x \right\|_{X_\alpha} = \sup_{s>0} \left\| \frac{1}{s^\alpha} \frac{1}{t^\beta} \big(T(s) - I\big)\big(T(t) - I\big)x \right\| \le \varepsilon$$

for every $0 < t \le \delta$, and so $x \in (X_\alpha)_\beta$. This finishes the proof in the case $\alpha + \beta < 1$.

We now suppose that $\alpha + \beta > 1$ and take $x \in (X_\alpha)_\beta$. It follows from Proposition 5.33 that $x \in X_1$. By Proposition 5.12, we have

$$\lim_{\lambda \to \infty} \sup_{\mu > 0} \left\| \mu^\alpha \lambda^\beta A^2 R(\mu, A) R(\lambda, A)x \right\| = 0$$

and, in particular,

$$\lim_{\lambda \to \infty} \left\| \lambda^{\alpha+\beta} A^2 R(2\lambda, A) R(\lambda, A)x \right\| = 0.$$

From this it follows that

$$\lim_{\lambda \to \infty} \left\| \lambda^{\alpha+\beta-1} A^2 R(2\lambda, A)x - \lambda^{\alpha+\beta-1} A^2 R(\lambda, A)x \right\| = 0.$$

Now let $\varepsilon > 0$ be given. Then there is a real number $K$ such that

$$\left\| A^2 R(2\lambda, A)x - A^2 R(\lambda, A)x \right\| \le \frac{2^{\alpha+\beta-1} - 1}{2^{\alpha+\beta-1}} \cdot \frac{\varepsilon}{\lambda^{\alpha+\beta-1}}$$

for all $\lambda \geq K$. This implies

$$\left\|A^2R(2\lambda, A)x\right\| \geq \left\|A^2R(\lambda, A)x\right\| - \frac{2^{\alpha+\beta-1}-1}{2^{\alpha+\beta-1}} \cdot \frac{\varepsilon}{\lambda^{\alpha+\beta-1}}$$

for all $\lambda \geq K$. Iterating this argument, we obtain

$$\left\|A^2R(2^n\lambda, A)x\right\| \geq \left\|A^2R(\lambda, A)x\right\| - \frac{2^{n(\alpha+\beta-1)}-1}{2^{n(\alpha+\beta-1)}} \cdot \frac{\varepsilon}{\lambda^{\alpha+\beta-1}}$$

for all $\lambda \geq K$ and $n \in \mathbb{N}$. From this we conclude that

$$0 \geq \left\|A^2R(\lambda, A)x\right\| - \frac{\varepsilon}{\lambda^{\alpha+\beta-1}},$$

that is,

$$\varepsilon \geq \left\|\lambda^{\alpha+\beta-1}A^2R(\lambda, A)x\right\|.$$

Therefore, we obtain

$$\lim_{\lambda\to\infty} \left\|\lambda^{\alpha+\beta-1}A^2R(\lambda, A)x\right\| = 0,$$

and hence $x \in X_{\alpha+\beta}$.

Conversely, let $x \in X_{\alpha+\beta}$ and $\varepsilon > 0$. Then there is a real number $K$ such that

$$\left\|\lambda^{\alpha+\beta-1}A^2R(\lambda, A)x\right\| \leq \frac{\varepsilon}{M}$$

for all $\lambda \geq K$. Then we have

$$\left\|\mu^\alpha\lambda^\beta A^2R(\mu, A)R(\lambda, A)x\right\| \leq M \left\|\lambda^{\alpha+\beta-1}A^2R(\lambda, A)x\right\| \leq \varepsilon$$

for $\mu \geq \lambda \geq K$ and, similarly,

$$\left\|\mu^\alpha\lambda^\beta A^2R(\mu, A)R(\lambda, A)x\right\| \leq M \left\|\mu^{\alpha+\beta-1}A^2R(\mu, A)x\right\| \leq \varepsilon$$

for $\lambda \geq \mu \geq K$. This implies

$$\sup_{\mu\geq K} \left\|\mu^\alpha\lambda^\beta A^2R(\mu, A)R(\lambda, A)x\right\| \leq \varepsilon$$

for all $\lambda \geq K$. If $\lambda$ is large enough, we obtain

$$\sup_{\mu>0} \left\|\mu^\alpha\lambda^\beta A^2R(\mu, A)R(\lambda, A)x\right\| \leq \varepsilon.$$

Since $\varepsilon$ was arbitrary, we conclude that $x \in (X_\alpha)_\beta$. □

**5.36 Exercises.** (1) Prove Proposition 5.24. (Hint: Use the power series representation of the resolvent in Proposition IV.1.3.(i).)

(2) Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq0}$ with growth bound $\omega_0 < 0$. Show that $A^{-\alpha} = {}^{(-1)^{-\alpha}}\!/_{\Gamma(\alpha)} \int_0^\infty r^{\alpha-1}T(r)\,dr$ for every $\alpha \in (0,1)$. (Hint: Use Corollary 5.28 and the integral representation of the resolvent. Note that $\Gamma(\alpha)\Gamma(1-\alpha) = {}^{\pi}\!/_{\sin\pi\alpha}$ for $\alpha \in (0,1)$.)

(3) Work out the details of the proof of Theorem 5.32.

(4) Let $A := {}^{d^2}\!/_{dx^2}$ with domain $D(A) := \mathrm{H}_0^2[0,1]$ on $X := \mathrm{L}^2[0,1]$. Show that $D(A^{1/2}) = \mathrm{H}_0^1[0,1]$. (Hint: Show that for $f \in D(A)$ one has $\|A^{1/2}f\| = \|f\|_{\mathrm{H}_0^1[0,1]}$. Then use the inclusion $D(A) \subset D(A^{1/2}) \cap \mathrm{H}_0^1[0,1]$ to obtain the assertion.)

## 6. Well-Posedness for Evolution Equations

Only now we turn our attention to what could have been, in a certain perspective, our starting point: We want to solve a differential equation. More precisely, we look at abstract (i.e., Banach-space-valued) linear initial value problems of the form

$$\begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = x, \end{cases}$$

where the independent variable $t$ represents time, $u(\cdot)$ is a function with values in a Banach space $X$, $A : D(A) \subset X \to X$ a linear operator, and $x \in X$ the initial value.

We start by introducing the necessary terminology.

**6.1 Definition.** (i) *The initial value problem*

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) & \textit{for } t \geq 0, \\ u(0) = x \end{cases}$$

*is called the abstract Cauchy problem associated to* $\big(A, D(A)\big)$ *and the initial value* $x$.

(ii) *A function* $u : \mathbb{R}_+ \to X$ *is called a (classical) solution of* (ACP) *if* $u$ *is continuously differentiable with respect to* $X$, $u(t) \in D(A)$ *for all* $t \geq 0$, *and* (ACP) *holds.*

If the operator $A$ is the generator of a strongly continuous semigroup, it follows from Lemma 1.3.(ii) that the semigroup yields solutions of the associated abstract Cauchy problem.

**6.2 Proposition.** *Let* $\big(A, D(A)\big)$ *be the generator of the strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$. *Then, for every* $x \in D(A)$, *the function*

$$u : t \mapsto u(t) := T(t)x$$

*is the unique classical solution of* (ACP).

The important point is that (classical) solutions exist if (and, by the definition of $D(A)$, only if) the initial value $x$ belongs to $D(A)$. However, modifying slightly the concept of "solution" and requiring differentiability only for $t > 0$, we obtain such solutions for each $x \in X$ as soon as the semigroup $\big(T(t)\big)_{t\geq 0}$ is immediately differentiable. This already suggests that different concepts of "solutions" might be useful. The most important one renounces differentiability and substitutes the differential equation by an integral equation.

**6.3 Definition.** *A continuous function* $u : \mathbb{R}_+ \to X$ *is called a mild solution of* (ACP) *if* $\int_0^t u(s)\,ds \in D(A)$ *for all* $t \geq 0$ *and*

$$u(t) = A \int_0^t u(s)\,ds + x.$$

It follows from our previous (and elementary) results (use Lemma 1.3.(iv)) that for $A$ being the generator of a strongly continuous semigroup, mild solutions exist for every initial value $x \in X$ and are again given by the semigroup.

**6.4 Proposition.** *Let* $\big(A, D(A)\big)$ *be the generator of the strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$. *Then, for every* $x \in X$, *the orbit map*

$$u : t \mapsto u(t) := T(t)x$$

*is the unique mild solution of the associated abstract Cauchy problem* (ACP).

Proof. We only have to show the uniqueness of the zero solution for the initial value $x = 0$. To this end, assume $u$ to be a mild solution of (ACP) for $x = 0$ and take $t > 0$. Then, for each $s \in (0, t)$, we obtain

$$\tfrac{d}{ds}\left(T(t-s)\int_0^s u(r)\,dr\right) = T(t-s)u(s) - T(t-s)A\int_0^s u(r)\,dr = 0.$$

Integration of this equality from 0 to $t$ gives

$$\int_0^t u(r)\,dr = 0, \qquad \text{hence} \qquad u(t) = u(0) = 0$$

as claimed. □

The above two propositions are just reformulations of results on strongly continuous semigroups. They might suggest that the converse holds. The following example shows that this is not true.

**6.5 Example.** Let $\big(B, D(B)\big)$ be a closed and *unbounded* operator on $X$. On the product space $\mathcal{X} := X \times X$, consider the operator $\big(\mathcal{A}, D(\mathcal{A})\big)$ written in matrix form as

$$\mathcal{A} := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \qquad \text{with domain} \qquad D(\mathcal{A}) := X \times D(B).$$

Then $t \mapsto u(t) := \binom{x+tBy}{y}$ is the unique solution of (ACP) associated to $\mathcal{A}$ for every $\binom{x}{y} \in D(\mathcal{A})$. However, the operator $\mathcal{A}$ does not generate a strongly continuous semigroup, since for every $\lambda \in \mathbb{C}$, one has

$$(\lambda - \mathcal{A})D(\mathcal{A}) = \left\{\binom{\lambda x - By}{\lambda y} : x \in X, y \in D(B)\right\} \subset X \times D(B) \neq \mathcal{X},$$

and hence $\sigma(\mathcal{A}) = \mathbb{C}$.

Fortunately, as soon as we have existence and uniqueness of solutions of (ACP) for every $x \in D(A)$, we are not very far from semigroup generators.

**6.6 Proposition.** *For a closed operator $A : D(A) \subset X \to X$, the following properties are equivalent.*

(a) *There exists a unique solution of* (ACP) *for every $x \in D(A)$.*

(b) *The part $A_1 := A_{|X_1}$ of $A$ in $X_1 := \big(D(A), \|\cdot\|_A\big)$ is the generator of a strongly continuous semigroup on the Banach space $X_1$.*

Proof. Denote the solution of (ACP) for an initial value $x$ by $u(\cdot, x)$. The idea of the proof is to obtain an operator semigroup $\big(T(t)\big)_{t\geq 0}$ from the solutions by putting

$$T(t)x := u(t, x),$$

while, conversely, a given semigroup $\big(T(t)\big)_{t\geq 0}$ should yield solutions of (ACP) by

$$u(t, x) := T(t)x.$$

With these definitions, it remains to verify carefully all the required continuity and differentiability properties.

(a) $\Rightarrow$ (b). Let $u(\cdot, x) \in \mathrm{C}^1(\mathbb{R}_+, X)$ be the unique solution of (ACP) for $x \in X_1$ and define as indicated above

$$T_1(t)x := u(t, x) \qquad \text{for} \qquad t \geq 0.$$

The uniqueness of the solutions implies that each $T_1(t)$ is a linear operator defined on $X_1$ satisfying

$$\begin{aligned} & T_1(0) = I_{X_1} \\ \text{and} \qquad & \\ & T_1(t+s) = T_1(t)T_1(s) \end{aligned}$$

for $t, s \geq 0$. Moreover, $T_1(\cdot)x$ belongs to $\mathrm{C}^1(\mathbb{R}_+, X)$ and satisfies

$$\tfrac{d}{dt} T_1(t)x = AT_1(t)x.$$

Therefore, $t \mapsto T_1(t)x$ is continuous from $\mathbb{R}_+$ into the Banach space $X_1$. It remains to show that $T_1(t) \in \mathcal{L}(X_1)$ for each $t > 0$. To that purpose, define a linear operator

$$\begin{aligned} & \Phi : X_1 \to \mathrm{C}([0, t], X_1) \\ \text{by} \qquad & \\ & \Phi(x) := T_1(\cdot)x \quad \text{for } x \in X_1. \end{aligned}$$

We now show that $\Phi$ is closed.

Let $x_n \to x$ in $X_1$ and $\Phi(x_n) \to f$ in $\mathrm{C}([0,t], X_1)$. From the differential equation in (ACP), it follows that

$$T_1(s)x_n = x_n + \int_0^s AT_1(r)x_n\,dr, \quad n \in \mathbb{N},$$

and therefore

$$f(s) = x + \int_0^s Af(r)\,dr \quad \text{for } s \in [0,t].$$

Define

$$\widetilde{f}(s) := \begin{cases} T_1(s-t)f(t) & \text{for } s > t, \\ f(s) & \text{for } 0 \le s \le t. \end{cases}$$

It is easy to check that $\widetilde{f}(\cdot)$ is a solution of (ACP) and therefore $\widetilde{f}(s) = T_1(s)x$ for all $s \ge 0$, or, in other words, $f = \Phi(x)$. So, we have shown that $\Phi$ is closed and hence bounded on the Banach space $X_1$. This implies the boundedness of $T_1(t)$ for each $t > 0$.

In the final step we have to identify the generator $\big(B, D(B)\big)$ of the semigroup $\big(T_1(t)\big)_{t\ge0}$ with the operator $\big(A_1, D(A_1)\big)$. To start with, we show that

$$AT_1(t)x = T_1(t)Ax \tag{6.1}$$

for every $x \in D(A_1) = D(A^2)$. In fact, if we define

$$f(t) := x + \int_0^t T_1(s)Ax\,ds,$$

we obtain

$$\begin{aligned} \tfrac{d}{dt} f(t) = T_1(t)Ax &= Ax + \int_0^t AT_1(s)Ax\,ds \\ &= A\left(x + \int_0^t T_1(s)Ax\,ds\right) = Af(t). \end{aligned}$$

This implies $f(t) = T_1(t)x$, since $f(0) = x$, and

$$AT_1(t)x = Af(t) = \tfrac{d}{dt} f(t) = T_1(t)Ax$$

as stated above.

Let $x \in D(A^2)$. Then $\lim_{t\downarrow0} {}^1\!/\!{}_t\big(T_1(t)x - x\big) = Ax$ and, by (6.1),

$$\begin{aligned} \lim_{t\downarrow0} A\tfrac{1}{t}\big(T_1(t)x - x\big) &= \lim_{t\downarrow0} \tfrac{1}{t}\big(T_1(t)Ax - Ax\big) \\ &= A^2x \end{aligned}$$

in the norm of $X$. Both equalities imply

$$\|\cdot\|_A\text{-}\lim_{t\downarrow0} \tfrac{1}{t}\big(T_1(t)x - x\big) = Ax,$$

and hence $A_1 \subset B$. Assume now $x \in D(B)$, i.e., $\|\cdot\|_A\text{-}\lim_{t\downarrow0} {}^1\!/\!{}_t\big(T_1(t)x - x\big)$ exists. Then the limit $\|\cdot\|\text{-}\lim_{t\downarrow0} A\,{}^1\!/\!{}_t\big(T_1(t)x - x\big)$ exists in $X$, and we conclude that $\|\cdot\|\text{-}\lim_{t\downarrow0} {}^1\!/\!{}_t\big(T_1(t)x - x\big) = Ax$. Since $A$ is closed, this implies $Ax \in D(A)$ and therefore $x \in D(A^2) = D(A_1)$.

(b) $\Rightarrow$ (a). Let $\big(T_1(t)\big)_{t\geq 0}$ be the strongly continuous semigroup on $X_1$ generated by $A_1$. As indicated above, we set

$$u(t,x) := T_1(t)x \qquad \text{for } x \in X_1$$

and verify that $u(\cdot,x)$ solves (ACP) uniquely. It is immediate that $u(\cdot,x)$ and $Au(\cdot,x)$ belong to $\mathrm{C}(\mathbb{R}_+,X)$. Moreover, one has

$$\begin{aligned} \int_0^t u(s,x)\,ds &= \int_0^t T_1(s)x\,ds \in D(A_1) = D(A^2) \\ \text{and} \qquad A\int_0^t u(s,x)\,ds &= u(t,x) - u(0,x) = u(t,x) - x \end{aligned}$$

(see Lemma 1.3.(iv)). This gives

$$u(t,x) = x + A\int_0^t u(s,x)\,ds = x + \int_0^t Au(s,x)\,ds$$

showing that $u(\cdot,x) \in \mathrm{C}^1(\mathbb{R}_+,X)$ and

$$\tfrac{d}{dt}u(t,x) = Au(t,x).$$

Having obtained a solution of (ACP), it remains to show its uniqueness.

Let $u(\cdot)$ be a solution of (ACP) with initial value $x = 0$ and define $f(t) := \int_0^t u(s)\,ds$. Then $f(t) \in D(A)$ and $Af(t) = \int_0^t Au(s)\,ds = \int_0^t {}^d\!/\!_{ds}u(s)\,ds = u(t) \in D(A)$. We conclude that $f(t) \in D(A^2)$ for all $t \geq 0$ and that

$$\tfrac{d}{dt}f(t) = u(t) = Af(t)$$

and

$$\tfrac{d}{dt}Af(t) = Au(t) = A\big(\tfrac{d}{dt}f(t)\big) = A^2 f(t).$$

This shows that $f \in \mathrm{C}^1(\mathbb{R}_+,X_1)$ and ${}^d\!/\!_{dt}f(t) = A_1 f(t)$. Since $f(0) = 0$, it follows that $f \equiv 0$ and therefore $u \equiv 0$. $\square$

The statement above can be verified explicitly for Example 6.5, where $\mathcal{A}$ becomes a bounded operator if restricted to $\mathcal{X}_1 := X \times D(B)$, hence generates a semigroup on $\mathcal{X}_1$.

In the following theorem we show which properties of the solutions $u(\cdot,x)$ or of the operator $\big(A,D(A)\big)$ have to be added in order to characterize semigroup generators.

**6.7 Theorem.** *Let $A : D(A) \subset X \to X$ be a closed operator. For the associated abstract Cauchy problem*

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = x, \end{cases}$$

*we consider the following existence and uniqueness condition:*

$$\text{(EU)} \qquad \begin{array}{l} \text{For every } x \in D(A), \text{ there exists} \\ \text{a unique solution } u(\cdot, x) \text{ of (ACP).} \end{array}$$

*Then the following properties are equivalent.*

(a) *$A$ generates a strongly continuous semigroup.*
(b) *$A$ satisfies (EU) and $\rho(A) \neq \emptyset$.*
(c) *$A$ satisfies (EU), and there exist a sequence $\lambda_n \uparrow \infty$ such that the ranges $(\lambda_n - A)D(A)$ equal $X$ for all $n \in \mathbb{N}$.*
(d) *$A$ satisfies (EU), has dense domain, and for every sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ satisfying $\lim_{n\to\infty} x_n = 0$, one has $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly in compact intervals $[0, t_0]$.*

Proof. From the basic properties of semigroup generators and, in particular, Proposition 6.2, it follows that (a) implies (b), (c), and (d). So, for the proof of the remaining implications, we assume that (EU) holds and consider the operator $A_1$ on $X_1$ as defined in Proposition 6.6.

(b) ⇒ (a). If there exists $\lambda \in \rho(A)$, we have

$$D(A) = \{x \in X : (\lambda - A)^{-1}x \in D(A_1)\}$$

and

$$Ax = (\lambda - A)A_1(\lambda - A)^{-1}x \qquad \text{for all} \qquad x \in D(A),$$

i.e., $\bigl(A, D(A)\bigr)$ and $\bigl(A_1, D(A_1)\bigr)$ are similar (see Paragraph 2.1). Since $\bigl(A_1, D(A_1)\bigr)$ is a semigroup generator by Proposition 6.6, the same holds for $\bigl(A, D(A)\bigr)$.

(c) ⇒ (b). By assumption, we can find $\lambda > \mathrm{s}(A_1)$, the spectral bound of $A_1$, such that $(\lambda - A)D(A) = X$. Assume $Ax = \lambda x$ for some $x \in D(A)$. Then $x$ even belongs to $D(A^2) = D(A_1)$, and therefore $x = 0$, since $\lambda \in \rho(A_1)$. This shows that $\lambda - A$ is injective; hence $\lambda \in \rho(A)$.

(d) ⇒ (a). The assumption implies the existence of bounded operators $T(t) \in \mathcal{L}(X)$ satisfying

$$T(t)x := u(t, x)$$

for each $x \in D(A)$. Moreover, we claim that $\sup_{0\leq t\leq 1} \|T(t)\| < \infty$. By contradiction, assume that there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset [0, 1]$ such that $\|T(t_n)\| \to \infty$ as $n \to \infty$. Then we can choose $x_n \in D(A)$ such that $\lim_{n\to\infty} x_n = 0$ and $\|T(t_n)x_n\| \geq 1$. Since $u(t_n, x_n) = T(t_n)x_n$, this contradicts the assumption in (d), and therefore $\|T(t)\|$ is uniformly bounded for $t \in [0, 1]$. Now, $t \mapsto T(t)x$ is continuous for each $x$ in the dense domain $D(A)$, and we obtain continuity for each $x \in X$ by Lemma I.5.2.

Finally, the uniqueness of the solutions implies $T(t+s) = T(t)T(s)x$ for each $x \in D(A)$ and all $t,s \geq 0$. Thus $\big(T(t)\big)_{t\geq 0}$ is a strongly continuous semigroup on $X$. Its generator $\big(B, D(B)\big)$ certainly satisfies $A \subset B$. Moreover, the semigroup $\big(T(t)\big)_{t\geq 0}$ leaves $D(A)$ invariant, which, by Proposition 1.7, is a core of $B$. Since $A$ is closed, we obtain $A = B$. $\square$

Observe that (b) and (c) imply that $D(A)$ is dense, while this property cannot be omitted in (d): Take the restriction $\widetilde{A}$ of a closed operator $A$ to the domain $D(\widetilde{A}) := \{0\}$.

Intuitively, property (d) expresses what we expect for a "well-posed" problem and its solutions:

*existence + uniqueness + continuous dependence on the data.*

Therefore we introduce a name for this property.

**6.8 Definition.** *The abstract Cauchy problem*

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = x \end{cases}$$

*associated to a closed operator $A : D(A) \subset X \to X$ is called well-posed if condition* (d) *in Theorem* 6.7 *holds.*

With this terminology, we can rephrase Theorem 6.7.

**6.9 Corollary.** *For a closed operator $A : D(A) \subset X \to X$, the associated abstract Cauchy problem* (ACP) *is well-posed if and only if $A$ generates a strongly continuous semigroup on $X$.*

Once we agree on the well-posedness concept from Definition 6.8, strongly continuous semigroups emerge as the perfect tool for the study of abstract Cauchy problems (ACP). In addition, this explains why in this book we

- study semigroups systematically in Chapters I–V and only then
- solve Cauchy problems in Chapter VI.

However, we have to point out that our definition of "well-posedness" is not the only possible one. In particular, in many situations arising from physically perfectly "well-posed" problems one does not obtain a semigroup on a given Banach space. We refer to [Are87a], [deL94] and [Neu88] for weaker concepts of "well-posedness" and show here how to produce, for the same operator by simply varying the underlying Banach space, a series of different "well-posedness" properties.

**6.10 Example.** Consider the left translation group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $\mathrm{L}^1(\mathbb{R})$ with generator $Af := f'$ and $D(A) := \mathrm{W}^{1,1}(\mathbb{R})$. Decompose this space as

$$\mathrm{L}^1(\mathbb{R}) = \mathrm{L}^1(\mathbb{R}_-) \oplus \mathrm{L}^1(\mathbb{R}_+),$$

and take any translation-invariant Banach space $Y$ continuously embedded in $\mathrm{L}^1(\mathbb{R}_-)$. Then the part $A_|$ of $A$ in $X := Y \oplus \mathrm{L}^1(\mathbb{R}_+)$ has domain $D(A_|) := \{f \in \mathrm{W}^{1,1}(\mathbb{R}) : f'_{|\mathbb{R}_-} \in Y\}$. The abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= A_| u(t) \quad \text{for } t \geq 0, \\ u(0) &= f \in D(A_|) \subset X \end{aligned}$$

formally has the solution $t \mapsto u(t) := T(t)f$ with $\big(T(t)\big)f(s) = f(s+t)$, $s \in \mathbb{R}$. This is a classical solution if and only if $u(t) \in D(A_|)$ for all $t \geq 0$. As concrete examples, we suggest to take $Y := \mathrm{W}^{n,1}(\mathbb{R}_-)$, or even $Y := \{0\}$, and leave the details as Exercise 6.11.(1).

After this mathematical discussion of "well-posed" Cauchy problems we add some historical and philosophical comments anticipating some of the arguments from Chapter VI and the Epilogue.

As emphasized before, we usually start from a strongly continuous semigroup $(T(t))_{t\geq 0}$ and then obtain, by Corollary 6.9, a well-posed Cauchy problem for its generator $A$. By taking $A$ to be a partial differential operator as in Paragraph 2.13 or Section VI.5 we can thus solve initial value problems for partial differential equations. However, the historical process followed the inverse order. Partial differential equations were solved long before the notion of a semigroup emerged. While obtaining these solutions, the great masters of the eighteenth and nineteenth centuries struggled to find basic principles behind the formulas. It was J. Hadamard who, quoting earlier ideas of E. Picard [Pic95] and V. Volterra [Vol13], first isolated certain principles governing the solutions of the wave equation and more general time-dependent partial differential equations. In his famous paper "Le principe de Huygens" [Had24] he states three principles with Proposition A being the most fundamental ("a kind of truism"):

> *A. Pour déduire d'un phénomène connu à l'instant $t_0$ l'effet produit à un instant ultérieur $t_2$, on peut commencer par calculer l'effet à un instant intermédiaire $t_1$, puis partir de celui-là pour en déduire l'effet en $t_2$. (See* [Had24, p. 613]*.)*[4]

In his discussion of Proposition A he comes close to the definition of a (semi) group.

> *La proposition A, elle ... doit être considérée comme d'évidence immédiate. Elle n'est pas distincte du principe même de notre déterminisme scientifique. Ce principe exprime en effet que, connaissant l'état du monde à un instant déterminé $t_0$, on doit pouvoir en déduire l'état du monde à un instant ultérieur quelconque $t_0 + h$, où $h$ est n'importe quel temps positif.*
>
> *On peut donc aussi, connaissant l'état relatif à $t_0$, en déduire celui qui est relatif à l'instant $t_0 + h + k$; mais ce même état dont nous venons de parler doit aussi pouvoir ($k$ étant positif) se calculer à l'aide de l'état à l'instant $t_0 + h$, lequel a été lui-même supposé calculable à partir de l'état en $t_0$. Les deux modes de calcul doivent conduire au même résultat (...).*
>
> *A est donc une sorte de truisme (...).*

[4] In order to deduce from a phenomenon known at time $t_0$ the effect produced at a later time $t_2$, one can start by computing the effect at an intermediate time $t_1$ and then go on from there to deduce the effect at $t_2$.

> *Il se rattache d'une manière tout à fait étroite à la notion de groupe. Il est clair en effet que le calcul par lequel de l'état en $t_0$, on passe à l'état en $t_0 + h$, constitue une transformation, laquelle dépend du paramètre $h$. La proposition A exprime que l'ensemble de toutes ces transformations, lorsque $h$ prend toutes les valeurs positives possibles (et l'on pourrait même y joindre les valeurs négatives, tout au moins si l'on admettait qu'il y a réversibilité), constitue un groupe; la transformation du paramètre $h+k$ coïncide avec le produit des deux transformations de paramètres respectives $h$ et $k$. (See* [Had24, p. 622].*)* [5]

His concluding remark (see [Had24, p. 640])

> *Alors ... la majeure A est un de ces "principes directeurs de la connaissance" suivant la terminologie des philosophes—en dehors desquels nous ne saurions penser et raisonner* [6]

is another argument for our approach leading from the functional equation (FE) and semigroups to well-posed Cauchy problems.

**6.11 Exercises.** (1) On $X := \mathrm{W}^{1,1}(\mathbb{R}_-) \oplus \mathrm{L}^1(\mathbb{R}_+)$, consider the operator $Af := f'$ with $D(A) := \{(f,g) \in \mathrm{W}^{2,1}(\mathbb{R}_-) \oplus \mathrm{W}^{1,1}(\mathbb{R}_+) : f(0) = g(0)\}$.

- (i) Which conditions of Generation Theorem 3.8 are fulfilled by the operator $(A, D(A))$? (Hint: Use (2.3) in Section 2.b to represent $R(\lambda, A)$.)
- (ii) Show that the abstract Cauchy problem associated to $(A, D(A))$ has a classical solution only for initial values $(f,g) \in D(A)$ such that $g \in \mathrm{W}^{2,1}(\mathbb{R}_+)$.
- (iii) Replace $\mathrm{W}^{2,1}(\mathbb{R}_-)$ by other translation-invariant Banach function spaces on $\mathbb{R}_-$ and find the initial values for which classical solutions exist.

(2) Consider the extrapolated Sobolev space $X_{-1}$ associated to a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on a Banach space $X_0$ and assume that $Y$ is a Banach space satisfying $X_0 \hookrightarrow Y \hookrightarrow X_{-1}$.

- (i) Define operators $S(t) \in \mathcal{L}(Y)$ by $S(t)y := \int_0^t T_{-1}(s)y\,ds$ for $y \in Y$ and $t \geq 0$ and show that $(S(t))_{t\geq 0}$ is an exponentially bounded *integrated semigroup* on $Y$ (see [KH89] and [ANS92]) whose generator is the part of $A_{-1}$ in $Y$.
- (ii*) Show that every exponentially bounded integrated semigroup can be obtained in this way. (Hint: See [ANS92].)
- (iii) Discuss the consequences for the associated abstract Cauchy problem.

---

[5] The proposition A must be considered to be of immediate evidence. It is not distinct from our principle of scientific determinism. In fact, this principle expresses that, knowing the state of the world at a certain time $t_0$, one must be able to deduce the state of the world at an arbitrary later time $t_0 + h$, where $h$ is any positive time.
Therefore, it is possible, knowing the state relatively to $t_0$, to deduce the state relatively to $t_0 + h + k$; however, this very state must be obtained from the state at time $t_0 + h$, which was supposed to be computable from the state at $t_0$. *The two ways of calculating must lead to the same result* (...).
Therefore, A is a kind of truism (...).
It is connected quite closely to the notion of a group. In fact, it is clear that the law by which one passes from the state at $t_0$ to the state at $t_0 + h$ constitutes a transformation depending on the parameter $h$. The proposition A expresses that the set of all these transformations, if $h$ takes all possible positive values (and one could even take negative values permitting reversibility), forms a *group*; the transformation of the parameter $h+k$ coincides with the product of the two transformations of the parameters $h$ and $k$, respectively.

[6] The major proposition A is one of the "directing principles of our knowledge" outside of which we cannot think nor reason

## Notes to Chapter II

**Section** 1**.** In order to develop a theory of one-parameter semigroups on infinite-dimensional Banach spaces, it was crucial to find continuity assumptions allowing the definition of a generator. Strong continuity seems to be most appropriate, even if, e.g. on dual Banach spaces, one can consider *weak*-continuous semigroups* and define *weak* generators* (see [Nee92]). In Yosida's classic [Yos65] semigroups and generators are studied on locally convex vector spaces (see also [Kom64], [Kōm68]).

The integral representation (1.13) of the resolvent of a generator is nothing else than the (Banach-space-valued) Laplace transform of the semigroup. This was already noted and extensively used by Hille–Phillips [HP57]. In fact, many of the theorems on semigroups can be viewed as results on the Laplace transform and their inverse. This became particularly clear when Arendt [Are87b] extended Widder's theorem to Banach spaces and characterized Laplace transforms of bounded vector-valued functions. Based on this result, authors like Arendt [Are87b], Neubrander [Neu88], and deLaubenfels [deL94] studied strongly continuous semigroups as a special case of the theory of the vector-valued Laplace transform. This alternative approach is the leitmotif in the monograph [ABHN99].

**Section** 2.a**.** Most of our standard constructions are common tools of modern operator theory. The adjoint and sun dual semigroups have been studied and used extensively by the Dutch school (see [CHA$^+$87]), and a complete treatment can be found in van Neerven's lecture notes [Nee92]. As additional and less elementary constructions, we mention *tensor product semigroups* ([Nag86, A-I, 3.7 and A-III, Cor. 6.8], [RS78, Sec. XIII.9]) and *ultrapower semigroups* as a generalization of Exercise I.5.16.(3) and Exercise 2.8.(3) (see [Nag86, A-I, 3.6]).

**Section** 2.b**.** In our text we emphasized multiplication and translation semigroups. However, diffusion semigroups generated by the (one- or $n$-dimensional) Laplace operator or more general elliptic operators provide perhaps the most important classes of examples; see the references in Section VI.5. Already Yosida [Yos49] realized that these semigroups describe Markov processes, and Dynkin [Dyn65] and Feller [Fel71] studied this aspect in their classic monographs. More recent presentations, using Dirichlet forms and $L^2$-theory, are the books by Bouleau–Hirsch [BH91], Fukushima–Oshima–Takeda [FOT94], and by Ma–Röckner [MR92]. Closer to our semigroup theory on Banach spaces are [Jac96], [Jac99], and [Tai88], where pseudodifferential operators are shown to generate semigroups describing Markov processes.

**Section** 3.a**.** Our proof of Generation Theorem 3.5 goes back to Yosida [Yos48]. Hille's proof in [Hil48, Thm. 12.2.1] is based on the exponential formula (3.3). For more historical information we refer to Chapter VII.

**Section** 3.b**.** Phillips [Phi59] introduced dissipative operators on Banach spaces, while Theorem 3.15 is from [LP61]. The same idea applied in the context of ordered Banach spaces (or Banach lattices) leads to *dispersive* operators as generators of positive contraction semigroups (see [Phi62], [ACK82], or [Nag86, C-II]). The extension of many results from generators to Hille–Yosida operators started with [DPS85] and [DPS87].

**Section** 3.c**.** A complete characterization of the duality set $J(f)$ in Example 3.26.(i) can be found, e.g., in [Sin76].

For more information on the delay differential operator in Paragraph 3.29 see Section VI.6, while the second-order differential operators from Paragraph 3.30 will reappear in Section VI.4.

**Section** 4.a. For many reasons, analytic semigroups represent the most important subclass of strongly continuous semigroups. They arise from parabolic partial differential equations, and books like [Lun95] and [Tai95] are entirely devoted to them. The results we present are standard with the exception of condition (b) in Theorem 4.6, where strong continuity in a sector implies analyticity (see also [Eng92, Lem. 8] and [Kan96]).

**Section** 4.b. Differentiable semigroups go back to Hille [Hil50] and Yosida [Yos48], while their generators have been characterized in [Paz68]. Exercise 4.16 is taken from [Paz83, Sec. 2.4] where also more results on differentiable semigroups can be found.

**Section** 4.c. Eventually norm-continuous semigroups include the previous classes, but are not yet characterized in a satisfactory way. While the necessary condition in Theorem 4.18 was known to Hille and Phillips [HP57, Thm. 16.4.2], the result in Paragraph 4.20 for Hilbert spaces is due to You [You92]. Our proof, however, is taken from [EME94]. Very recently it was shown by Goersmeyer–Weis [GW99] that for positive semigroups on $\mathrm{L}^p$-spaces the conditions (a) and (b) in the theorem in Paragraph 4.20 are equivalent as well. See also [EME96] and [BM96] for extensions to general Banach spaces.

**Section** 4.d. Compactness of the resolvent and/or the semigroup operators appears, e.g., for parabolic partial differential equations in bounded spatial domains (see [RR93, §8.3]) or for delay equations (see, e.g., Section VI.6). The results given in this subsection are standard, and the diagram in (4.26), indicating the implications between the various regularity properties, is taken from [Nag86, A.II].

**Section** 4.e. The examples in Paragraphs 4.32 and 4.33 should convince the reader of the value of such simple objects as multiplication operators. Clearly, the same examples can be produced, via the Spectral Theorem I.4.9, for normal operators on Hilbert spaces.

**Section** 5.a. The definition of the extrapolation spaces in Definition 5.4 is due to Nagel [Nag83]. A different definition was given by Da Prato–Grisvard in [DPG82] and [DPG84]. They are equivalent for operators with dense domain (for details see [Nee92]). Other references for extrapolation spaces are [Pal65], [Har86], [Ama87], [Liu89], [NS93], [Ama95, Chap. V]. A survey with applications can be found in [Sin96]. For further examples see [NNR96].

**Section** 5.b. The standard reference for Favard and abstract Hölder spaces, and many more intermediate spaces, is the monograph by Butzer–Berens [BB67]. The Favard spaces $F_\alpha$ for $0 < \alpha < 1$ introduced in this section are the real interpolation spaces $(X, D(A))_{\alpha,\infty}$ according to [Tri78], while for $\alpha = 1$ they are the Favard class $X_{1,1,\infty}$ from [BB67]. Our abstract Hölder space $X_\alpha$ is $X_{\alpha,1,\infty}$ in [BB67] and $D_A(\alpha)$ in [DPG79]. In this paper it is shown that these spaces, called there *continuous interpolation spaces*, are necessary to obtain optimal regularity for the abstract parabolic equations as in Corollary VI.7.17. For the use of these interpolation spaces in the study of evolution equations see [Lun95] and [Ama95].

**Section** 5.c. Fractional powers were introduced by Balakrishnan [Bal60]. They have been systematically studied by many authors including Kato [Kat61]. An extensive treatment can be found in a series of papers by Komatsu (e.g., [Kom66]). For recent developments see Jacob [Jac99], Schilling [Sch98], and Straub [Str94].

**Section 6.** It was already known to Hille and Phillips [HP57] and Kreĭn [Kre71] that the various well-posedness conditions for (ACP) in Theorem 6.7 characterize semigroup generators (see Fattorini [Fat83, Chap. 2] or Neubrander [Neu84]). However, that existence and uniqueness of the solutions already yield a semigroup on $D(A)$ was observed by Arendt [Nag86, A-II, Thm. 1.1]. The mild solution as defined in Definition 6.3 is called *integral solution* by Da Prato–Sinestrari [DPS87] when $A$ is a Hille–Yosida operator.

For many other well-posedness concepts related to integrated semigroups, regularized semigroups, or other structures, we refer to [Are87a], [deL94], [FY99], [Lum94], [LN99], [Lun95], [MA97], [Neu88], [Neu89], and [XL98, Chap. 1].

# Chapter III

# Perturbation and Approximation of Semigroups

The verification of the conditions in the various generation theorems from Chapter II is not an easy task and for many important operators cannot be performed in a direct way. Therefore, one tries to build up the given operator (and its semigroup) from simpler ones. Perturbation and approximation are the standard methods for this approach and will be discussed in the following sections.

## 1. Bounded Perturbations

In many concrete situations, the evolution equation (or the associated linear operator) is given as a (formal) sum of several terms having different physical meaning and different mathematical properties. While the mathematical analysis may be easy for each single term, it is not at all clear what happens after the formation of sums. In the context of generators of semigroups we take this as our point of departure.

**1.1 Problem.** *Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$. Find conditions such that the sum $A + B$ generates a strongly continuous semigroup $\big(S(t)\big)_{t\geq 0}$.*

We say that the generator $A$ is *perturbed* by the operator $B$ or that $B$ is a *perturbation* of $A$. However, before answering the above problem, we

have to realize that—at this stage—the sum $A+B$ is defined as

$$(A+B)x := Ax + Bx$$

only for

$$x \in D(A+B) := D(A) \cap D(B),$$

a subspace that might be trivial in general. To emphasize this and other difficulties caused by the addition of unbounded operators, we first discuss some examples.

**1.2 Examples.** (i) Let $\bigl(A, D(A)\bigr)$ be an unbounded generator of a strongly continuous semigroup. If we take $B := -A$, then the sum $A+B$ is the zero operator, defined on the dense subspace $D(A)$, hence not closed.

If we take $B := -2A$, then the sum is

$$A + B = -A \qquad \text{with domain} \qquad D(A+B) = D(A),$$

which is a generator only if $A$ generates a strongly continuous group (see Paragraph II.3.11).

(ii) Let $A : D(A) \subseteq X \to X$ be an unbounded generator of a strongly continuous semigroup and take an isomorphism $S \in \mathcal{L}(X)$ such that $D(A) \cap S\bigl(D(A)\bigr) = \{0\}$. Then $B := SAS^{-1}$ is a generator as well (see Paragraph II.2.1), but $A+B$ is defined only on $D(A+B) = D(A) \cap D(B) = D(A) \cap S\bigl(D(A)\bigr) = \{0\}$.

A concrete example for this situation is given on $X := \mathrm{C}_0(\mathbb{R}_+)$ by

$$Af := f' \qquad \text{with its canonical domain } D(A) := \mathrm{C}_0^1(\mathbb{R}_+)$$

and

$$Sf := q \cdot f$$

for some continuous, positive function $q$ such that $q$ and $q^{-1}$ are bounded and nowhere differentiable. Defining the operator $B$ as

$$Bf := q \cdot (q^{-1} \cdot f)' \qquad \text{on} \qquad D(B) := \bigl\{ f \in X : q^{-1} \cdot f \in D(A) \bigr\},$$

we obtain that the sum $A+B$ is defined only on $\{0\}$.

The above examples show that the addition of unbounded operators is a delicate operation and should be studied carefully. We start with a situation in which we avoid the pitfall due to the differing domains of the operators involved. More precisely, we assume one of the two operators to be bounded.

**1.3 Bounded Perturbation Theorem.** *Let $\bigl(A, D(A)\bigr)$ be the generator of a strongly continuous semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ on a Banach space $X$ satisfying*

$$\|T(t)\| \leq M\mathrm{e}^{wt} \text{ for all } t \geq 0$$

*and some $w \in \mathbb{R}, M \geq 1$. If $B \in \mathcal{L}(X)$, then*

$$C := A + B \qquad \text{with} \qquad D(C) := D(A)$$

*generates a strongly continuous semigroup $\bigl(S(t)\bigr)_{t\geq 0}$ satisfying*

$$\|S(t)\| \leq M\mathrm{e}^{(w+M\|B\|)t} \qquad \text{for all } t \geq 0.$$

Proof. In the first and essential step, we assume $w = 0$ and $M = 1$. Then $\lambda \in \rho(A)$ for all $\lambda > 0$, and $\lambda - C$ can be decomposed as

$$\lambda - C = \lambda - A - B = \big(I - BR(\lambda, A)\big)(\lambda - A). \tag{1.1}$$

Since $\lambda - A$ is bijective, we conclude that $\lambda - C$ is bijective, i.e., $\lambda \in \rho(C)$, if and only if

$$I - BR(\lambda, A)$$

is invertible in $\mathcal{L}(X)$. If this is the case, we obtain

$$R(\lambda, C) = R(\lambda, A)[I - BR(\lambda, A)]^{-1}. \tag{1.2}$$

Now choose $\operatorname{Re}\lambda > \|B\|$. Then $\|BR(\lambda, A)\| \le {\|B\|}/{\operatorname{Re}\lambda} < 1$ by Generation Theorem II.3.5.(c), and hence $\lambda \in \rho(C)$ with

$$R(\lambda, C) = R(\lambda, A) \sum_{n=0}^{\infty} (BR(\lambda, A))^n. \tag{1.3}$$

We now estimate

$$\|R(\lambda, C)\| \le \frac{1}{\operatorname{Re}\lambda} \cdot \frac{1}{1 - {\|B\|}/{\operatorname{Re}\lambda}} = \frac{1}{\operatorname{Re}\lambda - \|B\|}$$

for all $\operatorname{Re}\lambda > \|B\|$ and obtain from Corollary II.3.6 that $C$ generates a strongly continuous semigroup $\big(S(t)\big)_{t\ge 0}$ satisfying

$$\|S(t)\| \le \mathrm{e}^{\|B\| t} \text{ for } t \ge 0.$$

For general $w \in \mathbb{R}$ and $M \ge 1$, we first do a rescaling (see Paragraph II.2.2) to obtain $w = 0$. As in Lemma II.3.10, we then introduce a new norm

$$|||x||| := \sup_{t\ge 0} \|T(t)x\|$$

on $X$. This norm satisfies

$$\|x\| \le |||x||| \le M\,\|x\|,$$

makes $\big(T(t)\big)_{t\ge 0}$ a contraction semigroup, and yields

$$|||Bx||| \le M\,\|B\| \cdot \|x\| \le M\,\|B\| \cdot |||x|||$$

for all $x \in X$. By part one of this proof, the sum $C = A + B$ generates a strongly continuous semigroup $\big(S(t)\big)_{t\ge 0}$ satisfying the estimate

$$|||S(t)||| \le \mathrm{e}^{|||B||| t} \le \mathrm{e}^{M\|B\| t}.$$

Hence

$$\|S(t)x\| \le |||S(t)x||| \le \mathrm{e}^{M\|B\|t}\,|||x||| \le M\mathrm{e}^{M\|B\|t}\,\|x\|$$

for all $t \ge 0$, which is the assertion for $w = 0$. □

The identities (1.1) and (1.3) are not only the basis of this proof, but are also the key to many more perturbation results. Here, we use them to make the following observation using the terminology of Sobolev towers from Section II.5.a.

For sufficiently large $\lambda$, the generator $A$ of a strongly continuous semigroup, and an operator $B \in \mathcal{L}(X)$, the operators

$$\big(I - BR(\lambda, A)\big) \qquad \text{and} \qquad \big(I - BR(\lambda, A)\big)^{-1} = \sum_{n=0}^{\infty} \big(BR(\lambda, A)\big)^n$$

are isomorphisms of the Banach space $X$. Therefore, for large $\lambda$, the 1-norms with respect to $\lambda - A$ and $\lambda - A - B$, i.e.,

$$\|x\|_1^A := \|(\lambda - A)x\|$$

and

$$\|x\|_1^{A+B} := \|(\lambda - A - B)x\| = \Big\|\big(I - BR(\lambda, A)\big)(\lambda - A)x\Big\|$$

are equivalent on $X_1 := D(A) = D(A+B)$.

Similarly, the corresponding $(-1)$-norms

$$\|x\|_{-1}^A := \|R(\lambda, A)x\|$$

and

$$\|x\|_{-1}^{A+B} := \|R(\lambda, A+B)x\|$$

are equivalent on $X$ (use the identity

$$R(\lambda, A) = [I + R(\lambda, A+B)B]^{-1} R(\lambda, A+B) \tag{1.4}$$

and (1.2)), and hence the Sobolev spaces $X_{-1}^A$ for $A$ and $X_{-1}^{A+B}$ for $A+B$ from Definition II.5.4 coincide.

Since we know from Theorem 1.3 that $A+B$ is a generator, we obtain the following conclusion.

**1.4 Corollary.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach space $X_0$ and take $B \in \mathcal{L}(X_0)$. Then the operator*

$$A + B \qquad \textit{with domain } D(A+B) := D(A)$$

*is a generator, and the Sobolev spaces*

$$X_i^A \qquad \text{and} \qquad X_i^{A+B}$$

*corresponding to $A$ and $A+B$, resp., coincide for $i = -1, 0, 1$.*

We show in Exercise 1.17.(6) that this result is optimal in the sense that in general, only these three "floors" of the corresponding Sobolev towers coincide. Here, the above corollary immediately yields a first perturbation result for operators that are not bounded on the given Banach space.

**1.5 Corollary.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on the Banach space $X_0$. If $B$ is a bounded operator on $X_1^A := (D(A), \|\cdot\|_1)$, then $A+B$ with domain $D(A+B) = D(A)$ generates a strongly continuous semigroup on $X_0$.*

Proof. Consider the restriction $A_1$ of $A$ as a generator on $X_1^A$. Then $A_1 + B$ generates a strongly continuous semigroup on $X_1^A$ by Theorem 1.3. This perturbed semigroup can be extended to its extrapolation space $(X_1^A)_{-1}^{A_1+B}$, which by Corollary 1.4 coincides with the extrapolation space $(X_1^A)_{-1}^{A_1}$. However, this is the original Banach space $X_0$. The generator of the extended semigroup on $X_0$ is the continuous extension of $A_1 + B$, hence is $A + B$. □

**1.6 Example.** Take $Af := f'$ on $X := \mathrm{C}_0(\mathbb{R})$ with domain $\mathrm{C}_0^1(\mathbb{R})$. For some $h \in \mathrm{C}_0^1(\mathbb{R})$ define the operator $B$ by

$$Bf := f'(0) \cdot h, \quad f \in \mathrm{C}_0^1(\mathbb{R}).$$

Then $B$ is unbounded on $X$ but bounded on $D(A) = \mathrm{C}_0^1(\mathbb{R})$, and hence $A + B$ is a generator on $X$.

A more interesting application of this corollary will be made to operators arising from second-order Cauchy problems or from integro-differential equations; see Corollary VI.3.4 and Proposition VI.7.21.

Returning to Theorem 1.3, we recall that we have the series representation (1.3) for the resolvent $R(\lambda, A + B)$ of the perturbed operator $A + B$, while for the new semigroup $\big(S(t)\big)_{t\geq 0}$ we could prove only its existence. In order to prepare for a representation formula for this new semigroup, we show first that it satisfies an integral equation.

**1.7 Corollary.** *Consider two strongly continuous semigroups $\big(T(t)\big)_{t\geq 0}$ with generator $A$ and $\big(S(t)\big)_{t\geq 0}$ with generator $C$ on the Banach space $X$ and assume that*

$$C = A + B$$

*for some bounded operator $B \in \mathcal{L}(X)$. Then*

$$\text{(IE)} \qquad S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\, ds$$

*holds for every $t \geq 0$ and $x \in X$.*

Proof. Take $x \in D(A)$ and consider the functions

$$[0, t] \ni s \mapsto \xi_x(s) := T(t-s)S(s)x \in X.$$

Since $D(A) = D(C)$ is invariant under both semigroups, it follows that $\xi_x(\cdot)$ is continuously differentiable (use Lemma B.16) with derivative

$$\tfrac{d}{ds}\xi_x(s) = T(t-s)CS(s)x - T(t-s)AS(s)x = T(t-s)BS(s)x.$$

This implies

$$S(t)x - T(t)x = \xi_x(t) - \xi_x(0) = \int_0^t \xi_x'(s)\, ds = \int_0^t T(t-s)BS(s)x\, ds.$$

Finally, the density of $D(A)$ and the boundedness of the operators involved yield that this integral equation holds for all $x \in X$. □

If we replace the above functions $\xi_x$ by

$$\eta_x(s) := S(s)T(t-s)x$$

and use the same arguments, we obtain the analogous integral equation

$$\text{(IE}^*\text{)} \qquad S(t)x = T(t)x + \int_0^t S(s)BT(t-s)x\, ds$$

for $x \in X$ and $t \geq 0$.

Both equations (IE) and (IE*) will frequently be called the *variation of parameters formula* for the perturbed semigroup (see also Section 3.a and Section 3.c).

In order to solve the integral equation (IE) we rewrite it in operator form and introduce the operator-valued function space

$$\mathfrak{X}_{t_0} := \mathrm{C}\big([0,t_0],\mathcal{L}_s(X)\big)$$

of all continuous functions from $[0,t_0]$ into $\mathcal{L}_s(X)$, i.e., $F \in \mathfrak{X}_{t_0}$ if and only if $F(t) \in \mathcal{L}(X)$ and $t \mapsto F(t)x$ is continuous for each $x \in X$. This space becomes a Banach space for the norm

$$\|F\|_\infty := \sup_{s\in[0,t_0]} \|F(s)\|, \qquad F \in \mathfrak{X}_{t_0}$$

(see Proposition A.7). We now define a "Volterra-type" operator on it.

**1.8 Definition.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on $X$ and take $B \in \mathcal{L}(X)$. For any $t_0 > 0$, we call the operator defined by*

$$VF(t)x := \int_0^t T(t-s)BF(s)x\,ds$$

*for $x \in X$, $F \in \mathrm{C}\big([0,t_0],\mathcal{L}_s(X)\big)$ and $0 \leq t \leq t_0$ the associated abstract Volterra operator.*

The following properties of $V$ should be no surprise to anyone familiar with Volterra operators in the scalar-valued situation. In fact, the proof is just a repetition of the estimates there and will be omitted (see Exercise 1.17.(2)).

**1.9 Lemma.** *The abstract Volterra operator $V$ associated to the strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and the bounded operator $B \in \mathcal{L}(X)$ is a bounded operator in $\mathrm{C}\big([0,t_0],\mathcal{L}_s(X)\big)$ and satisfies*

$$\|V^n\| \leq \frac{\big(M\,\|B\|\,t_0\big)^n}{n!} \tag{1.5}$$

*for all $n \in \mathbb{N}$ and with $M := \sup_{s\in[0,t_0]} \|T(s)\|$. In particular, for its spectral radius we have*

$$\mathrm{r}(V) = 0. \tag{1.6}$$

From this last assertion it follows that the resolvent of $V$ at $\lambda = 1$ exists and is given by the Neumann series, i.e.,

$$R(1,V) = (I-V)^{-1} = \sum_{n=0}^{\infty} V^n.$$

We now turn back to our integral equation (IE), which becomes, in terms of our Volterra operator, the equation

$$T(\cdot) = (I - V)S(\cdot)$$

for the functions $T(\cdot)$, $S(\cdot) \in \mathrm{C}\big([0, t_0], \mathcal{L}_s(X)\big)$. Therefore,

$$S(\cdot) = R(1, V)T(\cdot) = \sum_{n=0}^{\infty} V^n T(\cdot), \tag{1.7}$$

where the series converges in the Banach space $\mathrm{C}\big([0, t_0], \mathcal{L}_s(X)\big)$. Rewriting (1.7) for each $t \geq 0$, we obtain the following representation for the semigroup $\big(S(t)\big)_{t\geq 0}$. This *Dyson–Phillips series* was found by F.J. Dyson in his work [Dys49] on quantum electrodynamics and then by R.S. Phillips in his first systematic treatment [Phi53] of perturbation theory for semigroups.

**1.10 Theorem.** *The strongly continuous semigroup $\big(S(t)\big)_{t\geq 0}$ generated by $C := A + B$, where $A$ is the generator of $\big(T(t)\big)_{t\geq 0}$ and $B \in \mathcal{L}(X)$, can be obtained as*

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{1.8}$$

*where $S_0(t) := T(t)$ and*

$$S_{n+1}(t) := V S_n(t) = \int_0^t T(t-s) B S_n(s)\, ds. \tag{1.9}$$

Here, the series (1.8) converges in the operator norm on $\mathcal{L}(X)$ and, since we may choose $t_0$ in Lemma 1.9 arbitrarily large, uniformly on compact intervals of $\mathbb{R}_+$. In contrast, the operators $S_{n+1}(t)$ in (1.9) are defined by an integral defined in the strong operator topology.

The Dyson–Phillips series and the integral equation (IE) from Corollary 1.7 will be very useful when we want to compare qualitative properties of the two semigroups. Here is a first example of such a comparison.

**1.11 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ and $\big(S(t)\big)_{t\geq 0}$ be two strongly continuous semigroups, where the generator of $\big(S(t)\big)_{t\geq 0}$ is a bounded perturbation of the generator of $\big(T(t)\big)_{t\geq 0}$. Then*

$$\|T(t) - S(t)\| \leq t\, M \tag{1.10}$$

*for $t \in [0, 1]$ and some constant $M$.*

PROOF. From the integral equation (IE), we obtain

$$\begin{aligned}\|T(t)x - S(t)x\| &\le \int_0^t \|T(t-s)BS(s)x\|\,ds \\ &\le t \sup_{r\in[0,1]} \|T(r)\| \sup_{s\in[0,1]} \|S(s)\| \cdot \|B\| \cdot \|x\|\end{aligned}$$

for all $x \in X$ and $t \in [0,1]$. □

Later, in Section 3.b, we will see that an estimate like (1.10) for the difference of two semigroups implies a close relation between their generators.

In the final part of this section we discuss some regularity properties from Section II.4 that are preserved under bounded perturbation. As a first sample we state the following simple result.

**1.12 Proposition.** *Let $(T(t))_{t\ge0}$ be a strongly continuous semigroup with generator $A$ on the Banach space $X$ and take $B \in \mathcal{L}(X)$.*

(i) *If $(T(t))_{t\ge0}$ is analytic, then so is the semigroup $(S(t))_{t\ge0}$ generated by $A+B$.*

(ii) *If $A$ has compact resolvent, then so has $A+B$.*

PROOF. (i) This assertion follows from Theorem II.4.6.(b) and the Bounded Perturbation Theorem 1.3.

(ii) As seen in (1.2), the resolvent $R(\lambda, A+B)$ for large $\lambda$ is a product of the compact operator $R(\lambda, A)$ and the bounded operator $[I - BR(\lambda, A)]^{-1}$, hence is compact. □

In order to obtain deeper results on regularity properties of perturbed strongly continuous semigroups, we look again at the Volterra operator from Definition 1.8 and rewrite it as a "convolution operator."

For a Banach space $X$ we consider the vector space

$$\mathcal{X} := \mathrm{C}\big(\mathbb{R}_+, \mathcal{L}_s(X)\big)$$

and define the *convolution* of two functions $F$, $G \in \mathcal{X}$ by

$$(F * G)(t)x := \int_0^t F(t-s)G(s)x\,ds \tag{1.11}$$

for $x \in X$ and $t \ge 0$. By Exercise 1.17.(1) it follows that $* : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is associative. In the following lemma we show that certain regularity properties are preserved under convolution in $\mathcal{X}$.

**1.13 Lemma.** *For $F$, $G \in \mathcal{X}$ the following assertions are true.*

(i) *If $F$ is norm continuous (resp. compact[1]) on $(0,\infty)$, then $F*G$ and $G*F$ are norm continuous (resp. compact) on $(0,\infty)$.*

(ii) *If $F$ is norm continuous (resp. compact) on $(\alpha,\infty)$ and $G$ is norm continuous (resp. compact) on $(\beta,\infty)$, then $F*G$ and $G*F$ are norm continuous (resp. compact) on $(\alpha+\beta,\infty)$.*

[1] This means that $F(t)$ is a compact operator for $t > 0$.

Proof. (i) Let $t > 0$, $h > 0$ and $x \in X$. Then we have

$$\begin{aligned}
&\lim_{h\downarrow 0}\big\|F * G\,(t+h)x - F * G\,(t)x\big\| \\
&\qquad = \lim_{h\downarrow 0}\bigg\|\int_0^t \big(F(t+h-s) - F(t-s)\big)G(s)x\,ds \\
&\qquad\qquad + \int_t^{t+h} F(t+h-s)G(s)x\,ds\bigg\| \\
&\qquad \le \lim_{h\downarrow 0}\int_0^t \|F(t+h-s) - F(t-s)\| \sup_{\tau\in[0,t]}\|G(\tau)\|\cdot\|x\|\,ds \\
&\qquad\qquad + \lim_{h\downarrow 0}\int_t^{t+h} \sup_{\tau\in[0,t]}\|F(\tau)\| \sup_{\tau\in[0,t]}\|G(\tau)\|\cdot\|x\|\,ds = 0
\end{aligned}$$

uniformly for $\|x\| \le 1$. Hence, the map $t \mapsto F * G(t)$ is right continuous in the uniform operator topology for $t > 0$. Similarly, we can show that it is also left continuous. For the assertion concerning $G * F$ we use the identity

$$\int_0^t G(t-s)F(s)x\,ds = \int_0^t G(s)F(t-s)x\,ds. \tag{1.12}$$

This proves the norm continuity.

The assertions on compactness follow immediately from Theorem C.7 combined with the fact that the compact operators form an ideal in $\mathcal{L}(X)$.

(ii) Let $t > \alpha + \beta$, $0 < h < \min\{t - \alpha - \beta, \alpha\}$ and $x \in X$. Then we have

$$\begin{aligned}
\big\|F * G\,(t+h)x - F * G\,(t)x\big\| &\le \bigg\|\int_0^t \big(F(t+h-s) - F(t-s)\big)G(s)x\,ds\bigg\| \\
&\quad + \int_t^{t+h} \sup_{\tau\in[0,t]}\|F(\tau)\| \sup_{\tau\in[0,t]}\|G(\tau)\|\cdot\|x\|\,ds \\
&=: I_1 + I_2.
\end{aligned}$$

It is obvious that $I_2$ tends to 0 uniformly for $\|x\| \le 1$ as $h \downarrow 0$. Hence, we only have to estimate $I_1$ and obtain

$$\begin{aligned}
I_1 &\le \int_0^{t-\alpha} \|F(t+h-s) - F(t-s)\|\cdot \sup_{\tau\in[0,t]}\|G(\tau)\|\cdot\|x\|\,ds \\
&\quad + \bigg\|\int_{t-\alpha}^t \big(F(t+h-s) - F(t-s)\big)G(s)x\,ds\bigg\| =: I_3 + I_4.
\end{aligned}$$

We have that $I_3$ tends to 0 uniformly for $\|x\| \le 1$ as $h \downarrow 0$, since $t+h-s > t-s > \alpha$ for $s \in (0, t-\alpha)$. Hence, it remains to estimate $I_4$, which is

$$\begin{aligned}
I_4 &= \bigg\|\int_{t-\alpha-h}^{t-h} F(t-s)\,G(s+h)x\,ds - \int_{t-\alpha}^t F(t-s)\,G(s)x\,ds\bigg\| \\
&\le \int_{t-\alpha-h}^{t-\alpha} \|F(t-s)\,G(s+h)x\|\,ds + \int_{t-\alpha}^{t-h} \big\|F(t-s)\,\big(G(s+h) - G(s)\big)x\big\|\,ds \\
&\quad + \int_{t-h}^t \|F(t-s)\,G(s)x\|\,ds =: I_5 + I_6 + I_7.
\end{aligned}$$

It is obvious that $I_5$ and $I_7$ tend to 0 uniformly for $\|x\| \le 1$ as $h \downarrow 0$. The same holds for $I_6$, since $t-\alpha > \beta$. Hence, $F*G$ is norm right continuous on $(\alpha+\beta, \infty)$.

In the same way we can show that $F * G$ is norm left continuous.

To prove the assertion on $G * F$, we again use the identity (1.12). This proves the statements concerning norm continuity.

It remains only to prove the assertions concerning compactness. To this end, we take $t > \alpha + \beta$ and note that, if $0 < s < t - \alpha$, then $t - s > \alpha$, and that if $t - \alpha < s < t$, then $s > \beta$. We can now split the convolution integral and obtain

$$\begin{aligned} F * G(t)x &= \int_0^t F(t-s)G(s)x\,ds \\ &= \int_0^{t-\alpha} F(t-s)G(s)x\,ds + \int_{t-\alpha}^t F(t-s)G(s)x\,ds. \end{aligned}$$

If we now apply Theorem C.7 twice, we see that our claim is true. □

Now let $\mathcal{T} = (T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ and take a bounded operator $B$ such that $A + B$ generates the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$.

In order to apply the above lemma to our problem, it suffices to realize that $\mathcal{T}$, $\mathcal{S} \in \mathcal{X}$. The Volterra operator $V$ from Definition 1.8 can be extended to $\mathcal{X}$ and is just the convolution operator

$$VF = \mathcal{T} * BF = \mathcal{T}B * F$$

for $F \in \mathcal{X}$. In particular, one has from (IE) that

$$\mathcal{S} = \mathcal{T} + \mathcal{T} * B\mathcal{S} = \mathcal{T} + \mathcal{T}B * \mathcal{S}.$$

As a first consequence, we obtain the following result.

**1.14 Proposition.** *If $\mathcal{T} = (T(t))_{t\geq 0}$ is eventually norm continuous (resp. eventually compact) with generator $A$ and if $B$ is compact, then the semigroup $\mathcal{S} = (S(t))_{t\geq 0}$ generated by $A + B$ is eventually norm continuous (resp. eventually compact).*

Proof. Since $B$ is compact, $\mathcal{T}B$ is norm continuous on $(0, \infty)$. Since $\mathcal{S} = \mathcal{T} + \mathcal{T}B * \mathcal{S}$, the assertion follows from Lemma 1.13.(i). □

This proposition does not hold without assuming $B$ to be compact.

**1.15 Example.** Take $(T(t))_{t\geq 0}$ to be the nilpotent right translation semigroup on $X := \mathrm{L}^1[0, 2]$, which is eventually norm continuous and eventually compact. Its generator $(A, D(A))$ is given by

$$Af = -f' \qquad \text{for} \qquad f \in D(A) = \left\{g \in \mathrm{W}^{1,1}[0, 2] : g(0) = 0\right\}$$

(see Paragraph II.2.11). Define the bounded operator $B \in \mathcal{L}(X)$ by

$$(Bf)(s) := \begin{cases} f(s+1) & \text{for } 0 \leq s \leq 1, \\ 0 & \text{for } 1 < s \leq 2 \end{cases}$$

and for $n \in \mathbb{N}$ the functions $f_n \in D(A)$ by

$$f_n(s) := \begin{cases} s\mathrm{e}^{-2\pi i n s} & \text{for } 0 \leq s \leq 1, \\ \mathrm{e}^{-2\pi i n s} & \text{for } 1 < s \leq 2. \end{cases}$$

It follows that

$$(A + B)f_n = 2\pi i n f_n,$$

and hence $2\pi i n \in P\sigma(A + B)$ for each $n \in \mathbb{N}$ and $\sigma(A + B) \cap i\mathbb{R}$ is unbounded. Therefore, Theorem II.4.18 implies that the semigroup $(S(t))_{t\geq 0}$ generated by $A + B$ is not eventually norm continuous.

However, if the "eventual" properties in Proposition 1.14 are replaced by the corresponding "immediate" ones, the compactness assumption on $B$ can be dropped. This will be shown in the first part of our next result. In its second part we will relax the compactness property on $B$ and make instead an assumption on the Volterra operator $V$.

**1.16 Theorem.** *For the semigroups $\mathcal{T} = (T(t))_{t\geq 0}$ and $\mathcal{S} = (S(t))_{t\geq 0}$ with generators $A$ and $A+B$ with $B \in \mathcal{L}(X)$ the following assertions are true.*

(i) *If $\mathcal{T}$ is immediately norm continuous (resp. immediately compact), then the same holds for the perturbed semigroup $\mathcal{S}$.*

(ii) *If $\mathcal{T}$ is norm continuous (resp. compact) on $(\alpha, \infty)$ and if there exists $k \in \mathbb{N}$ such that $V^k\mathcal{T}$ is norm continuous (resp. compact) on $(0,\infty)$, then $\mathcal{S}$ is norm continuous (resp. compact) on $(k\alpha, \infty)$.*

Proof. (i) follows from Lemma 1.13.(i), since $\mathcal{S} = \mathcal{T} + \mathcal{T} * B\mathcal{S}$.

(ii) By the Dyson–Phillips series (1.7) we have

$$\mathcal{S} = \sum_{n=0}^{\infty} V^n\mathcal{T} = \sum_{n=0}^{k} V^n\mathcal{T} + \sum_{n=1}^{\infty} V^n(V^k\mathcal{T}),$$

where the series converges uniformly on compact intervals of $\mathbb{R}_+$. The terms in the first sum of the right-hand side are $\mathcal{T}$, $\mathcal{T} * B\mathcal{T}$, $\mathcal{T} * B(\mathcal{T} * B\mathcal{T}), \ldots$, which are norm continuous (resp. compact) at least on $(k\alpha, \infty)$ by Lemma 1.13.(ii). The series

$$\sum_{n=1}^{\infty} V^n(V^k\mathcal{T}) = \mathcal{T} * B(V^k\mathcal{T}) + \mathcal{T} * B\big(\mathcal{T} * B(V^k\mathcal{T})\big) + \cdots$$

converges uniformly on compact intervals, and each term is norm continuous (resp. compact) on $(0,\infty)$ by Lemma 1.13.(i). This implies the assertion. □

**1.17 Exercises.** (1) Show that the convolution $F * G$ of two strongly continuous functions $F,\, G : \mathbb{R}_+ \to X$ is again strongly continuous. Moreover, "$*$" is associative, i.e., $(F * G) * H = F * (G * H)$ for all $F, G, H \in \mathcal{X} := \mathrm{C}(\mathbb{R}_+, \mathcal{L}_s(X))$. (Hint: Use Lemma B.15.)

(2) Prove Lemma 1.9. (Hint: By Exercise (1), $V$ is a linear operator on the space $\mathrm{C}([0,t_0], \mathcal{L}_s(X))$. Use induction on $n \in \mathbb{N}$ to verify (1.5). Equation (1.6) then follows from the Hadamard formula $\mathrm{r}(V) = \lim_{n\to\infty} \|V^n\|^{1/n}$ for the spectral radius.)

(3) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on the Banach space $X$ and $(S(t))_{t\geq 0}$ the semigroup with generator $A+B$ for $B \in \mathcal{L}(X)$.

(i) Show that instead of the integral equations (IE) and (IE*) we can write

$$S(t)x = T(t)x + \int_0^t T(s)BS(t-s)x\,ds$$

and

$$S(t)x = T(t)x + \int_0^t S(t-s)BT(s)x\,ds$$

for $x \in X$, $t \geq 0$.

(ii) Define a Volterra operator $V^*$ based on the integral equation (IE$^*$) and show that

$$S(t) = \sum_{n=0}^{\infty} S_n^*(t),$$

where $S_0^*(t) := T(t)$ and

$$S_{n+1}^*(t)x := V^* S_n^*(t)x = \int_0^t S_n^*(s) B T(t-s)x \, ds$$

for $x \in X$, $t \geq 0$.

(4) Show that the variation of parameters formulas (IE) and (IE$^*$) also hold for perturbations $B \in \mathcal{L}(X_1)$ and $x \in D(A)$.

(5) Take the Banach space $X := \mathrm{C}_0(\mathbb{R})$ and a function $q \in \mathrm{C_b}(\mathbb{R})$, and define

$$T(t)f(s) := \mathrm{e}^{\int_{s-t}^{s} q(\tau)\, d\tau} \cdot f(s-t)$$

for $s \in \mathbb{R}$, $t \geq 0$, and $f \in X$.

(i) Show that $(T(t))_{t\geq 0}$ is a strongly continuous semigroup on $X$.

(ii) Compute its generator.

(iii) What happens if the function $q$ is taken in $\mathrm{L}^\infty(\mathbb{R})$?

(iv) Can one allow the function $q$ to be unbounded such that $(T(t))_{t\geq 0}$ still becomes a strongly continuous semigroup on $X$?

(v) Assume that

$$u(t,s) := \mathrm{e}^{\int_s^t q(\tau)\, d\tau}$$

is uniformly bounded for $s, t \in \mathbb{R}$. Show that the semigroup $(T(t))_{t\geq 0}$ is similar to the left translation semigroup on $X$. (Hint: Use the multiplication operator $M_{u(\cdot,0)}$ as a similarity transformation.)

(6) Let $(A, D(A))$ be an unbounded generator on the Banach space $X$. On the product space $\mathcal{X} := X \times X$ define

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \quad \text{with domain} \quad D(\mathcal{A}) := D(A) \times X$$

and the bounded operator $\mathcal{B} := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$.

(i) Show that $\mathcal{X}_2^{\mathcal{A}+\mathcal{B}} = D((\mathcal{A}+\mathcal{B})^2) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in D(A) \times X : Ax + y \in D(A) \right\}$, hence is different from $\mathcal{X}_2^{\mathcal{A}} = D(A^2) \times X$.

(ii) Prove a similar statement for the extrapolation spaces of order 2. (Hint: Consider $\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ with domain $D(\mathcal{A}) := D(A) \times X$ and $\mathcal{B} := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$.)

This confirms the statement following Corollary 1.4.

(7) Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups with generators $A$ and $A+B$, respectively, where $B \in \mathcal{L}(X)$.

(i) Show that for $\lambda$ sufficiently large

$$S(t) = \lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} S(s)\, ds = \lim_{h\downarrow 0} \frac{1}{h} R(\lambda, A+B)\big(S(t) - \mathrm{e}^{-\lambda h} S(t+h)\big)$$

in the strong operator topology. (Hint: Use (1.6) in Lemma II.1.3.)

(ii) Show that $R(\lambda, A+B)S(t)$ for $\lambda$ large is given by

$$\left[\sum_{n=0}^{\infty}\big(R(\lambda,A)B\big)^n\right]\left(R(\lambda,A)T(t)+\int_0^t T(t-s)R(\lambda,A)BS(s)\,ds\right).$$

(Hint: Use the identity $(\lambda - A - B) = (\lambda - A)(I - R(\lambda, A)B)$ and (IE).)

(iii) If $(T(t))_{t\geq 0}$ is eventually compact, $(S(t))_{t\geq 0}$ is eventually norm continuous, and $R(\lambda, A)B$ is compact for large $\lambda$, then $(S(t))_{t\geq 0}$ is eventually compact. (Hint: Apply (i) and (ii) for the norm topology and use Theorem C.7.)

## 2. Perturbations of Contractive and Analytic Semigroups

Addition of two unbounded operators is a very delicate operation and can destroy many of the good properties the single operators may have. This is, in part, due to the fact that the "naive" domain

$$D(A+B) := D(A) \cap D(B)$$

for the sum $A+B$ of the operators $\big(A, D(A)\big)$ and $\big(B, D(B)\big)$ can be too small (see Example 1.2.(ii)). In order to avoid this, we assume in this section that the perturbing operator $B$ behaves well with respect to the unperturbed operator $A$. More precisely, we assume the following property.

**2.1 Definition.** *Let $A : D(A) \subset X \to X$ be a linear operator on the Banach space $X$. An operator $B : D(B) \subset X \to X$ is called (relatively) $A$-bounded if $D(A) \subseteq D(B)$ and if there exist constants $a, b \in \mathbb{R}_+$ such that*

$$\|Bx\| \leq a\,\|Ax\| + b\,\|x\| \tag{2.1}$$

*for all $x \in D(A)$. The $A$-bound of $B$ is*

$$a_0 := \inf\{a \geq 0 : \text{there exists } b \in \mathbb{R}_+ \text{ such that (2.1) holds}\}.$$

Before applying this notion to the perturbation problem for generators we discuss a concrete example.

**2.2 Example.** For an interval $I \subseteq \mathbb{R}$ we consider on $X := \mathrm{L}^p(I)$, $1 \leq p \leq \infty$, the operators

$$A := \tfrac{d^2}{dx^2}, \qquad D(A) := W^{2,p}(I),$$
$$B := \tfrac{d}{dx}, \qquad D(B) := W^{1,p}(I).$$

**Proposition.** *The operator $B$ is $A$-bounded with $A$-bound $a_0 = 0$.*

PROOF. We choose an arbitrary bounded interval $J := (\alpha, \beta) \subset I$, and set $\varepsilon := \beta - \alpha$,

$$J_1 := (\alpha, \alpha + {}^{\varepsilon}/_{3}), \quad J_2 := (\alpha + {}^{\varepsilon}/_{3}, \beta - {}^{\varepsilon}/_{3}), \quad J_3 := (\beta - {}^{\varepsilon}/_{3}, \beta).$$

Then, for all $f \in D(A)$ and $s \in J_1$, $t \in J_3$ there exists, by the mean value theorem, a point $x_0 = x_0(s,t) \in J$ such that

$$f'(x_0) = \frac{f(t) - f(s)}{t - s}.$$

Using this and $t - s \geq {}^{\varepsilon}/_{3}$, we obtain

$$\text{(2.2)} \quad |f'(x)| = \left| f'(x_0) + \int_{x_0}^{x} f''(y)\, dy \right| \leq \frac{3}{\varepsilon}\big(|f(s)| + |f(t)|\big) + \int_J |f''(y)|\, dy$$

for all $x \in J$, $s \in J_1$, and $t \in J_3$. If we denote by $\|\cdot\|_{p,J}$ the $p$-norm in $\mathrm{L}^p(J)$ and integrate inequality (2.2) on both sides with respect to $s \in J_1$ and $t \in J_3$, we obtain

$$\begin{aligned} \frac{\varepsilon^2}{9}|f'(x)| &\leq \int_{J_1} |f(s)|\, ds + \int_{J_3} |f(t)|\, dt + \frac{\varepsilon^2}{9}\int_J |f''(y)|\, dy \\ &\leq \|f\|_{1,J} + \frac{\varepsilon^2}{9}\|f''\|_{1,J} \\ &\leq \varepsilon^{1/q}\|f\|_{p,J} + \frac{\varepsilon^{2+1/q}}{9}\|f''\|_{p,J}, \end{aligned}$$

where we used Hölder's inequality for ${}^1/_p + {}^1/_q = 1$. From this estimate, it then follows that

$$\begin{aligned} \frac{\varepsilon^2}{9}\|f'\|_{p,J} &\leq \varepsilon^{1/p}\varepsilon^{1/q}\|f\|_{p,J} + \varepsilon^{1/p}\frac{\varepsilon^{2+1/q}}{9}\|f''\|_{p,J} \\ &= \varepsilon\,\|f\|_{p,J} + \frac{\varepsilon^3}{9}\|f''\|_{p,J}, \end{aligned}$$

that is

$$\|f'\|_{p,J} \leq \frac{9}{\varepsilon}\|f\|_{p,J} + \varepsilon\|f''\|_{p,J}.$$

By splitting the interval $I$ in finitely or countable many (depending on whether $I$ is bounded or not) disjoint subintervals $I_n$, $n \in N \subseteq \mathbb{N}$, of length $\varepsilon$, we obtain by Minkowski's inequality

$$\begin{aligned} \|Bf\|_p &= \Big(\sum_{n\in N} \|f'\|_{p,I_n}^p\Big)^{1/p} \leq \frac{9}{\varepsilon}\Big(\sum_{n\in N} \|f\|_{p,I_n}^p\Big)^{1/p} + \varepsilon\Big(\sum_{n\in N} \|f''\|_{p,I_n}^p\Big)^{1/p} \\ &= \frac{9}{\varepsilon}\|f\|_p + \varepsilon\|Af\|_p. \end{aligned}$$

Since we can choose $\varepsilon > 0$ arbitrarily small, the proof of our claim is complete. □

Note that from (2.2) we immediately obtain an analogous result for the second and first derivative on $X := \mathrm{C}_0(I)$. More precisely, if $I \subseteq \mathbb{R}$ is an arbitrary interval and

$$\begin{aligned} A &:= \tfrac{d^2}{dx^2}, && D(A) := \bigl\{f \in \mathrm{C}_0^2(I) : f', f'' \in \mathrm{C}_0(I)\bigr\},\\ B &:= \tfrac{d}{dx}, && D(B) := \bigl\{f \in \mathrm{C}_0^1(I) : f' \in \mathrm{C}_0(I)\bigr\}, \end{aligned}$$

then $B$ is $A$-bounded with $A$-bound $a_0 = 0$.

We now return to the abstract situation and observe that for an $A$-bounded operator $B$ the sum $A + B$ is defined on $D(A + B) := D(A)$. However, many desirable properties may get lost.

**2.3 Examples.** Take $A : D(A) \subset X \to X$ to be the generator of a strongly continuous semigroup such that $\sigma(A) = \mathbb{C}_- := \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$ (e.g., take the generator of the translation semigroup on $\mathrm{C}_0(\mathbb{R}_+)$, see Example IV.2.6.(i)).

(i) If we take $B := \alpha A$ for $\alpha \in \mathbb{C}$, then $A + B$ is not a generator for $\alpha \in \mathbb{C} \setminus (-1, \infty)$, and is not even closed for $\alpha = -1$.

(ii) Consider the new operator $\mathcal{A} := \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$ with $D(\mathcal{A}) := D(A) \times D(A)$ on the product space $\mathfrak{X} := X \times X$. If we take

$$\mathcal{B}_1 := \begin{pmatrix} 0 & \varepsilon A \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad D(\mathcal{B}_1) := X \times D(A),$$

then $\mathcal{A}+\mathcal{B}_1$ is not a generator for every $0 \ne \varepsilon \in \mathbb{C}$ (use Exercise II.4.12.(7)). For

$$\mathcal{B}_2 := \begin{pmatrix} 0 & -A \\ A & -2A \end{pmatrix} \qquad \text{with} \qquad D(\mathcal{B}_2) := D(A) \times D(A),$$

the sum $\mathcal{A} + \mathcal{B}_2$ is not closed, and its closure is not a generator.

We now proceed with a series of lemmas showing which assumptions on the unperturbed operator $A$ and the $A$-bounded perturbation $B$ are needed such that the sum $A + B$

- is closed,
- has nonempty resolvent set, and, finally,
- becomes the generator of a strongly continuous semigroup.

**2.4 Lemma.** *If $\bigl(A, D(A)\bigr)$ is closed and $\bigl(B, D(B)\bigr)$ is $A$-bounded with $A$-bound $a_0 < 1$, then*

$$\bigl(A + B, D(A)\bigr)$$

*is a closed operator.*

PROOF. Since an operator is closed if and only if its domain is a Banach space for the graph norm, it suffices to show that the graph norm $\|\cdot\|_{A+B}$ of $A+B$ is equivalent to the graph norm $\|\cdot\|_A$ of $A$. By assumption, there exist constants $0 \le a < 1$ and $0 < b$ such that

$$\|Bx\| \le a\,\|Ax\| + b\,\|x\|$$

for all $x \in D(A)$. Therefore, one has

$$\|Ax\| = \|(A+B)x - Bx\| \le \|(A+B)x\| + a\,\|Ax\| + b\,\|x\|$$

and, consequently,

$$-b\,\|x\| + (1-a)\,\|Ax\| \le \|(A+B)x\| \le \|Ax\| + \|Bx\| \le (1+a)\,\|Ax\| + b\,\|x\|.$$

This yields the estimate

$$b\,\|x\| + (1-a)\,\|Ax\| \le \|(A+B)x\| + 2b\,\|x\| \le (1+a)\,\|Ax\| + 3b\,\|x\|,$$

proving the equivalence of the two graph norms. □

**2.5 Lemma.** *Let $(A, D(A))$ be closed with $\rho(A) \ne \emptyset$ and assume $(B, D(B))$ to be $A$-bounded with constants $0 \le a, b$ in estimate (2.1). If $\lambda_0 \in \rho(A)$ and*

(2.3) $$c := a\,\|AR(\lambda_0, A)\| + b\,\|R(\lambda_0, A)\| < 1,$$

*then $A+B$ is closed, and one has $\lambda_0 \in \rho(A+B)$ with*

(2.4). $$\|R(\lambda_0, A+B)\| \le (1-c)^{-1}\,\|R(\lambda_0, A)\|.$$

PROOF. As in the proof of Theorem 1.3, we decompose $\lambda_0 - A - B$ as the product

$$\lambda_0 - A - B = [I - BR(\lambda_0, A)](\lambda_0 - A)$$

and observe that $\lambda_0 - A$ is a bijection from $D(A)$ onto $X$, while $BR(\lambda_0, A)$ is bounded on $X$ (use Exercise 2.18.(1.i)). If we can show that $\|BR(\lambda_0, A)\| < 1$, we obtain that $[I - BR(\lambda_0, A)]$, hence $\lambda_0 - A - B$, is invertible with inverse

(2.5) $$R(\lambda_0, A+B) = R(\lambda_0, A)\sum_{n=0}^{\infty}\big(BR(\lambda_0, A)\big)^n$$

satisfying

$$\|R(\lambda_0, A+B)\| \le \|R(\lambda_0, A)\|\,(1 - \|BR(\lambda_0, A)\|)^{-1}.$$

To that purpose, take $x \in X$ and use (2.1) to obtain

$$\begin{aligned}\|BR(\lambda_0, A)x\| &\le a\,\|AR(\lambda_0, A)x\| + b\,\|R(\lambda_0, A)x\| \\ &\le \big(a\,\|AR(\lambda_0, A)\| + b\,\|R(\lambda_0, A)\|\big)\cdot\|x\|,\end{aligned}$$

whence $\|BR(\lambda_0, A)\| \le c < 1$ by assumption (2.3). □

In the last preparatory lemma, we consider operators satisfying a Hille–Yosida type estimate for the resolvent (but not for all its powers as required in Generation Theorem II.3.8). It is shown that this class of operators remains invariant under $A$-bounded perturbations with *small* $A$-bound.

**2.6 Lemma.** *Let $\big(A, D(A)\big)$ be an operator whose resolvent exists for all*

$$0 \neq \lambda \in \overline{\Sigma}_\delta := \{z \in \mathbb{C} : |\arg z| \leq \delta\}$$

*and satisfies*

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}$$

*for some constants $\delta \geq 0$ and $M \geq 1$. Moreover, assume $\big(B, D(B)\big)$ to be $A$-bounded with $A$-bound*

$$a_0 < \frac{1}{M+1}.$$

*Then there exist constants $r \geq 0$ and $\widetilde{M} \geq 1$ such that*

$$\overline{\Sigma}_\delta \cap \{z \in \mathbb{C} : |z| > r\} \subset \rho(A+B) \qquad \text{and} \qquad \|R(\lambda, A+B)\| \leq \frac{\widetilde{M}}{|\lambda|}$$

*for all $\lambda \in \overline{\Sigma}_\delta \cap \{z \in \mathbb{C} : |z| > r\}$.*

Proof. Choose constants $0 \leq a < {}^1\!/_{M+1}$ and $0 \leq b$ satisfying the estimate (2.1). From this we obtain

$$\begin{aligned} c := {} & a\,\|AR(\lambda, A)\| + b\,\|R(\lambda, A)\| \\ = {} & a\,\|\lambda R(\lambda, A) - I\| + b\,\|R(\lambda, A)\| \\ \leq {} & a(M+1) + \frac{bM}{|\lambda|} < 1, \end{aligned}$$

whenever $|\lambda| > r := \frac{bM}{1-a(M+1)}$. The assertion now follows from Lemma 2.5. $\square$

If we now assume the constants to be $M = \widetilde{M} = 1$, we obtain a perturbation theorem for generators of contraction semigroups. The surprising fact is that the relative bound $a_0$, which in Lemma 2.6 and for $M = 1$ should be smaller than $\frac{1}{2}$, must only satisfy $a_0 < 1$. The dissipativity (see Definition II.3.13) of the operators involved makes this possible.

**2.7 Theorem.** *Let $\big(A, D(A)\big)$ be the generator of a contraction semigroup and assume $\big(B, D(B)\big)$ to be dissipative and $A$-bounded with $A$-bound $a_0 < 1$. Then $\big(A+B, D(A)\big)$ generates a contraction semigroup.*

Proof. We first assume that $a_0 < {}^1\!/_2$. From the criterion in Proposition II.3.23, it follows that the sum of a generator of a contraction semigroup and a dissipative operator is again dissipative. Therefore, $A+B$ is a densely defined, dissipative operator, and by Theorem II.3.15 it suffices to find $\lambda_0 > 0$ such that $\lambda_0 \in \rho(A+B)$. This, however, follows from Lemma 2.6 by choosing $\delta = 0$, i.e., $\overline{\Sigma}_\delta = [0, \infty)$.

In order to extend this to the case $0 \le a_0 < 1$, we define for $0 \le \alpha \le 1$ the operators
$$C_\alpha := A + \alpha B, \quad D(C_\alpha) := D(A).$$
Then, for $x \in D(A)$, one has
$$\begin{aligned} \|Bx\| &\le a\,\|Ax\| + b\,\|x\| \le a(\|C_\alpha x\| + \alpha\,\|Bx\|) + b\,\|x\| \\ &\le a\,\|C_\alpha x\| + a\,\|Bx\| + b\,\|x\|; \end{aligned}$$
and hence
$$\|Bx\| \le \frac{a}{1-a}\,\|C_\alpha x\| + \frac{b}{1-a}\,\|x\| \qquad \text{for all} \qquad 0 \le \alpha \le 1.$$
Next, we choose $k \in \mathbb{N}$ such that
$$c := \frac{a}{k(1-a)} < \frac{1}{2}.$$
Then the estimate
$$\left\|\tfrac{1}{k}Bx\right\| \le c\,\|C_\alpha x\| + \frac{b}{k(1-a)}\,\|x\|$$
shows that for each $0 \le \alpha \le 1$ the operator ${}^1\!/_k B$ is $C_\alpha$-bounded with $C_\alpha$-bound less than ${}^1\!/_2$. As observed above, this implies that
$$C_\alpha + \tfrac{1}{k}B = A + (\alpha + \tfrac{1}{k})B$$
generates a contraction semigroup whenever $C_\alpha = A + \alpha B$ does. However, $A$ generates a contraction semigroup, hence $A + {}^1\!/_k B$ does. Repeating this argument $k$ times shows that $(A + {}^{(k-1)}\!/_k B) + {}^1\!/_k B = A + B$ generates a contraction semigroup as claimed. □

In the limit case, i.e., if one has $a = 1$ in the estimate (2.1), the result remains essentially true, provided that the adjoint of $B$ is densely defined.

**2.8 Corollary.** *Let $(A, D(A))$ be the generator of a contraction semigroup on $X$ and assume that $(B, D(B))$ is dissipative, $A$-bounded, and satisfies*

$$\|Bx\| \le \|Ax\| + b\|x\| \tag{2.6}$$

*for all $x \in D(A)$ and some constant $b \ge 0$. If the adjoint $B'$ is densely defined on $X'$, then the closure of $(A + B, D(A))$ generates a contraction semigroup on $X$.*

PROOF. The sum $A + B$ remains dissipative and densely defined. Hence, by the Lumer–Phillips Theorem II.3.15, it suffices to show that $\mathrm{rg}(I - A - B)$ is dense in $X$.

Choose $y' \in X'$ satisfying $\langle z, y' \rangle = 0$ for all $z \in \mathrm{rg}(I - A - B)$ and then $y \in X$ such that $\langle y, y' \rangle = \|y'\|$. The perturbed operators $A + \varepsilon B$ with domain $D(A)$ are generators of contraction semigroups for each $0 \le \varepsilon < 1$ by Theorem 2.7. From Generation Theorem II.3.5 we obtain $1 \in \rho(A + \varepsilon B)$, and hence there exists a unique $x_\varepsilon \in D(A)$ such that $\|x_\varepsilon\| \le \|y\|$ and

$$x_\varepsilon - (A + \varepsilon B)x_\varepsilon = y.$$

From the estimate

$$\begin{aligned}\|Bx_\varepsilon\| &\le \|Ax_\varepsilon\| + b\|x_\varepsilon\| \\ &\le \|(A + \varepsilon B)x_\varepsilon\| + \varepsilon\|Bx_\varepsilon\| + b\|x_\varepsilon\| \\ &\le \|x_\varepsilon - y\| + \varepsilon\|Bx_\varepsilon\| + b\|x_\varepsilon\|\end{aligned}$$

we deduce

$$(1 - \varepsilon)\|Bx_\varepsilon\| \le \|x_\varepsilon - y\| + b\|x_\varepsilon\| \le (2 + b)\|y\| \tag{2.7}$$

for all $0 \le \varepsilon < 1$.

We now use the density of $D(B')$. In fact, for $z' \in D(B')$ it follows that

$$\begin{aligned}\left|\left\langle (1 - \varepsilon)Bx_\varepsilon, z' \right\rangle\right| &\le (1 - \varepsilon)\|x_\varepsilon\| \cdot \|B'z'\| \\ &\le (1 - \varepsilon)\|y\| \cdot \|B'z'\|,\end{aligned}$$

and hence

$$\lim_{\varepsilon \uparrow 1} \left\langle (1 - \varepsilon)Bx_\varepsilon, z' \right\rangle = 0.$$

Our assumption and the norm boundedness of the elements $(1 - \varepsilon)Bx_\varepsilon$ (see (2.7)) then implies

$$\lim_{\varepsilon \uparrow 1} \left\langle (1 - \varepsilon)Bx_\varepsilon, y' \right\rangle = 0,$$

and therefore

$$\begin{aligned}\|y'\| = \left\langle y, y' \right\rangle &= \left\langle x_\varepsilon - (A + \varepsilon B)x_\varepsilon, y' \right\rangle \\ &= \left\langle (1 - \varepsilon)Bx_\varepsilon, y' \right\rangle + \left\langle (I - A - B)x_\varepsilon, y' \right\rangle \\ &\to 0 \text{ as } \varepsilon \uparrow 1.\end{aligned}$$

From the Hahn–Banach theorem we then conclude that $\mathrm{rg}(I - A - B)$ is dense in $X$. ☐

If $X$ is reflexive, the adjoint of every closable, densely defined operator is again densely defined on the dual space (see Proposition B.10). Since densely defined, dissipative operators are always closable (see Proposition II.3.14.(iv)), we arrive at the following result.

**2.9 Corollary.** *Let $(A, D(A))$ be the generator of a contraction semigroup on a reflexive Banach space $X$. If $(B, D(B))$ is dissipative, $A$-bounded, and satisfies the estimate (2.6), then the closure of $(A + B, D(A))$ generates a contraction semigroup on $X$.*

In order to obtain the previous perturbation results, we used Lemma 2.6 and could estimate *only* the resolvent of the perturbed operator $A + B$ and *not all* its powers. Due to the Lumer–Phillips Theorem II.3.15, this was sufficient if $A$ was the generator of a contraction semigroup and $B$ was dissipative. There is, however, another case where an estimate on the resolvent alone forces an operator to generate a semigroup. Such a result has been proved in Theorem II.4.6 for analytic semigroups and now easily leads to another perturbation theorem.

**2.10 Theorem.** *Let the operator $\big(A, D(A)\big)$ generate an analytic semigroup $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}}$ on a Banach space $X$. Then there exists a constant $\alpha > 0$ such that $\big(A + B, D(A)\big)$ generates an analytic semigroup for every $A$-bounded operator $B$ having $A$-bound $a_0 < \alpha$.*

Proof. We first assume that $\big(T(z)\big)_{z\in\Sigma_\delta\cup\{0\}}$ is bounded, which means, by Theorem II.4.6, that $A$ is sectorial. Hence, there exist constants $\delta' \in (0, {}^{\pi}\!/_{2}]$ and $C \geq 1$ such that for every

$$0 \neq \lambda \in \overline{\Sigma}_{\pi/2+\delta'} := \Big\{z \in \mathbb{C} : |\arg z| \leq \frac{\pi}{2} + \delta'\Big\}$$

we have

$$\lambda \in \rho(A) \qquad \text{and} \qquad \|R(\lambda, A)\| \leq \frac{C}{|\lambda|}.$$

If we define $\alpha := {}^{1}\!/_{C+1}$, we can apply Lemma 2.6 and obtain constants $r \geq 0$ and $M \geq 1$ such that

$$\Sigma := \overline{\Sigma}_{\pi/2+\delta'} \cap \{z \in \mathbb{C} : |z| > r\} \subseteq \rho(A + B)$$

and

$$\|R(\lambda, A + B)\| \leq \frac{M}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma.$$

By Exercise II.4.12.(6), this implies that $A+B$ generates an analytic semigroup, proving the assertion in the bounded case.

In order to treat the general case, we take $w \in \mathbb{R}$ and conclude from

$$\|Bx\| \leq a\,\|Ax\| + b\,\|x\| \leq a\,\|(A - w)x\| + (aw + b)\,\|x\|$$

for all $x \in D(A)$ that $B$ is also $A - w$ bounded with the same bound $a_0$. Since the semigroup generated by $A - w$ is analytic and bounded in $\Sigma_\delta$ for $w$ sufficiently large, the first part of the proof implies that $A + B - w$; hence $A + B$ generates an analytic semigroup. □

Before proceeding and in particular before presenting examples, we want to look back at the results obtained so far. They can be distinguished according to the following modification of Problem 1.1.

**2.11 Problem.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on a Banach space $X$ and let $(B, D(B))$ be an $A$-bounded perturbation. For which constants $c \in \mathbb{C}$ is it true that $(A + cB, D(A))$ is a generator?*

The above results provide the following answers.

(i) If both operators $A$ and $B$ are dissipative, then $A+cB$ is a generator whenever $0 \le c < 1$ (Theorem 2.7).

(ii) If $A$ generates an analytic semigroup, then there exists $c_0 > 0$ such that $A + cB$ is a generator whenever $|c| < c_0$ (Theorem 2.10).

(iii) If $B$ is bounded on $X$, then $A + cB$ is a generator for all $c \in \mathbb{C}$ (Theorem 1.3).

Clearly, this last situation is most desirable and can also be achieved if in Theorem 2.10 the $A$-bound $a_0$ of $B$ satisfies $a_0 = 0$.

**2.12 Examples.** (i) In Example II.4.8 we showed that the second derivative

$$A := \tfrac{d^2}{dx^2}, \qquad D(A) := \big\{f \in \mathrm{H}^2[0,1] : f(0) = f(1) = 0\big\}$$

generates an analytic semigroup on $H := \mathrm{L}^2[0,1]$. Since by Example 2.2 the first derivative ${}^{d}\!/\!{}_{dx}$ with maximal domain $\mathrm{H}^1[0,1]$ is $A$-bounded with $A$-bound $a_0 = 0$, we conclude by Theorem 2.10 and Exercise 2.18.(1) that for all $B \in \mathcal{L}\big(\mathrm{H}^1[0,1], \mathrm{L}^2[0,1]\big)$ the operator

$$C := A + B, \qquad D(C) := D(A)$$

generates an analytic semigroup on $H$.

(ii) As in Paragraph II.2.13, we consider the diffusion semigroup on $\mathrm{L}^1(\mathbb{R}^n)$ given by

$$\big(T(t)f\big)(s) := (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|s-r|^2/4t} f(r)\, dr =: \int_{\mathbb{R}^n} K_t(s-r) f(r)\, dr.$$

It is generated by the closure of the Laplacian $\Delta$ defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. In Example II.4.10 we have seen that $\big(T(t)\big)_{t\ge 0}$ is a bounded analytic semigroup. As a perturbation we take the multiplication operator

$$(M_q f)(s) := q(s) f(s) \quad \text{for} \quad f \in D(M_q) := \big\{g \in \mathrm{L}^1(\mathbb{R}^n) : qg \in \mathrm{L}^1(\mathbb{R}^n)\big\}$$

induced by a function $q \in \mathrm{L}^p(\mathbb{R}^n)$ for $p > \max\{1, {}^{n}\!/\!{}_{2}\}$.

We now show that $B := M_q$ is $\Delta$-bounded with $\Delta$-bound zero. To this end, we estimate for $f \in \mathrm{L}^1(\mathbb{R}^n)$ and $\lambda > 0$

$$\begin{aligned}
\|BR(\lambda,\Delta)f\|_{\mathrm{L}^1(\mathbb{R}^n)} &= \left\| B\int_0^\infty \mathrm{e}^{-\lambda t}T(t)f\,dt\right\|_{\mathrm{L}^1(\mathbb{R}^n)} \\
&\le \int_{\mathbb{R}^n} |q(s)| \int_0^\infty \mathrm{e}^{-\lambda t}\int_{\mathbb{R}^n} K_t(s-r)|f(r)|\,dr\,dt\,ds \\
&= \int_{\mathbb{R}^n} |f(r)| \int_0^\infty \mathrm{e}^{-\lambda t}\int_{\mathbb{R}^n} K_t(s-r)|q(s)|\,ds\,dt\,dr \\
&\le \|f\|_{\mathrm{L}^1(\mathbb{R}^n)} \sup_{r\in\mathbb{R}^n} \int_0^\infty \mathrm{e}^{-\lambda t}\int_{\mathbb{R}^n} K_t(s-r)|q(s)|\,ds\,dt \\
&\le \|f\|_{\mathrm{L}^1(\mathbb{R}^n)} \cdot \|q\|_{\mathrm{L}^p(\mathbb{R}^n)} \int_0^\infty \mathrm{e}^{-\lambda t}\Big(\int_{\mathbb{R}^n} K_t(s)^{p'} ds\Big)^{1/p'} dt
\end{aligned}$$

with $1/p + 1/p' = 1$, where we used Fubini's theorem and Hölder's inequality. It is now easy to verify that $\|K_t\|_{\mathrm{L}^{p'}(\mathbb{R}^n)} = ct^{-n/2p}$ for a constant $c > 0$. Hence, we conclude that $D(\Delta) \subset D(B)$ and

$$\begin{aligned}
\|Bf\|_{\mathrm{L}^1(\mathbb{R}^n)} &\le c\|q\|_{\mathrm{L}^p(\mathbb{R}^n)} \int_0^\infty \mathrm{e}^{-\lambda t} t^{-n/2p}\,dt\; \|(\lambda-\Delta)f\|_{\mathrm{L}^1(\mathbb{R}^n)} \\
&=: a_\lambda \|(\lambda-\Delta)f\|_{\mathrm{L}^1(\mathbb{R}^n)} \le \lambda a_\lambda \|f\|_{\mathrm{L}^1(\mathbb{R}^n)} + a_\lambda \|\Delta f\|_{\mathrm{L}^1(\mathbb{R}^n)}
\end{aligned}$$

for all $f \in D(\Delta)$. Since $a_\lambda := c\|q\|_{\mathrm{L}^p(\mathbb{R}^n)} \int_0^\infty \mathrm{e}^{-\lambda t} t^{-n/2p}\,dt$ converges to zero as $\lambda \to \infty$, this proves our claim. Thus, by Theorem 2.10, the operator $\big(\Delta + M_q, D(\Delta)\big)$ generates an analytic semigroup for every $q \in \mathrm{L}^p(\mathbb{R}^n)$ with $p > \max\{1, n/2\}$.

We now introduce two classes of operators always having $A$-bound zero with respect to a given operator $A$.

First, we will use the notions from Section II.5.b and assume that the domain of the perturbing operator contains a Favard space $F_\alpha$.

**2.13 Lemma.** *Let $A$ be the generator of a strongly continuous semigroup. Moreover, let $B : D(B) \subseteq X \to X$ be closed and assume that $F_\alpha \subseteq D(B)$ for some $0 < \alpha < 1$. Then $B$ is $A$-bounded with $A$-bound zero.*

Proof. We proceed in two steps, where, without loss of generality, we assume that $A$ generates a semigroup $(T(t))_{t\ge 0}$ having negative growth bound $\omega_0$.

First, we show that $B : F_\alpha \to X$ is bounded, i.e., that there exists $K > 0$ such that

$$\|Bx\| \le K\|x\|_{F_\alpha} \qquad \text{for all } x \in F_\alpha. \tag{2.8}$$

To this end, we observe that by Theorem II.5.15.(i), $F_\alpha$ is a Banach space that by Proposition II.5.14 is continuously embedded in $X$. Hence, $B : F_\alpha \to X$ is closed and therefore bounded by the closed graph theorem.

In the second step, we show that for every $\varepsilon > 0$ there exists $b_\varepsilon > 0$ such that

$$\|x\|_{F_\alpha} \le \varepsilon \, \|Ax\| + b_\varepsilon \, \|x\| \qquad \text{for all } x \in D(A). \tag{2.9}$$

In fact, for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left\| \frac{1}{t^\alpha} \int_0^t T(s)\, ds \right\| \le \varepsilon \qquad \text{for all } t \in (0, \delta].$$

Then, for $b_\varepsilon := \sup_{t \ge \delta} \left\| \frac{T(t)-I}{t^\alpha} \right\|$, we obtain

$$\begin{aligned} \sup_{t>0} \left\| \frac{T(t)x - x}{t^\alpha} \right\| &\le \sup_{\delta \ge t > 0} \left\| \frac{1}{t^\alpha} \int_0^t T(s)Ax\, ds \right\| + \sup_{t \ge \delta} \left\| \frac{T(t)x - x}{t^\alpha} \right\| \\ &\le \varepsilon \cdot \|Ax\| + b_\varepsilon \cdot \|x\| \end{aligned}$$

for all $x \in D(A)$. By the definition of the Favard norm $\| \cdot \|_{F_\alpha}$, this proves (2.9). Since $D(A) \subseteq F_\alpha$, the assertion then follows by combining the estimates (2.8) and (2.9). □

We point out that by Proposition II.5.33 the domain condition on $B$ in the previous lemma is satisfied if the abstract Hölder space $X_\alpha$ or the domain $D(A^\alpha)$ of the fractional power $A^\alpha$ of $A$ is contained in $D(B)$ for some $\alpha \in (0, 1)$.

As an immediate consequence of Theorem 2.10 and Lemma 2.13 we obtain the following perturbation result.

**2.14 Corollary.** *If $A$ generates an analytic semigroup and $B : D(B) \subset X \to X$ is closed and satisfies $F_\alpha \subseteq D(B)$ for some $0 < \alpha < 1$, then $(A + B, D(A))$ generates an analytic semigroup.*

While in the above situation we made an assumption on the domain of the perturbing operator $B$, we now require a property concerning its range.

**2.15 Definition.** *Let $\big(A, D(A)\big)$ be a closed operator on a Banach space $X$. An operator $\big(B, D(B)\big)$ is called (relatively) $A$-compact if $D(A) \subseteq D(B)$ and $B : X_1 \to X$ is compact, where $X_1$ denotes the domain $D(A)$ equipped with the graph norm $\| \cdot \|_A$.*

If $\rho(A)$ is nonempty, one can show that an $A$-bounded operator $B$ is $A$-compact if and only if $BR(\lambda, A) \in \mathcal{L}(X)$ is compact for some/all $\lambda \in \rho(A)$, see Exercise 2.18.(1). Since compact operators are "small" in some sense, one might hope that an $A$-compact operator is $A$-bounded with bound 0. This is, however, not true in general (see [Hes70]), and we need some additional conditions to ensure it.

**2.16 Lemma.** *Let $\big(A, D(A)\big)$ be a closed operator on a Banach space $X$ and assume $\big(B, D(B)\big)$ to be $A$-compact. If*

(i) *$A$ is a generator and $X$ is reflexive, or if*

(ii) *$\big(B, D(B)\big)$ is closable in $X$,*

*then $B$ is $A$-bounded with $A$-bound $a_0 = 0$.*

Proof. (i) For $0 < \mu$ sufficiently large and $x \in D(A)$, we write

$$\begin{aligned} Bx &= BR(\mu, A)(\mu - A)x \\ &= \mu BR(\mu, A)x - BR(\mu, A)AR(\lambda, A)(\lambda - A)x \\ &= \mu BR(\mu, A)x - BR(\mu, A)A\lambda R(\lambda, A)x + BR(\mu, A)AR(\lambda, A)Ax \end{aligned}$$

for all $\lambda > \mu$. Since the operators appearing in the first two terms are bounded, it suffices to show that for each $\varepsilon > 0$ there exist $\lambda > \mu$ such that

$$\begin{aligned} \varepsilon > \|BR(\mu, A)AR(\lambda, A)\| &= \big\|BR(\mu, A)\big(\lambda R(\lambda, A) - I\big)\big\| \\ &= \big\|\big(\lambda R(\lambda, A') - I\big)\big(BR(\mu, A)\big)'\big\|. \end{aligned}$$

If $X$ is reflexive, then the adjoint operator $A'$ is again a generator (see Paragraph I.5.14). Therefore, by Lemma II.3.4, $\lambda R(\lambda, A')$ converges strongly to $I$ as $\lambda \to \infty$. Moreover, $BR(\mu, A)$ and therefore its adjoint $\big(BR(\mu, A)\big)'$ are compact operators. Combining these two properties and applying Proposition A.3 yields

$$\lim_{\lambda \to \infty} \big\|\big(\lambda R(\lambda, A') - I\big)\big(BR(\mu, A)\big)'\big\| = 0.$$

(ii) Assume the assertion to be false. Then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbb{N}} \subset D(A)$ such that

$$\|Bx_n\| > \varepsilon\|Ax_n\| + n\|x_n\| \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

For $y_n := {}^{x_n}/_{\|x_n\|_A}$ this means

$$\|By_n\| > \varepsilon\|Ay_n\| + n\|y_n\|. \tag{2.11}$$

Since $\|y_n\|_A = 1$ for all $n \in \mathbb{N}$ and since $B$ is $A$-compact, there exists a subsequence $(z_n)_{n \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ such that $(Bz_n)_{n \in \mathbb{N}}$ converges in $X$. Moreover, $\|z_n\| < {}^{\|Bz_n\|}/_n$ and $(Bz_n)_{n \in \mathbb{N}}$ is bounded in $X$; hence $\lim_{n\to\infty} \|z_n\| = 0$. Using the assumption that $B$ is closable, this implies $\lim_{n\to\infty} \|Bz_n\| = 0$ and therefore $\lim_{n\to\infty} \|Az_n\| = 0$ by (2.11). This, however, yields a contradiction, since

$$1 = \|z_n\|_A = \|z_n\| + \|Az_n\| \qquad \text{for all } n \in \mathbb{N}.$$

□

We again combine this lemma with our previous perturbation results.

**2.17 Corollary.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup on a Banach space $X$ and assume the operator $\big(B, D(B)\big)$ to be $A$-compact. If $X$ is reflexive or if $B$ is closable, then the following assertions are true.*

(i) *If $A$ and $B$ are dissipative, then $\big(A + cB, D(A)\big)$ generates a contraction semigroup on $X$ for all $c \in \mathbb{R}_+$.*
(ii) *If the semigroup generated by $A$ is analytic, then $\big(A + cB, D(A)\big)$ generates an analytic semigroup on $X$ for all $c \in \mathbb{C}$.*

One can show that Corollary 2.17.(ii) holds without the extra assumptions that $B$ is closable or that $X$ is reflexive (see [DS88]).

**2.18 Exercises.** (1) Let $A$ be an operator on a Banach space $X$ having nonempty resolvent set $\rho(A)$. Show that for a linear operator $B : D(A) \to X$ the following assertions are true.

(i) $B$ is $A$-bounded if and only if $B \in \mathcal{L}(X_1, X)$ if and only if $BR(\lambda, A) \in \mathcal{L}(X)$ for some/all $\lambda \in \rho(A)$.

(ii) $B$ is $A$-compact if and only if $BR(\lambda, A)$ is compact for some/all $\lambda \in \rho(A)$.

(2) Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and let $(B, D(B))$ be a closed operator on $X$. If there exists

(i) a $(T(t))_{t\geq 0}$-invariant dense subspace $D \subset D(A) \cap D(B)$ such that the map $t \mapsto BT(t)x$ is continuous for all $x \in D$ and

(ii) constants $t_0 > 0$ and $q \geq 0$ such that

$$\int_0^{t_0} \|BT(t)x\|\, dt \leq q\|x\| \qquad \text{for all } x \in D,$$

then $B$ is $A$-bounded with $A$-bound less than or equal to $q$. (Hint: Use the formula

$$BR(\lambda, A)x = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n t_0} \int_0^{t_0} \mathrm{e}^{-\lambda r} BT(r)T(nt_0)x\, dr, \qquad x \in D, \tag{2.12}$$

in order to show that $BR(\lambda, A)$ is bounded on $D$. Then it follows from Proposition B.2.(i) and Theorem B.6 that $D(A) \subseteq D(B)$. Finally, take in (2.12) the limit as $\lambda \to \infty$ to estimate the $A$-bound of $B$. Compare this with part (iv) of the proof of Theorem 3.14 on p. 197.)

(3) Assume $(A, D(A))$ to generate an analytic semigroup of angle $\delta \in (0, \pi]$. Show that in the situation of Theorem 2.10 the semigroup generated by $A + B$ is analytic of angle at least $\delta$.

(4) Take the operators $Af := f''$ and $Bf := f'$ with maximal domains in $X := \mathrm{C}_0(\mathbb{R})$. Show that $A+\alpha B-\beta$ generates a contraction semigroup for $\alpha \in \mathbb{R}$, $\beta \geq 0$. Can one replace the constants $\alpha$ and $\beta$ by certain functions?

(5) Let $(A, D(A))$ be the generator of a contraction semigroup on the Banach space $X$.

(i) If $(B, D(B))$ is dissipative, then $(A+B, D(A)\cap D(B))$ is again dissipative.

(ii) If $B$ is dissipative and bounded, then $(A+B, D(A))$ generates a contraction semigroup.

(6) Take $X := c_0$ and define $A(x_n) := (inx_n)$ with domain $D(A)$ consisting of all finite sequences.

(i) Show that the closure $\overline{A}$ of $A$ generates a group of isometries on $X$.

(ii) Construct a different semigroup generator $(B, D(B))$ on $X$ such that $A$ and $B$ coincide on $D(A)$.

(7) Let $B$ be an operator on a Banach space $X$ such that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(B)$ satisfying $\lim_{n\to\infty} \|R(\lambda_n, B)\| = 0$. Show that $B$ is $A := B^2$-bounded with $A$-bound $a_0 = 0$. (Hint: Compute $B^2R(\lambda, B)$ using the formula $BR(\lambda, B) = \lambda R(\lambda, B) - I$.)

# 3. More Perturbations

The perturbation results in Sections 1 and 2 are based on an explicit series representation for the resolvent of $A+B$ (see (2.5)) and some version of the Hille–Yosida theorem. However, this series does not allow one to estimate all powers of the resolvent of $A+B$ as needed in the general form of the Hille–Yosida theorem. Therefore, this approach is limited to cases where an estimate of the resolvent and not of all its powers was sufficient, i.e., to perturbations of contractive and analytic semigroups.

In Sections 3.a and 3.c below we will therefore use a different approach based on the variation of parameters formulas (IE) and (IE$^*$), respectively, from Section 1. While both work equally well for bounded perturbations, they allow different generalizations to the unbounded case. In fact, an extension of (IE) gives rise to "Desch–Schappacher" perturbations in Section 3.a. On the other hand, in Section 3.c we will use a generalization of (IE$^*$) to study "Miyadera–Voigt" perturbations.

Section 3.b uses the results of Section 3.a in order to determine the relation between the generators of semigroups being "close" to each other at $t=0$. Finally, in Section 3.d we compare additive and so-called "multiplicative" perturbations of generators.

The results in this section are more advanced than the preceding ones and may therefore be skipped at a first reading.

## *a. The Perturbation Theorem of Desch–Schappacher*

As already mentioned above, we now will use a "direct" approach to the perturbation problem based on a generalization of the *variation of parameters formula* (IE) from Corollary 1.7.

We start by considering a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and the corresponding extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ on $X_{-1}$, cf. Section II.5.a. Then, we recall from Proposition A.7 that for each $t_0>0$ the space

$$\mathfrak{X}_{t_0} := \mathrm{C}\big([0,t_0],\mathcal{L}_s(X)\big)$$

of all strongly continuous, $\mathcal{L}(X)$-valued functions equipped with the norm

$$\|F\|_\infty := \sup_{r\in[0,t_0]} \|F(r)\|$$

is a Banach space. On this space we define for a given operator $B\in\mathcal{L}(X,X_{-1})$ the *abstract Volterra operator* $V_B$ (cf. Definition 1.8) by

$$F\mapsto V_BF \quad\text{with}\quad (V_BF)(t) := \int_0^t T_{-1}(t-r)BF(r)\,dr \in \mathcal{L}(X,X_{-1})$$

for $0\leq t\leq t_0$ and $F\in\mathfrak{X}_{t_0}$, where the integral converges in $X_{-1}$ in the strong sense. We now assume that for each $F\in\mathfrak{X}_{t_0}$

(1) the range $\operatorname{rg}((V_BF)(t))$ is contained in $X$ for all $t\in[0,t_0]$,

(2) the map $[0,t_0]\ni t\mapsto (V_BF)(t)$ is strongly continuous on $X$.

Then, by assumption (1) and Corollary B.7, the operator $(V_B F)(t) : X \to X$ is bounded; hence by (2) the map $V_B : \mathfrak{X}_{t_0} \to \mathfrak{X}_{t_0}$ is well-defined, and we assume that

(3) $V_B$ defines a bounded operator on $\mathfrak{X}_{t_0}$ satisfying $\|V_B\| < 1$.

Using this notation, we introduce the class of *Desch–Schappacher perturbations* (of the generator $A$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$) consisting of those operators $B$ satisfying assumptions (1)–(3), i.e., we define

$$\mathcal{S}_{t_0}^{\mathrm{DS}} := \big\{ B \in \mathcal{L}(X, X_{-1}) : V_B \in \mathcal{L}(\mathfrak{X}_{t_0}) \text{ and } \|V_B\| < 1 \big\}.$$

Later, see Corollaries 3.3, 3.4, and 3.6, we will give several sufficient conditions for an operator $B$ to belong to $\mathcal{S}_{t_0}^{\mathrm{DS}}$. Combined with the following general theorem, this will open the door for a treatment of so-called "boundary perturbations" (see, e.g., Example 3.5, Theorem VI.6.1, and [Gre87].)

**3.1 Theorem.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. If $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ for some $t_0 > 0$, then the operator*

$$(A_{-1}+B)_{|X} \qquad \textit{with domain} \qquad D\big((A_{-1}+B)_{|X}\big) := \big\{ x \in X : A_{-1}x + Bx \in X \big\}$$

*generates a strongly continuous semigroup on $X$.*

Proof. For brevity, we write in the sequel $V := V_B$. Since by assumption the operator $V$ satisfies $\|V\| < 1$, it follows that $I - V \in \mathcal{L}(\mathfrak{X}_{t_0})$ is invertible. Therefore, we can define the operator-valued function

$$S(\cdot) := (I - V)^{-1} T(\cdot),$$

i.e., $S(\cdot)$ is the unique solution in $\mathfrak{X}_{t_0}$ of the equation

$$S(t) = T(t) + \int_0^t T_{-1}(t-r) B S(r)\, dr, \qquad t \in [0, t_0]. \tag{3.1}$$

In analogy to the terminology from Section 1, we call this identity the *variation of parameters formula*. We now proceed in several steps by verifying the following assertions.

(i) The operators $S(t)$ satisfy the identity

$$S(s+t) = S(s)S(t) \quad \text{for all } 0 \leq s, t \leq s+t \leq t_0.$$

(ii) If $t \geq 0$, take some $n \in \mathbb{N}$ satisfying $t/n \leq t_0$. Then the operator

$$S(t) := S(t/n)^n \tag{3.2}$$

is well-defined, and $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on $X$.

(iii) The semigroup $(S(t))_{t\geq 0}$ satisfies (3.1) for all $t \geq 0$.

(iv) The resolvent set $\rho((A_{-1} + B)_{|X})$ is nonempty.

(v) The generator $C$ of $(S(t))_{t\geq 0}$ is given by $C = (A_{-1} + B)_{|X}$.

In order to show (i), we first claim that

$$[V^nT](s+t) = \sum_{k=0}^{n} \left[V^{n-k}T\right](s) \cdot \left[V^kT\right](t) \tag{3.3}$$

for all $n \in \mathbb{N}_0$ and $s, t \in [0, t_0]$ satisfying $s + t \le t_0$. Since $V^0 = I$, we see that equation (3.3) is trivially satisfied for $n = 0$. If it is true for some $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n+1} \left[V^{n+1-k}T\right](s) \cdot \left[V^kT\right](t) \\
&\quad = \sum_{k=0}^{n} \int_0^s T_{-1}(s-r)B\left[V^{n-k}T\right](r)\,dr \cdot \left[V^kT\right](t) \\
&\qquad + T(s)\int_0^t T_{-1}(t-r)B\,[V^nT](r)\,dr \\
&\quad = \int_0^s T_{-1}(s-r)B \sum_{k=0}^{n} \left[V^{n-k}T\right](r) \cdot \left[V^kT\right](t)\,dr \\
&\qquad + \int_0^t T_{-1}(s+t-r)B\,[V^nT](r)\,dr \\
&\quad = \int_0^s T_{-1}(s-r)B\,[V^nT](r+t)\,dr + \int_0^t T_{-1}(s+t-r)B\,[V^nT](r)\,dr \\
&\quad = \int_t^{s+t} T_{-1}(s+t-r)B\,[V^nT](r)\,dr + \int_0^t T_{-1}(s+t-r)B\,[V^nT](r)\,dr \\
&\quad = \left[V^{n+1}T\right](s+t),
\end{aligned}$$

which, by induction, proves (3.3).

We now observe that for all $t \in [0, t_0]$ the point evaluation $\delta_t : \mathfrak{X}_{t_0} \to \mathcal{L}(X)$ is a contraction. Moreover, we have $\|V\| < 1$, and therefore the inverse of $I - V$ is given by the Neumann series. Hence, we obtain

$$S(t) = \delta_t\left(\sum_{n=0}^{\infty} V^nT\right) = \sum_{n=0}^{\infty} [V^nT](t), \qquad t \in [0, t_0]. \tag{3.4}$$

From the estimate

$$\|\,[V^nT](t)\| = \|\delta_t V^nT\| \le \|V\|^n \cdot \|T\|$$

we see that the second series in (3.4) converges absolutely in norm. Therefore, we conclude, by using the Cauchy product and formula (3.3), that

$$\begin{aligned}
S(s)S(t) &= \sum_{n=0}^{\infty} [V^nT](s) \cdot \sum_{n=0}^{\infty} [V^nT](t) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[V^{n-k}T\right](s)\left[V^kT\right](t) \\
&= \sum_{n=0}^{\infty} [V^nT](s+t) = S(s+t),
\end{aligned}$$

which proves (i).

(ii) By equation (3.1), we have $S(0) = T(0) = I$. In order to show that $S(t)$ is well-defined, we take $t \geq 0$ and $m, n \in \mathbb{N}$ such that ${}^t/_n, {}^t/_m \leq t_0$. Then (i) implies

$$S({}^t/_n)^n = \big([S({}^1/_m {}^t/_n)]^m\big)^n = \cdots = \big([S({}^1/_n {}^t/_m)]^n\big)^m = S({}^t/_m)^m.$$

Hence, the definition of $S(t)$ in (3.2) is independent of the special choice of $n \in \mathbb{N}$, which shows the first claim. Next, we verify the semigroup property and choose for $s, t \geq 0$ an integer $n \in \mathbb{N}$ such that ${}^{s+t}/_n \leq t_0$. Then, again by (i), we obtain

$$\begin{aligned} S(s+t) &= S({}^{s+t}/_n)^n = \big(S({}^s/_n)S({}^t/_n)\big)^n \\ &= S({}^s/_n)^n S({}^t/_n)^n = S(s)S(t). \end{aligned}$$

Finally, since $S(\cdot)$ belongs to the space $\mathcal{X}_{t_0}$, it is clear that the operator family $(S(t))_{t\geq 0}$ is strongly continuous, and (ii) is proved.

We proceed by verifying (iii). For $t = nt_0 + \tau$, $n \in \mathbb{N}$, and $\tau \in [0, t_0)$, we have

$$\begin{aligned} \int_0^t T_{-1}(t-r)BS(r)\,dr &= \sum_{k=0}^{n-1} \int_{kt_0}^{(k+1)t_0} T_{-1}(t-r)BS(r)\,dr \\ &\quad + \int_{nt_0}^{t} T_{-1}(t-r)BS(r)\,dr \\ &= \sum_{k=0}^{n-1} T\big(t-(k+1)t_0\big) \int_0^{t_0} T_{-1}(t_0-r)BS(r)\,dr \cdot S(kt_0) \\ &\quad + \int_0^{\tau} T_{-1}(\tau-r)BS(r)\,dr \cdot S(nt_0). \end{aligned}$$

From (3.1), it then follows that

$$\begin{aligned} \int_0^t T_{-1}(t-r)BS(r)\,dr &= \sum_{k=0}^{n-1} T\big(t-(k+1)t_0\big)\big(S(t_0)-T(t_0)\big)S(kt_0) \\ &\quad + \big(S(\tau)-T(\tau)\big)S(nt_0) \\ &= S(t)-T(t). \end{aligned}$$

This proves (iii).

(iv) We first claim that $R(\lambda, A_{-1})B$ is bounded and satisfies $\|R(\lambda, A_{-1})B\| < 1$ for $\lambda$ sufficiently large. To this end we choose constants $M \geq 1$ and $w \geq 0$ such that $\|T(t)\| \leq M e^{wt}$. Then for $\lambda > w$ the resolvent of $A_{-1}$ is given by the integral representation, and we obtain

$$\begin{aligned} R(\lambda, A_{-1})B &= \int_0^{\infty} e^{-\lambda r} T_{-1}(r)B\,dr \\ &= \sum_{n=0}^{\infty} e^{-\lambda n t_0} T_{-1}(nt_0) \cdot \int_0^{t_0} e^{-\lambda r} T_{-1}(r)B\,dr \\ &= \sum_{n=0}^{\infty} e^{-\lambda n t_0} T(nt_0) \cdot \big[VF_\lambda\big](t_0), \end{aligned}$$

where $F_\lambda \in \mathfrak{X}_{t_0}$ is defined by $F_\lambda(r) := \mathrm{e}^{-\lambda(t_0-r)}I$. Next, we estimate

$$\begin{aligned}\|R(\lambda, A_{-1})B\| &\le \left(\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda n t_0}\|T(nt_0)\|\right) \cdot \|V\| \cdot \|F_\lambda\|_\infty \\ &\le \|V\| + M\sum_{n=1}^{\infty} \mathrm{e}^{(w-\lambda)nt_0} = \|V\| + \frac{M\mathrm{e}^{(w-\lambda)t_0}}{1-\mathrm{e}^{(w-\lambda)t_0}}.\end{aligned} \tag{3.5}$$

Since by assumption $\|V\| < 1$, this implies that

$$\mathrm{r}\big(R(\lambda, A_{-1})B\big) \le \|R(\lambda, A_{-1})B\| < 1 \tag{3.6}$$

for $\lambda$ sufficiently large. Now, if $\lambda \in \rho(A) = \rho(A_{-1})$, we have

$$\lambda - (A_{-1} + B)_{|X} = (\lambda - A)\big(I - R(\lambda, A_{-1})B\big), \tag{3.7}$$

and hence $\lambda - (A_{-1}+B)_{|X}$ is invertible whenever $\lambda \in \rho(A)$ and $1 \in \rho(R(\lambda, A_{-1})B)$. This proves (iv).

(v) We finish the proof by verifying that the generator $C$ of $(S(t))_{t\ge0}$ is given by $(A_{-1}+B)_{|X}$. To this end, we apply the Laplace transform to the Variation of Parameters Formula (3.1) and obtain, using the Convolution Theorem C.17,

$$R(\lambda, C) = R(\lambda, A) + R(\lambda, A_{-1})BR(\lambda, C) \tag{3.8}$$

for all $\lambda > \max\{\omega_0(A), \omega_0(C)\}$. This yields

$$\big(I - R(\lambda, A_{-1})B\big)R(\lambda, C) = R(\lambda, A) \tag{3.9}$$

and therefore

$$\begin{aligned}I &= (\lambda - A)\big(I - R(\lambda, A_{-1})B\big)R(\lambda, C) \\ &= \big(\lambda - (A_{-1} + B)_{|X}\big)R(\lambda, C).\end{aligned}$$

Hence, $R(\lambda, C)$ is a right inverse of $\lambda - (A_{-1} + B)_{|X}$, which shows that $C \subseteq (A_{-1}+B)_{|X}$. However, by (iv), we know that $\lambda-(A_{-1}+B)_{|X}$ and $\lambda - C$ are both invertible for $\lambda$ sufficiently large, and we obtain (v) (use Exercise IV.1.21.(5)). ☐

From the above proof we immediately deduce the following representation formulas for the semigroup $(S(t))_{t\ge0}$ generated by the perturbed operator.

**3.2 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\ge0}$ and let $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ for some $t_0 > 0$. Then the semigroup $(S(t))_{t\ge0}$ generated by $(A_{-1} + B)_{|X}$ is given by*

(i) *the variation of parameters formula*

$$S(t) = T(t) + \int_0^t T_{-1}(t-r)BS(r)\,dr, \quad t \ge 0, \tag{3.10}$$

(ii) *and by the Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \quad t \ge 0, \tag{3.11}$$

*where $S_0(t) := T(t)$ and*

$$S_n(t) := \int_0^t T_{-1}(t-r)BS_{n-1}(r)\,dr. \tag{3.12}$$

*Here, the series (3.11) converges in $\mathcal{L}(X)$ uniformly on compact intervals of $\mathbb{R}_+$ (cf. Exercise 3.8.(3)), while the integral in (3.12) is defined in the strong operator topology in $X_{-1}$.*

The Perturbation Theorem 3.1 is rather abstract, and one might be uncertain how to verify the property $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ in concrete examples. We therefore present some variations of Theorem 3.1 imposing simpler conditions on the operator $B$ implying $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ and hence that $(A_{-1}+B)_{|X}$ is a generator on $X$.

**3.3 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exists $t_0 > 0$ and $q \in [0,1)$ such that*

(i) $\displaystyle\int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\,dr \in X$ *and*

(ii) $\displaystyle\left\|\int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\,dr\right\| \leq q\,\|f\|_\infty$

*for all continuous functions $f \in \mathrm{C}([0,t_0],X)$. Then $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, and therefore $(A_{-1}+B)_{|X}$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $X$.*

Proof. We start by showing that $V := V_B$ defines a linear operator on $\mathfrak{X}_{t_0}$. To this end, we define for $f \in \mathrm{C}([0,t_0],X)$ and $t \in [0,t_0]$ the function $f_t : [0,t_0] \to X$ by

$$f_t(r) := \begin{cases} f(0) & \text{if } 0 \leq r \leq t_0 - t, \\ f(r + t - t_0) & \text{if } t_0 - t \leq r \leq t_0. \end{cases}$$

Then $f_t \in \mathrm{C}([0,t_0],X)$, and one easily verifies that

$$\int_0^t T_{-1}(t-r)Bf(r)\,dr = \int_0^{t_0} T_{-1}(t_0-r)Bf_t(r)\,dr - \int_t^{t_0} T_{-1}(r)Bf(0)\,dr. \tag{3.13}$$

Since by Lemma II.1.3

$$\int_t^{t_0} T_{-1}(r)Bf(0)\,dr = T(t)\int_0^{t_0-t} T_{-1}(r)Bf(0)\,dr \in D(A_{-1}) = X,$$

this shows that

$$\int_0^t T_{-1}(t-r)Bf(r)\,dr \in X \qquad \text{for all } t \in [0,t_0].$$

Therefore, the function $g : [0,t_0] \to X$ given by

$$g(t) := \int_0^t T_{-1}(t-r)Bf(r)\,dr$$

is well-defined. We now claim that $g$ is continuous. Indeed, applying Lemma II.1.3 to (3.13) and using the fact that the graph norm $\|\cdot\|_{A_{-1}}$ on $D(A_{-1}) = X$ is equivalent to $\|\cdot\|$, we obtain

$$\begin{aligned}\|g(t)-g(s)\| &\leq \left\|\int_0^{t_0} T_{-1}(t_0-r)B\big(f_t(r)-f_s(r)\big)\,dr\right\| \\ &\quad + \left\|\int_s^t T_{-1}(r)Bf(0)\,dr\right\| \\ &\leq q\,\|f_t - f_s\|_\infty + K\cdot\left(\left\|\int_s^t T_{-1}(r)Bf(0)\,dr\right\|_{-1}\right. \\ &\qquad\qquad \left. + \big\|\big(T_{-1}(t)-T_{-1}(s)\big)Bf(0)\big\|_{-1}\right)\end{aligned}$$

for all $s, t \in [0, t_0]$ and a suitable constant $K > 0$. Since $(T_{-1}(t))_{t\geq 0}$ is strongly continuous and $f$ is uniformly continuous on the compact interval $[0, t_0]$, this implies

$$\lim_{s\to t} \|g(t) - g(s)\| = 0,$$

i.e., $g$ is continuous. Now, by choosing $f := F(\cdot)x$ for $F \in \mathfrak{X}_{t_0}$ and $x \in X$, we conclude that the conditions (1) and (2) at the beginning of Section 3.a are satisfied, i.e., $V$ defines a linear operator on $\mathfrak{X}_{t_0}$.

Next, we show that $\|V\| \leq q$. To this end, we write $f \in \mathrm{C}([0, t_0], X)$ as $f = \widetilde{f}_\delta + h_\delta$, where

$$h_\delta(r) := \begin{cases} (1 - {}^r\!/_\delta)\, f(0) & \text{if } r \in [0, \delta), \\ 0 & \text{if } r \in [\delta, t_0] \end{cases}$$

for some $\delta \in (0, t_0)$. Then $\widetilde{f}_\delta$, $h_\delta$ are continuous and $\widetilde{f}_\delta(0) = 0$. Hence, using again the fact that on $X$ the graph norm $\|\cdot\|_{A_{-1}}$ is equivalent to $\|\cdot\|$, we obtain from (ii) and (3.13) for $\delta < t < t_0$ the estimate

$$\begin{aligned}
&\left\| \int_0^t T_{-1}(t-r)Bf(r)\, dr \right\| \\
&\qquad \leq \left\| \int_0^t T_{-1}(t-r)B\widetilde{f}_\delta(r)\, dr \right\| + \left\| \int_0^t T_{-1}(t-r)Bh_\delta(r)\, dr \right\| \\
&\qquad \leq q \left\| \widetilde{f}_\delta \right\|_\infty + K\Bigg( \left\| \int_0^t T_{-1}(t-r)Bh_\delta(r)\, dr \right\|_{-1} \\
&\qquad\quad + \left\| A_{-1} \int_0^t T_{-1}(t-r)Bh_\delta(r)\, dr \right\|_{-1} \Bigg) \\
&\qquad \leq q \left\| \widetilde{f}_\delta \right\|_\infty + K \left\| \int_0^\delta T_{-1}(t-r)\, (1 - {}^r\!/_\delta)\, Bf(0)\, dr \right\|_{-1} \\
&\qquad\quad + K \left\| T_{-1}(t)Bf(0) - \frac{1}{\delta} \int_0^\delta T_{-1}(t-r)Bf(0)\, dr \right\|_{-1}.
\end{aligned}$$

By taking the limit as $\delta \downarrow 0$, we then conclude that

$$\left\| \int_0^t T_{-1}(t-r)Bf(r)\, dr \right\| \leq q\, \|f\|_\infty$$

for all $f \in \mathrm{C}([0, t_0], X)$ and all $t \in [0, t_0]$. Clearly, this implies $\|V\| \leq q < 1$, i.e., $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, and the assertion follows from Theorem 3.1. ☐

If condition (i) in Corollary 3.3 is satisfied not only for continuous functions but also for $\mathrm{L}^p$-functions, then the estimate in 3.3.(ii) is superfluous. More precisely, we can prove the following result.

**3.4 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and let $B \in \mathcal{L}(X, X_{-1})$. Moreover, assume that there exist $t_0 > 0$ and $p \in [1, \infty)$ such that*

$$\int_0^{t_0} T_{-1}(t_0 - r)Bf(r)\, dr \in X$$

*for all functions $f \in \mathrm{L}^p([0, t_0], X)$. Then $(A_{-1} + B)_{|X}$ generates a strongly continuous semigroup on $X$.*

PROOF. We first define, for $t \in [0, t_0]$ and $f \in \mathrm{L}^p([0,t], X)$, the function $\widetilde{f}_t : [0, t_0] \to X$ by

$$\widetilde{f}_t(r) := \begin{cases} 0 & \text{if } 0 \le r \le t_0 - t, \\ f(r + t - t_0) & \text{if } t_0 - t < r \le t_0. \end{cases}$$

Then $\widetilde{f}_t \in \mathrm{L}^p([0,t_0], X)$, and one has

$$\int_0^t T_{-1}(t-r)Bf(r)\,dr = \int_0^{t_0} T_{-1}(t_0 - r)B\widetilde{f}_t(r)\,dr \in X. \tag{3.14}$$

Hence, the operator $Q_t \in \mathcal{L}\left(\mathrm{L}^p([0,t], X), X_{-1}\right)$ defined by

$$Q_t f := \int_0^t T_{-1}(t-r)Bf(r)\,dr, \qquad f \in \mathrm{L}^p\big([0,t], X\big),$$

satisfies $\mathrm{rg}(Q_t) \subseteq X$ for all $t \in [0, t_0]$. Since $X$ is continuously embedded in $X_{-1}$, the closed graph theorem then implies that $Q_t$ is bounded from $\mathrm{L}^p([0,t], X)$ to $X$ (cf. Corollary B.7). Hence, there exists a constant $M \ge 0$ such that

$$\|Q_{t_0} g\|_X \le M\, \|g\|_p \qquad \text{for all } g \in \mathrm{L}^p\big([0,t_0], X\big),$$

where $\|g\|_p := \left(\int_0^{t_0} \|g(r)\|^p\,dr\right)^{1/p}$. Combining this estimate with (3.14), we obtain

$$\begin{aligned} \left\| \int_0^t T_{-1}(t-r)Bf(r)\,dr \right\|_X^p &= \left\| Q_{t_0}\widetilde{f}_t \right\|_X^p \le M^p \left\| \widetilde{f}_t \right\|_p^p \\ &= M^p \int_0^t \|f(r)\|^p\,dr \le t\, M^p \cdot \|f\|_\infty^p \end{aligned} \tag{3.15}$$

for all $f \in \mathrm{C}([0,t], X)$. Hence, by choosing $t < 1/M^p$, we see from (3.14) and (3.15) that the assumptions of Corollary 3.3 are satisfied for $t_0 = t$, and the assertion follows. □

**3.5 Example.** To give a typical application of the previous corollary, we take the Banach space $X := \mathrm{L}^p[a,b]$, $1 \le p < \infty$, and the first derivative

$$Ch := h' \qquad \text{with domain} \qquad D(C) := \left\{ h \in \mathrm{W}^{1,p}[a,b] : h(b) = \Phi(h) \right\},$$

where $\Phi \in \mathrm{L}^p[a,b]'$ is a bounded linear functional. See also Section VI.6, where similar operators are treated on spaces of vector-valued continuous functions.

We claim that $C$ is the generator of a strongly continuous semigroup on $X$. In order to verify this assertion, we consider $C$ as a perturbation of the generator

$$Ah := h' \qquad \text{with domain} \qquad D(A) := \left\{ h \in \mathrm{W}^{1,p}[a,b] : h(b) = 0 \right\}$$

of the nilpotent translation semigroup $(T(t))_{t\ge 0}$ given by

$$\big(T(t)h\big)(s) := \begin{cases} h(s+t) & \text{for } s + t \le b, \\ 0 & \text{for } s + t > b, \end{cases}$$

cf. Paragraph I.4.17. We then define the operator

$$B := -A_{-1}(\mathbb{1} \otimes \Phi) \in \mathcal{L}(X, X_{-1}),$$

that is, $Bh := -\Phi(h) \cdot A_{-1}\mathbb{1}$ for $h \in X$. Then our operator $C$ coincides with the part of $A_{-1} + B$ in $X$, i.e.,

$$C = (A_{-1} + B)_{|X}.$$

By Corollary 3.4, $C$ is a generator if we can show that

$$\int_0^{b-a} T_{-1}(b-a-r)A_{-1}(\mathbb{1} \otimes \Phi)f(r)\,dr \in X,$$

or, equivalently,

$$\int_0^{b-a} T(b-a-r)(\mathbb{1} \otimes \Phi)f(r)\,dr \in D(A)$$

for all $f \in \mathrm{L}^p([0, b-a], X)$. In fact, for such $f$ we have

$$\begin{aligned}\int_0^{b-a} T(b-a-r)(\mathbb{1} \otimes \Phi)f(r)\,dr &= \int_0^{b-a} \Phi\big(f(r)\big) \cdot T(b-a-r)\mathbb{1}(\cdot)\,dr \\ &= \int_{\cdot-a}^{b-a} \Phi\big(f(r)\big)\,dr =: g(\cdot).\end{aligned}$$

Since $\Phi \circ f \in \mathrm{L}^p[0, b-a]$, this implies $g \in \mathrm{W}^{1,p}[a,b]$ and $g(b) = 0$, i.e., $g \in D(A)$. Hence, the operator $C$ is a generator on $X$.

If the perturbing operator $B \in \mathcal{L}(X, X_{-1})$ satisfies the assumptions of Corollary 3.3, then it even follows that $BP \in \mathcal{S}^{\mathrm{DS}}_{t_0}$ for all $P \in \mathcal{L}(X)$ with $\|P\| \le 1$. If $B$ also satisfies the assumptions of Corollary 3.4, then $BP \in \mathcal{S}^{\mathrm{DS}}_{t_0}$ for all $P \in \mathcal{L}(X)$. Since $\operatorname{rg}(BP) \subseteq \operatorname{rg}(B)$, this indicates that an appropriate condition on $\operatorname{rg}(B)$ might already force the perturbation $B$ to belong to $\mathcal{S}^{\mathrm{DS}}_{t_0}$. Such a condition can be obtained by using the "extrapolated" Favard space $F_0$ (see Section II.5.b).

**3.6 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\ge0}$ on a Banach space $X$ and let $B \in \mathcal{L}(X, X_{-1})$ satisfy $\operatorname{rg}(B) \subseteq F_0$. Then $(A_{-1} + B)_{|X}$ generates a strongly continuous semigroup on $X$.*

Proof. By Corollary 3.4, it suffices to show that

$$\int_0^1 T_{-1}(1-r)Bf(r)\,dr \in X \tag{3.16}$$

for all $f \in \mathrm{L}^1([0,1], X)$. Hence, we take such a function $f$ and choose a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}^1([0,1], X)$ converging to $f$ in $\mathrm{L}^1([0,1], X)$. For the continuously differentiable functions $Bf_n : [0,1] \to X_{-1}$, it follows from Corollary VI.7.6 that

$$\int_0^1 T_{-1}(1-r)Bf_n(r)\,dr \in D(A_{-1}) = X \qquad \text{for all } n \in \mathbb{N}.$$

Moreover, for $h > 0$, we obtain

$$\begin{aligned}\frac{T_{-1}(h)-I}{h} &\int_0^1 T_{-1}(1-r)Bf_n(r)\,dr \\ &= \int_0^1 T_{-1}(1-r)\,\frac{T_{-1}(h)Bf_n(r)-Bf_n(r)}{h}\,dr.\end{aligned} \tag{3.17}$$

Observe now that by the closed graph theorem, the operator $B$ is bounded between $X$ and $F_0$, i.e., there exists a constant $K \geq 0$ such that $\|Bx\|_{F_0} \leq K\,\|x\|_X$. Hence, taking in (3.17) the limit as $h \downarrow 0$, we obtain

$$\begin{aligned}&\left\| A_{-1} \int_0^1 T_{-1}(1-r)Bf_n(r)\,dr \right\|_{-1} \\ &\qquad \leq M \varlimsup_{h\downarrow 0} \int_0^1 \tfrac{1}{h}\,\|(T_{-1}(h)-I)Bf_n(r)\|_{-1}\,dr \\ &\qquad \leq M \int_0^1 \|Bf_n(r)\|_{F_0}\,dr \\ &\qquad \leq MK \int_0^1 \|f_n(r)\|\,dr = MK\,\|f_n\|_1\end{aligned} \tag{3.18}$$

for $M := \sup_{r\in[0,1]} \|T_{-1}(r)\|$. By replacing $f_n$ in (3.18) by $f_n - f_m$, we see that the sequence $\left(A_{-1}\int_0^1 T_{-1}(1-r)Bf_n(r)\,dr\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X_{-1}$. Since $A_{-1}$ is closed, this implies that

$$\lim_{n\to\infty} \int_0^1 T_{-1}(1-r)Bf_n(r)\,dr = \int_0^1 T_{-1}(1-r)Bf(r)\,dr \in D(A_{-1}) = X,$$

therefore proving (3.16). □

In Section 3.b we will see that generators of the form $C = (A_{-1}+B)_{|X}$ with $\operatorname{rg}(B) \subseteq F_0$ are exactly those yielding semigroups that are "close" to the unperturbed semigroup for small $t$.

**3.7 Remark.** Since the above perturbation results are all based on Theorem 3.1, both representation formulas from Corollary 3.2 hold for the semigroup generated by $(A_{-1}+B)_{|X}$. See also Exercise 3.8.(3).

**3.8 Exercises.** (1) Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and assume that $B \in \mathcal{L}(X)$. Show that $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$ if $t_0 > 0$ is sufficiently small.

(2) If $A$ generates an analytic semigroup and $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, then the semigroup generated by $(A_{-1}+B)_{|X}$ is analytic as well. (Hint: Use (3.6) and (3.7) in order to verify the resolvent estimate (4.10) in Theorem II.4.6 for $(A_{-1}+B)_{|X}$.)

(3) Show that the Dyson–Phillips series (3.11) converges in $\mathcal{L}(X)$ uniformly on compact $t$-intervals to the semigroup generated by $(A_{-1}+B)_{|X}$. (Hint: Since the assertion is true for the $t$-interval $[0,t_0]$, it suffices to show that uniform convergence on the interval $[0,t_1]$ implies uniform convergence on $[0,2t_1]$. To this end, prove first (by induction as in the proof of (3.3), Theorem 3.1) that $S_n(2t) = \sum_{k=0}^{n} S_{n-k}(t)\cdot S_k(t)$. Then use this fact to compute the Cauchy product for $S(t)S(t) = S(2t)$.)

(4) Let $A$ be the generator of a strongly continuous semigroup on a Banach space $X$. In the proof of Corollary 3.6, we showed that

$$\int_0^1 T_{-1}(1-r)Bf(r)\,dr \in X \qquad \text{for all } f \in \mathrm{L}^1\big([0,1],X\big)$$

if $\mathrm{rg}(B) \subseteq F_0$. Show that also the converse statement holds, i.e., if the integral $\int_0^1 T_{-1}(1-r)Bf(r)\,dr \in X$ for all $f \in \mathrm{L}^1([0,1],X)$, then $\mathrm{rg}(B) \subseteq F_0$. (Hint: Consider (3.15) in the proof of Corollary 3.4 for $p=1$ and $f(r) \equiv x$, $x \in X$.)

(5) Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on a Banach space $X$ and consider the nilpotent left translation semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^p([-1,0],X)$ for some $1 \leq p < \infty$.

(i) Show that the generator of $(T(t))_{t\geq 0}$ is

$$A_0 f := f' \text{ with } D(A_0) := \big\{f \in \mathrm{W}^{1,p}\big([-1,0],X\big) : f(0)=0\big\}.$$

(ii) Consider the product space $\mathcal{X} := X \times \mathrm{L}^p([-1,0],X)$ and show that the operator $(\mathcal{A}_0, D(\mathcal{A}_0))$ defined by

$$\mathcal{A}_0 := \begin{pmatrix} B & 0 \\ 0 & A_0 \end{pmatrix} \qquad \text{with} \qquad D(\mathcal{A}_0) := D(B) \times D(A_0)$$

generates the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ with

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ 0 & T(t) \end{pmatrix}, \quad t \geq 0.$$

(iii) Determine the extrapolated operator $\mathcal{A}_{-1}$ and define

$$\mathcal{B} := -\mathcal{A}_{-1} \cdot \begin{pmatrix} 0 & 0 \\ (\mathbb{1}\otimes Id) & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_{-1}).$$

Show that the part of $\mathcal{A}_{-1} + \mathcal{B}$ in $\mathcal{X}$ is the operator

$$\mathcal{A} := \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix},$$

with

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times \mathrm{W}^{1,p}([-1,0],X) : f(0) = x \right\},$$

where $Af := f'$ for $f \in \mathrm{W}^{1,p}([-1,0],X)$.

(iv) Use Corollary 3.4 to show that $(\mathcal{A}, D(\mathcal{A}))$ is a generator on $\mathcal{X}$.

(v) Compute the corresponding Dyson–Phillips series.

## b. Comparison of Semigroups

As a byproduct of the results of the previous section, we are able to solve the following problem, which appeared first in [Rob77]; see also [BR79, 3.1.5].

*Which semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ satisfy an estimate*

$$\|S(t) - T(t)\| \leq t\,M \tag{3.19}$$

*for some constant $M \geq 0$ and all $t \in [0,1]$?*

In Corollary 1.11 we already saw that (3.19) is true if the generators of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ differ only by a bounded perturbation. The general case is characterized by slightly "more unbounded" perturbations.

**3.9 Theorem.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on a Banach space $X$. Then for a linear operator $(C, D(C))$ the following assertions are equivalent.*

(a) *$(C, D(C))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $X$ such that*

$$\|S(t) - T(t)\| \leq t\, M \quad \text{for all } t \geq 0$$

*and some constant $M \geq 0$.*

(b) *There exists a linear operator $B \in \mathcal{L}(X, X_{-1}^A)$ such that $\mathrm{rg}(B) \subseteq F_0^A$ and*

$$C = (A_{-1} + B)_{|X}.$$

(c) *The domain $D(C)$ is dense in $X$, and there exist constants $w \geq \mathrm{s}(A)$ and $K \geq 0$ such that $(w, \infty) \subset \rho(C)$ and*

$$\lambda^2 \|R(\lambda, C) - R(\lambda, A)\| \leq K$$

*for all $\lambda > w$.*

Proof. We will show that (a) ⇒ (c) ⇒ (b) ⇒ (a). Here we suppose $A$ to be invertible; otherwise we take some $\mu \in \rho(A)$ and replace $A$ and $C$ by $A - \mu$ and $C - \mu$, respectively.

(a) ⇒ (c). If $\widetilde{w} > \max\{0, \omega_0(A), \omega_0(C)\}$, then by increasing the constant $M$ if necessary, we can assume that

$$\|S(t) - T(t)\| \leq t\, M \mathrm{e}^{\widetilde{w} t} \quad \text{for all } t \geq 0.$$

Then, from Theorem II.1.10.(i), we obtain

$$\begin{aligned} \|R(\lambda, C) - R(\lambda, A)\| &= \left\| \int_0^\infty \mathrm{e}^{-\lambda t} \big(S(t) - T(t)\big)\, dt \right\| \\ &\leq M \int_0^\infty t \mathrm{e}^{-(\lambda - \widetilde{w})t}\, dt = \frac{M}{(\lambda - \widetilde{w})^2} \leq \frac{K}{\lambda^2} \end{aligned}$$

for all $\lambda > w := \widetilde{w} + 1$ and $K := M \sup_{\lambda > \widetilde{w}+1} \left[ \lambda^2 / (\lambda - \widetilde{w})^2 \right]$.

(c) ⇒ (b). We first observe that

$$\|\lambda R(\lambda, C)x - x\| \leq \tfrac{K}{\lambda} \|x\| + \|\lambda R(\lambda, A)x - x\|$$

for all $x \in X$ and $\lambda > w$. Hence, by Lemma II.3.4, we conclude that

$$\lim_{\lambda \to \infty} \lambda R(\lambda, C)x = x \quad \text{for all } x \in X. \tag{3.20}$$

Next, for $w < n \in \mathbb{N}$, we define the operators

$$Q_n := n^2 A^{-1} \big(R(n, C) - R(n, A)\big) \in \mathcal{L}(X).$$

Then, for all $x \in D(C)$, we obtain, using (3.20) and the resolvent equation,

$$\begin{aligned} \lim_{n\to\infty} Q_n x &= A^{-1} \cdot \lim_{n\to\infty} n\big(nR(n, C) - I\big)x - \lim_{n\to\infty} n\big(nR(n, A) - I\big)A^{-1}x \\ &= A^{-1} \cdot \lim_{n\to\infty} nR(n, C)Cx - \lim_{n\to\infty} nR(n, A)x = A^{-1}Cx - x. \end{aligned}$$

Since by assumption $\|Q_n\| \le K \cdot \|A^{-1}\|$ and since $D(C)$ is dense in $X$, this implies that

$$\lim_{n\to\infty} Q_n x =: Qx$$

exists for all $x \in X$ and defines an operator $Q \in \mathcal{L}(X)$, which is the unique bounded extension of $A^{-1}C - I$. From

$$(Q_n x)_{n\in\mathbb{N}} \subset D(A), \qquad \lim_{n\to\infty} Q_n x = Qx \quad \text{and} \quad \overline{\lim_{n\to\infty}} \|AQ_n x\| \le K\,\|x\|$$

we conclude, by Exercise II.5.23.(2), that the range $\operatorname{rg}(Q)$ is contained in $F_1$. Hence, by Theorem II.5.15.(ii), the range of $B := A_{-1}Q \in \mathcal{L}(X, X_{-1})$ satisfies $\operatorname{rg} B \subseteq F_0$, and it follows from Corollary 3.6 that the operator

$$\widetilde{C} := (A_{-1} + B)_{|X}$$

is a generator on $X$. Next, we easily verify that $D(C) \subseteq D(\widetilde{C})$ and $Cx = \widetilde{C}x$ for all $x \in D(C)$. Since $\lambda - C$ and $\lambda - \widetilde{C}$ both are invertible for $\lambda$ large, this implies $C = \widetilde{C}$ (use Exercise IV.1.21), which proves (b).

(b) $\Rightarrow$ (a). By Corollary 3.6, we know that $C := (A_{-1} + B)_{|X}$ generates a strongly continuous semigroup $(S(t))_{t\ge0}$ on $X$. Now choose $M \ge 1$ such that $\|T(t)\| \le M$ and $\|S(t)\| \le M$ for all $t \in [0,1]$. Since by the closed graph theorem $A_{-1}^{-1}B$ is bounded from $X$ into $F_1$, we can choose $K \ge 0$ such that $\|A_{-1}^{-1}Bx\|_{F_1} \le K\|x\|$ for all $x \in X$. From Corollary 3.2.(i), we then obtain

$$\begin{aligned}
\|S(t)x - T(t)x\| &= \left\| A_{-1} \int_0^t T(t-r) A_{-1}^{-1} B S(r)x \, dr \right\| \\
&\le M \, \overline{\lim_{h\downarrow 0}} \int_0^t \tfrac{1}{h} \left\| \big(T(h) - I\big)\, A_{-1}^{-1} B S(r) x \right\| dr \\
&\le M \int_0^t \left\| A_{-1}^{-1} B S(r) x \right\|_{F_1} dr \le t\, K M^2 \cdot \|x\|
\end{aligned}$$

for all $x \in X$ and $t \in [0,1]$, which proves (a). □

If in Theorem 3.9 we make an extra assumption on the space $X$, we can strengthen condition (b) considerably.

**3.10 Corollary.** *In the situation of Theorem* 3.9 *assume $X$ to be reflexive. Then each of the conditions* 3.9.(a) *and* 3.9.(c) *implies that $C = A + B$ for some bounded operator $B \in \mathcal{L}(X)$.*

Proof. By Corollary II.5.21 we know that $F_1 = D(A)$, hence $F_0 = X$, and the assertion follows from Corollary B.7 combined with the previous result. □

**3.11 Example.** In Example 3.5 we showed that for each $p \ge 1$ and each $\Phi \in \mathrm{L}^p[a,b]'$ the operator

$$Cf := f' \qquad \text{with domain} \qquad D(C) := \big\{ f \in \mathrm{W}^{1,p}[a,b] : f(b) = \Phi(f) \big\}$$

generates a strongly continuous semigroup $(S_\Phi(t))_{t\ge0}$ on the Banach space $X := \mathrm{L}^p[a,b]$. We now denote by $(T(t))_{t\ge0}$ the nilpotent (left) translation semigroup on $X$, i.e., $T(t) = S_0(t)$, and assume that $\Phi \ne 0$. Then we obtain from Exercise 3.8.(4), Theorem 3.9, and Corollary 3.10 that

$$\|S_\Phi(t) - T(t)\| \le t\,M \quad \text{for all } t \in [0,1]$$

and a suitable constant $M \ge 0$ if and only if $p = 1$. See also Exercise 3.13.(2).

By imposing an additional condition on the domains of the operators $A$ and $C$, we also arrive at the conclusion of Corollary 3.10.

**3.12 Corollary.** *In the situation of Theorem* 3.9, *assume $D(A) \cap D(C)$ to be dense in $X$. Then each of the conditions* 3.9.(a) *and* 3.9.(c) *implies that $C = A+B$ for some bounded operator $B \in \mathcal{L}(X)$.*

PROOF. First, we assume 3.9.(a) to be true and define operators

$$B_n := n\big(S(1/n) - T(1/n)\big) \in \mathcal{L}(X) \qquad \text{for } n \in \mathbb{N}.$$

Then, for $x \in D(A) \cap D(C)$, we obtain

$$\begin{aligned}\lim_{n\to\infty} B_n x &= \lim_{n\to\infty} n\big(S(1/n) - I\big)x - \lim_{n\to\infty} n\left(T(1/n) - I\right)x \\ &= Cx - Ax.\end{aligned}$$

Since by assumption $\|B_n\| \le M$ and $D(A) \cap D(C)$ is dense in $X$, we conclude that

$$\lim_{n\to\infty} B_n x =: Bx$$

exists for all $x \in X$ and defines an operator $B \in \mathcal{L}(X) \subseteq \mathcal{L}(X, X_{-1})$ that is the unique bounded extension of $C - A$. On the other hand, we know by Theorem 3.9 that there exists $\widetilde{B} \in \mathcal{L}(X, X_{-1})$ such that $C = (A_{-1} + \widetilde{B})_{|X}$. Now, $B$ and $\widetilde{B}$ coincide on the dense subspace $D(A) \cap D(C) \subseteq X$; hence $B = \widetilde{B}$, which implies $C = A + B$.

Since we already know that 3.9.(a) is equivalent to 3.9.(c), the proof is complete. ☐

**3.13 Exercises.** (1) Show that the conditions (a)–(c) in Theorem 3.9 are equivalent to

(d) $D(A') = D(C')$ and $A' - C'$ is bounded from $X_1^{A'}$ to $X'$.

(2) With the notation from Example 3.11 show that $\|S_\Phi(t) - T(t)\| \le t^{1/p} M$ for suitable $M \ge 0$ and $t \in [0, 1]$. (Hint: Inspect the proof in Example 3.5 and then use (3.15) in the proof of Corollary 3.4 to show the assertion.)

## c. The Perturbation Theorem of Miyadera–Voigt

In this section we will give another perturbation result that is based on a generalization of the *variation of parameters formula* (IE$^*$) and, in some sense, is dual to the Desch–Schappacher results from Section 3.a. Our starting point is again the Banach space

$$\mathcal{X}_{t_0} := \mathrm{C}\big([0, t_0], \mathcal{L}_s(X)\big)$$

of all strongly continuous $\mathcal{L}(X)$-valued functions on $[0, t_0]$ equipped with the norm $\|F\|_\infty := \sup_{r\in[0,t_0]} \|F(r)\|$, cf. Proposition A.7. On the Banach space $X$ we consider a strongly continuous semigroup $(T(t))_{t\ge0}$ with generator $A$ and take now a perturbing operator $B \in \mathcal{L}(X_1, X)$. Under these hypotheses, we define an *abstract Volterra operator* $V_B^*$ by

$$F \mapsto V_B^* F \quad \text{with} \quad (V_B^* F)(t) := \int_0^t F(r) B T(t - r)\, dr \in \mathcal{L}(X_1, X)$$

for $0 \le t \le t_0$ and $F \in \mathcal{X}_{t_0}$, where the integral is understood in the strong sense in $X$.

We now assume that

(1) for all $t \in [0, t_0]$ and $F \in \mathfrak{X}_{t_0}$ the map $(V_B^* F)(t) : D(A) \subset X \to X$ can be extended to a bounded operator $\overline{(V_B^* F)(t)} : X \to X$,

(2) for all $F \in \mathfrak{X}_{t_0}$ the map $[0, t_0] \ni t \mapsto \overline{(V_B^* F)(t)}$ is strongly continuous on $X$, and

(3) $\overline{V_B^*}$ gives a bounded operator on $\mathfrak{X}_{t_0}$ satisfying $\left\|\overline{V_B^*}\right\| < 1$.

Using this notation, we introduce the class of *Miyadera–Voigt perturbations* consisting of all operators $B \in \mathcal{L}(X_1, X)$ satisfying the conditions (1)–(3), i.e., we define

$$\mathcal{S}_{t_0}^{\mathrm{MV}} := \left\{ B \in \mathcal{L}(X_1, X) : \overline{V_B^*} \in \mathcal{L}(\mathfrak{X}_{t_0}) \text{ and } \left\|\overline{V_B^*}\right\| < 1 \right\}.$$

In Corollary 3.16 below we will give a more concrete condition implying an operator to be in $\mathcal{S}_{t_0}^{\mathrm{MV}}$. First, however, we state the main perturbation result, which is analogous to Theorem 3.1.

**3.14 Theorem.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. If $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ for some $t_0 > 0$, then the operator*

$$A + B \qquad \text{with domain} \qquad D(A+B) := D(A)$$

*generates a strongly continuous semigroup on $X$.*

Proof. We closely follow the proof of Theorem 3.1 and therefore abbreviate similar calculations.

First, we write $V := \overline{V_B^*}$ and observe that by assumption $\|V\| < 1$. Therefore, $I - V \in \mathcal{L}(\mathfrak{X}_{t_0})$ is invertible, and we can define

$$S(\cdot) := (I - V)^{-1} T(\cdot). \tag{3.21}$$

Then $S(\cdot)$ is the unique solution in $\mathfrak{X}_{t_0}$ of the *variation of parameters formula*

$$S(t)x = T(t)x + \int_0^t S(r)BT(t-r)x\, dr, \tag{3.22}$$

valid for all $x \in D(A)$ and $t \in [0, t_0]$. We now proceed in several steps in order to verify the following assertions.

(i) The operators $S(t)$ satisfy the semigroup property

$$S(s+t) = S(s)S(t) \quad \text{for all } 0 \leq s, t \leq s+t \leq t_0.$$

(ii) If $t \geq 0$, take some $n \in \mathbb{N}$ satisfying $t/n \leq t_0$. Then the operator

$$S(t) := S(t/n)^n$$

is well-defined, and $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on $X$.

(iii) The semigroup $(S(t))_{t\geq 0}$ satisfies the Variation of Parameters Formula (3.22) for all $t \geq 0$ and $x \in D(A)$.

(iv) The resolvent set $\rho(A+B)$ is nonempty.

(v) The generator $C$ of $(S(t))_{t\geq 0}$ is given by $C = A + B$.

In order to show (i), we first claim that

$$[V^n T](s+t) = \sum_{k=0}^{n} \left[V^k T\right](s) \cdot \left[V^{n-k} T\right](t) \tag{3.23}$$

for all $n \in \mathbb{N}_0$ and $s, t \in [0, t_0]$ satisfying $s+t \le t_0$. Since $V^0 = I$, equation (3.23) is trivially satisfied for $n = 0$. Now assume that (3.23) holds for some $n \in \mathbb{N}$. Then, for all $x \in D(A)$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n+1} \left[V^k T\right](s) \cdot \left[V^{n+1-k} T\right](t)\, x \\
&\quad = \int_0^t \sum_{k=0}^{n} \left[V^k T\right](s) \cdot \left[V^{n-k} T\right](r) B T(t-r) x \, dr \\
&\qquad + \int_0^s [V^n T](r) B T(s+t-r) x \, dr \\
&\quad = \int_s^{s+t} [V^n T](r) B T(s+t-r) x \, dr + \int_0^s [V^n T](r) B T(s+t-r) x \, dr \\
&\quad = \left[V^{n+1} T\right](s+t) x.
\end{aligned}$$

Since $D(A)$ is dense in $X$, an induction argument gives (3.23). The semigroup property then follows as in the proof of (i) in Theorem 3.1 (cf. p. 184).

Next, assertion (ii) follows as in the proof of Theorem 3.1 (see p. 185) and we obtain that $(S(t))_{t \ge 0}$ is a strongly continuous semigroup on $X$.

(iii) Let $x \in D(A)$ and $t = nt_0 + \tau$ for $\tau \in [0, t_0)$, $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\int_0^t S(r) B T(t-r) x \, dr &= \sum_{k=0}^{n-1} S(kt_0) \int_0^{t_0} S(r) B T(t_0 - r) T\big(t - (k+1)t_0\big) x \, dr \\
&\qquad + S(nt_0) \int_0^{\tau} S(r) B T(\tau - r) x \, dr \\
&= \sum_{k=0}^{n-1} S(kt_0) \big(S(t_0) - T(t_0)\big) T\big(t - (k+1)t_0\big) x \\
&\qquad + S(nt_0) \big(S(\tau) - T(\tau)\big) x \\
&= S(t)x - T(t)x,
\end{aligned}$$

which proves (iii).

(iv) We first claim that $\|BR(\lambda, A)\| < 1$ for $\lambda$ sufficiently large. To this end, we choose constants $M \ge 1$ and $w \ge 0$ such that $\|T(t)\| \le M e^{wt}$ for all $t \ge 0$. Then, for $\lambda > w$, the resolvent of $A$ is given by the integral representation, and we obtain

$$\begin{aligned}
BR(\lambda, A)x &= \sum_{n=0}^{\infty} e^{-\lambda n t_0} \int_0^{t_0} e^{-\lambda r} B T(r) T(nt_0) x \, dr \\
&= \sum_{n=0}^{\infty} e^{-\lambda n t_0} \left[V F_\lambda\right](t_0)\, T(nt_0) x
\end{aligned}$$

for all $x \in D(A)$, where $F_\lambda \in \mathfrak{X}_{t_0}$ is defined by $F_\lambda(r) := \mathrm{e}^{-\lambda(t_0-r)}I$. Hence, we can estimate as in (3.5) and obtain

$$\|BR(\lambda, A)x\| \le \|V\| + \frac{M\mathrm{e}^{(w-\lambda)t_0}}{1-\mathrm{e}^{(w-\lambda)t_0}}$$

for all $x \in D(A)$ satisfying $\|x\| \le 1$. Since $D(A)$ is dense and $\|V\| < 1$, this implies that $\|BR(\lambda, A)\| < 1$ for $\lambda$ sufficiently large. Using this and the equation

$$\lambda - A - B = \big(I - BR(\lambda, A)\big)(\lambda - A), \tag{3.24}$$

we see that $\lambda - (A + B)$ is invertible whenever $\lambda \in \rho(A)$ and $1 \in \rho(BR(\lambda, A))$. This proves (iv).

(v) Finally, we will determine the generator $C$ of $(S(t))_{t\ge0}$. To this end, we apply the Laplace transform to the Variation of Parameters Formula (3.22) and obtain, using the Convolution Theorem C.17 and the density of $D(A)$ in $X$, that

$$R(\lambda, C) = R(\lambda, A) + R(\lambda, C)BR(\lambda, A)$$

for all $\lambda > \max\{\omega_0(A), \omega_0(C)\}$. This implies

$$R(\lambda, C)\big(I - BR(\lambda, A)\big) = R(\lambda, A),$$

and hence

$$R(\lambda, C)\big(\lambda - A - B\big) = I_{D(A)}.$$

From this we deduce that $R(\lambda, C)$ is a left inverse of $\lambda - (A + B)$ and therefore

$$\lambda - (A + B) \subseteq \lambda - C. \tag{3.25}$$

Since we verified in step (iv) that $\lambda - (A + B)$ is bijective for $\lambda$ sufficiently large, (3.25) implies $C = A + B$, which proves (v). □

Before giving a simpler condition on an operator $B \in \mathcal{L}(X_1, X)$ to be contained in $\mathcal{S}^{\mathrm{MV}}_{t_0}$, we state two representation formulas for the semigroup $(S(t))_{t\ge0}$ generated by $A + B$.

**3.15 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\ge0}$ and let $B \in \mathcal{S}^{\mathrm{MV}}_{t_0}$ for some $t_0 > 0$. Then the semigroup $(S(t))_{t\ge0}$ generated by $A + B$ is determined by*

(i) *the variation of parameters formula*

$$S(t)x = T(t)x + \int_0^t S(r)BT(t-r)x\,dr \qquad \text{for each } t \ge 0,\ x \in D(A),$$

(ii) *and by the abstract Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} (V^n T)(t)$$

*for $t \in [0, t_0]$, where $V := \overline{V_B^*}$ is defined as above.*

Proof. Assertion (i) has already been verified in the proof of Theorem 3.14, while (ii) follows from (3.21) and the Neumann series representation of $(I-V)^{-1}$. □

Based on Theorem 3.14, we now give a more concrete criterion for $B \in \mathcal{L}(X_1, X)$ implying $A+B$ to be a generator on $X$.

**3.16 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and let $B \in \mathcal{L}(X_1, X)$ satisfy*

$$\int_0^{t_0} \|BT(r)x\|\, dr \leq q\,\|x\| \qquad \text{for all } x \in D(A) \tag{3.26}$$

*for some $0 \leq q < 1$. Then $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, and therefore the sum $A+B$ with domain $D(A+B) := D(A)$, generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $X$. Moreover, $(S(t))_{t\geq 0}$ satisfies*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\, ds \qquad \text{and} \tag{3.27}$$

$$\int_0^{t_0} \|BS(t)x\|\, dt \leq \frac{q}{1-q}\,\|x\| \qquad \text{for } x \in D(A) \text{ and } t \geq 0, \tag{3.28}$$

*where $q$ and $t_0$ are given by (3.26). If, in addition, $(B, X_1)$ is closable in $X$ and $(B, D(B))$ denotes its closure, then we have $T(t)x, S(t)x \in D(B)$ for almost all $t \geq 0$ and all $x \in X$. Finally, the functions $BT(\cdot)x$ and $BS(\cdot)x$ are locally integrable, and Corollary 3.15.(i) and the integral equation (3.27) hold for all $x \in X$ and $t \geq 0$.*

Proof. We first show that $A+B$ is the generator of a strongly continuous semigroup on $X$. To do so, by Theorem 3.14, it suffices to verify the following assertions.

(i) For all $F \in \mathfrak{X}_{t_0}$ and $t \in [0, t_0]$, the map $(V_B^* F)(t) : D(A) \subset X \to X$ given by

$$x \mapsto \int_0^t F(r)BT(t-r)x\, dr$$

can be extended to a bounded operator $\overline{(V_B^* F)(t)}$ on $X$.

(ii) The operator $V$ defined by

$$VF := \overline{(V_B^* F)(\cdot)}$$

maps $\mathfrak{X}_{t_0}$ into $\mathfrak{X}_{t_0}$, i.e., the function $t \mapsto \overline{(V_B^* F)(t)}$ is strongly continuous for all $F \in \mathfrak{X}_{t_0}$.

(iii) The operator $V$ is bounded and satisfies $\|V\| \leq q < 1$.

In fact, from (3.26) we obtain

$$\left\| \int_0^t F(r)BT(t-r)x\, dr \right\| \leq \int_0^{t_0} \|BT(r)x\|\, dr \cdot \|F\|_\infty \leq q \cdot \|F\|_\infty \cdot \|x\|$$

for all $x \in D(A)$. Since $D(A)$ is dense in $X$, this shows (i) with

$$\left\| \overline{(V_B^* F)(t)} \right\| \leq q\,\|F\|_\infty \tag{3.29}$$

for all $t \in [0, t_0]$.

In order to verify (ii) it therefore suffices, by Lemma I.5.2, to show that

$$[0,t_0] \ni t \mapsto (V_B^* F)(t)x \in X$$

is continuous for all $x \in D(A)$. To this end, we define, for $F \in \mathfrak{X}_{t_0}$ and $t \in [0,t_0]$, the function $F_t : [0,t_0] \to \mathcal{L}(X)$ by

$$F_t(r) := \begin{cases} F(0) & \text{if } 0 \le r \le t_0 - t, \\ F(r+t-t_0) & \text{if } t_0 - t \le r \le t_0. \end{cases}$$

Then $F_t \in \mathfrak{X}_{t_0}$, and one readily verifies that

$$\int_0^t F(r)BT(t-r)x\,dr = \int_0^{t_0} F_t(r)BT(t_0-r)x\,dr - \int_t^{t_0} F(0)BT(r)x\,dr.$$

Hence, for $s,t \in [0,t_0]$ and $x \in D(A)$, we obtain

$$(V_B^* F)(t)x - (V_B^* F)(s)x = \int_0^{t_0} \bigl(F_t(r) - F_s(r)\bigr)BT(t_0-r)x\,dr + \int_s^t F(0)BT(r)x\,dr.$$

Note that the set $C := \bigl\{BT(t_0-r)x : r \in [0,t_0]\bigr\}$ is compact in $X$; hence by Lemma I.5.2 for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bigl\| \bigl(F_t(r) - F_s(r)\bigr) y \bigr\| < \varepsilon$$

for all $y \in C$, $r \in [0,t_0]$, and $s,t \in [0,t_0]$ satisfying $|t-s| < \delta$. Using this estimate, we finally conclude that

$$\|(V_B^* F)(t)x - (V_B^* F)(s)x\| \le \varepsilon \cdot t_0 + |t-s| \cdot \|F(0)BR(\lambda,A)\| \cdot \|T(\cdot)\|_\infty \cdot \|(\lambda - A)x\|$$

for some fixed $\lambda \in \rho(A)$. This implies that $\lim_{s\to t} \|(V_B^* F)(t)x - (V_B^* F)(s)x\| = 0$ for all $x \in D(A)$, proving (ii). Since (iii) follows immediately from (3.29), this proves that $A+B$ is the generator of a strongly continuous semigroup, which we denote by $(S(t))_{t\ge 0}$.

Since this semigroup satisfies

$$\frac{d}{ds} T(t-s)S(s)x = -T(t-s)AS(s)x + T(t-s)(A+B)S(s)x = T(t-s)BS(s)x$$

for all $x \in D(A)$ and $t \ge s \ge 0$, integration from 0 to $t$ gives (3.27).

For $x \in D(A)$ and $\lambda > \omega_0(A)$, we derive from (3.26) and (3.27) that

$$\begin{aligned} &\int_0^{t_0} \|B\lambda R(\lambda,A)S(t)x\|\,dt \\ &\quad \le \int_0^{t_0} \|B\lambda R(\lambda,A)T(t)x\|\,dt + \int_0^{t_0}\int_s^{t_0} \|BT(t-s)\lambda R(\lambda,A)BS(s)x\|\,dt\,ds \\ &\quad \le \int_0^{t_0} \|B\lambda R(\lambda,A)T(t)x\|\,dt + q\int_0^{t_0} \|\lambda R(\lambda,A)BS(s)x\|\,ds. \end{aligned}$$

Using Lemma II.3.4 and letting $\lambda \to \infty$, we then obtain

$$\int_0^{t_0} \|BS(t)x\|\,dt \le \int_0^{t_0} \|BT(t)x\|\,dt + q\int_0^{t_0} \|BS(s)x\|\,ds.$$

Another application of (3.26) establishes (3.28).

Assume now $(B, X_1)$ to be closable and denote its closure by $(B, D(B))$. Fix $x \in X$, $t \geq 0$ and choose $x_n \in D(A)$ tending to $x$ in $X$. Note that Corollary 3.15.(i) holds with $x$ replaced by $x_n$. Due to the Miyadera estimate (3.26) the sequence $(BT(\cdot)x_n)_{n\in\mathbb{N}}$ converges in $\mathrm{L}^1([0,t], X)$. Since $B$ is closed, this implies that $T(s)x \in D(B)$ for almost all $s \in [0,t]$, $BT(\cdot)x \in \mathrm{L}^1([0,t], X)$, and Corollary 3.15.(i) holds for all $x \in X$. The other assertions follow in a similar way from (3.27) and (3.28). □

Note that if $B \in \mathcal{L}(X_1, X)$ satisfies the assumption in Corollary 3.16, then $PB \in \mathcal{S}_{t_0}^{\mathrm{MV}}$ for all $P \in \mathcal{L}(X)$ satisfying $\|P\| \leq 1$. For an extension of Corollary 3.16 see Exercise 3.17.(2).

**3.17 Exercises.** (1) If $A$ generates an analytic semigroup and $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, then the semigroup generated by $A + B$ is analytic as well. (Hint: Use (3.24) in order to verify the resolvent estimate in Theorem II.4.6.(d) for $A + B$.

(2) Show that the conclusion of Corollary 3.16 remains true if the perturbation $(B, D(B))$ satisfies only the assumptions of Exercise 2.18.(2) for some $q < 1$.

## *d. Additive Versus Multiplicative Perturbations*

In the previous subsections we encountered two types of *additive perturbations* of a generator $A$ on a Banach space $X$. In fact, the perturbed operator $C$ was given as $C = A + B$ for an $A$-bounded operator $B$ or as $C = (A_{-1} + B)_{|X}$ for an operator $B$ on $X$ assuming values in the extrapolation space $X_{-1}$.

In this subsection we will introduce perturbations of a different type. For a given generator $A$ and a bounded operator $Q \in \mathcal{L}(X)$, the new operator $C$ is given either as a *left multiplicative perturbation*

$$C := QA, \quad D(C) := D(A) \tag{3.30}$$

or as a *right multiplicative perturbation*

$$C := AQ, \quad D(C) := \{x \in X : Qx \in D(A)\}. \tag{3.31}$$

The following result shows that an *additive* perturbation can always be written as a *multiplicative* perturbation and vice versa.

**3.18 Proposition.** *Let $A$ be an invertible operator on a Banach space $X$ and take $C : D(C) \subseteq X \to X$. Then the following assertions are true.*

(i) *$C = A + B$ with $D(C) = D(A)$ for some $B \in \mathcal{L}(X_1, X)$ if and only if $C = (I + K)A$ with $D(C) = D(A)$ for some $K \in \mathcal{L}(X)$. Here $B$ and $K$ are related by $B = KA$ and $K = BA^{-1}$, respectively.*

(ii) *$C = (A_{-1} + B)_{|X}$ with $D(C) = \{x \in X : A_{-1}x + Bx \in X\}$ for some $B \in \mathcal{L}(X, X_{-1})$ if and only if $C = A(I + K)$ with $D(C) = \{x \in X : (I + K)x \in D(A)\}$ for some $K \in \mathcal{L}(X)$. Here $B$ and $K$ are related by $B = A_{-1}K$ and $K = A_{-1}^{-1}B$, respectively.*

The simple proof is left to the reader; cf. Exercise 3.23.(1).

Our next aim is to clarify the relation between left and right multiplicative perturbations. To this end, we first consider a rather special case.

**3.19 Lemma.** *Let $A$ be the generator of a strongly continuous semigroup on a Banach space $X$. If $K \in \mathcal{L}(X)$ such that $KA$ admits a bounded extension $\overline{KA} \in \mathcal{L}(X)$, then also the operator*

$$A(I-K) \qquad \text{with domain} \qquad D\big(A(I-K)\big) := \big\{x \in X : (I-K)x \in D(A)\big\}$$

*is a generator on $X$.*

Proof. Let $\mu \in \rho(A)$. Then it follows from the resolvent equation that we can write

$$A(I-K) - \mu = (A-\mu)\big(I + AR(\mu, A)K\big). \tag{3.32}$$

From spectral theory (see, e.g., [GGK90, Sec. III.2,(3)]) we know that

$$-1 \in \rho\big(AR(\mu, A)K\big) \qquad \text{if and only if} \qquad -1 \in \rho\big(KAR(\mu, A)\big).$$

Since, by the Hille–Yosida Generation Theorem II.3.8

$$\|KAR(\mu, A)\| \le \big\|\overline{KA}\big\| \cdot \|R(\mu, A)\| \to 0 \quad \text{as } \mu \to \infty,$$

this shows that $I + AR(\mu, A)K$ is invertible for $\mu$ sufficiently large. Hence, we obtain from (3.32) that $A(I-K)$ is similar to the operator

$$\big(I + AR(\mu, A)K\big)(A-\mu) + \mu =: A + B$$

for $\mu$ large and $B := AR(\mu, A)\overline{K(A-\mu)} \in \mathcal{L}(X)$. The assertion now follows from Paragraph II.2.1 and the Bounded Perturbation Theorem 1.3. □

The following theorem relates, in combination with Proposition 3.18, additive and multiplicative perturbations.

**3.20 Theorem.** *Let $(A, D(A))$ be an operator with $\rho(A) \neq \emptyset$ on a Banach space $X$ and let $Q \in \mathcal{L}(X)$.*

(i) *If $(QA, D(A))$ is a generator on $X$, then*

$$AQ \quad \text{with domain} \quad D(AQ) := \big\{x \in X : Qx \in D(A)\big\}$$

*is a generator on $X$.*

(ii) *If $(AQ, D(AQ))$ is a generator on $X$, then*

$$(QA)_1 := QA_{|X_1^A} \quad \text{with domain} \quad D\big((QA)_1\big) := \big\{x \in D(A) : QAx \in D(A)\big\}$$

*is a generator on $X_1^A := (D(A), \|\cdot\|_A)$. If, in addition, the resolvent set $\rho(QA) \neq \emptyset$, then $(QA, D(A))$ is a generator on $X$.*

Proof. For the rest of the proof we fix some $\lambda \in \rho(A)$.

(i) By the Bounded Perturbation Theorem 1.3, the operator $(Q(A-\lambda), D(A))$ is a generator on $X$; hence (use Proposition II.5.2) $(Q(A-\lambda))_1 := Q(A-\lambda)_{|X_1^{Q(A-\lambda)}}$ is a generator on $X_1^{Q(A-\lambda)} := (D(A), \|\cdot\|_{Q(A-\lambda)})$. Since $Q$ is bounded, one easily verifies that on $D(A)$ the norm $\|\cdot\|_A$ is finer than $\|\cdot\|_{Q(A-\lambda)}$; hence these norms are equivalent by the open mapping theorem. Therefore, the operator $\lambda - A \in \mathcal{L}(X_1^{Q(A-\lambda)}, X)$ is an isomorphism, and from Paragraph II.2.1 we conclude that

$$(\lambda - A)\big(Q(A-\lambda)\big)_1 R(\lambda, A) = (A-\lambda)Q = AQ - \lambda Q$$

is a generator on $X$. The assertion now follows from the Bounded Perturbation Theorem 1.3.

(ii) We first note that the operator $(QA)_1 := QA_{|X_1^A}$ on $X_1^A := (D(A), \|\cdot\|_A)$ is similar to

$$C := (\lambda - A)(QA)_1 R(\lambda, A)$$

on $X$, where

$$\begin{aligned} D(C) :&= \{x \in X : R(\lambda, A)x \in D\big((QA)_1\big)\} \\ &= \{x \in X : \big(I - R(\lambda, A)\big)x \in D(AQ)\}. \end{aligned}$$

An easy computation now shows that

$$Cx = AQ\big(I - \lambda R(\lambda, A)\big) + B$$

for $B := \lambda QAR(\lambda, A) \in \mathcal{L}(X)$. Therefore, by Paragraph II.2.1 and the bounded perturbation theorem, we conclude that $(QA)_1$ is a generator on $X_1^A$ if $AQ(I - \lambda R(\lambda, A))$ is a generator on $X$. Since $\lambda R(\lambda, A)\, AQ$ has a bounded extension to $X$, the latter is true by Lemma 3.19, and the first part of assertion (ii) is proved.

In case $\rho(QA) \neq \emptyset$, we obtain as above that the norms $\|\cdot\|_A$ and $\|\cdot\|_{QA}$ are equivalent on $D(A)$. This implies that $R(\mu, QA) \in \mathcal{L}(X_1^A, X)$ is an isomorphism for every $\mu \in \rho(QA)$; hence $(QA)_1$ on $X_1^A$ and

$$(\mu - QA)(QA)_1 R(\mu, QA) = QA \qquad \text{on } X$$

are similar. The assertion then follows from Paragraph II.2.1. □

This result shows that additive and multiplicative perturbations are basically "equivalent." However, one might be surprised why the additional assumption "$\rho(QA) \neq \emptyset$" appears in part (ii).

A trivial counterexample shows that without this assumption, the additional assertion in Theorem 3.20.(ii) is not true anymore. Indeed, if $Q = A^{-1}$ for some unbounded, invertible operator $A$, then $AQ = I$ is a generator on $X$, while $QA = I_{D(A)}$ is not closed on $X$, hence does not generate a strongly continuous semigroup. The following example shows that even if $QA$ is closed on $X$, the assumption $\rho(QA) \neq \emptyset$ cannot be omitted in order to conclude that $QA$ is a generator on $X$.

**3.21 Example.** For an unbounded operator $\widetilde{A}$ on a Banach space $X$ with nonempty resolvent set consider

$$A := \begin{pmatrix} I & 0 \\ 0 & \widetilde{A} \end{pmatrix}, \quad D(A) := X \times D(\widetilde{A}), \qquad \text{and} \qquad Q := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

defined on the product space $X \times X$. Then $Q$ is bounded, and we obtain

$$QA = \begin{pmatrix} 0 & \widetilde{A} \\ 0 & 0 \end{pmatrix}, \quad D(QA) = D(A), \qquad \text{and} \qquad AQ = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(X \times X);$$

hence $AQ$ is a generator on $X \times X$, while $QA$ is not (compare Example II.6.5). However, the part $(QA)_1$ of $QA$ in $(D(A), \|\cdot\|_A)$ is bounded, hence generates a strongly continuous semigroup.

We now combine our previous results with the perturbation theorems from Sections 3.a and 3.c.

**3.22 Corollary.** *Let $A$ be the generator of a strongly continuous semigroup on a Banach space $X$. If $A$ is invertible, then the following assertions are true.*

(i) *If $B \in \mathcal{S}_{t_0}^{\mathrm{DS}}$, then the operator $C := A + A_{-1}^{-1}BA = (I + A_{-1}^{-1}B)A$ with domain $D(C) := D(A)$ generates a strongly continuous semigroup on $X$.*

(ii) *If $B \in \mathcal{S}_{t_0}^{\mathrm{MV}}$, then the operator $C := \big(A_{-1} + A_{-1}BA^{-1}\big)|_X = A(I+BA^{-1})$ with domain $D(C) := \big\{x \in X : \big(A_{-1} + A_{-1}BA^{-1}\big)x \in X\big\}$ generates a strongly continuous semigroup on $X$.*

Proof. (i) By Theorems 3.1 and 3.20.(ii), it suffices to show that $(I + A_{-1}^{-1}B)A$ has nonempty resolvent set. To this end, we first write for $\lambda \in \rho(A)$

$$\lambda - (I + A_{-1}^{-1}B)A = \big(I - A_{-1}^{-1}B\,AR(\lambda, A)\big)(\lambda - A).$$

This shows that $\lambda - (I + A_{-1}^{-1}B)A$ is invertible, provided that $\lambda \in \rho(A)$ and

$$1 \in \rho\big(A_{-1}^{-1}B\,AR(\lambda, A)\big). \tag{3.33}$$

Since by [GGK90, Sec. III.2,(3)] condition (3.33) is equivalent to

$$1 \in \rho\big(AR(\lambda, A)\,A_{-1}^{-1}B\big) = \rho\big(R(\lambda, A_{-1})B\big),$$

the assertion follows from step (iv) in the proof of Theorem 3.1 (cf. p. 185).

Assertion (ii) follows from Theorem 3.14 and Theorem 3.20.(i). □

**3.23 Exercises.** (1) Give a proof of Proposition 3.18.

(2) Let the assumptions of Theorem 3.20 be satisfied.

(i) If in 3.20.(i) the semigroup generated by $QA$ is denoted by $(U(t))_{t\ge 0}$, then the semigroup $(V(t))_{t\ge 0}$ generated by $AQ$ is given by

$$V(t)x = x + A\int_0^t U(s)Qx\,ds \quad \text{for all } x \in X \text{ and } t \ge 0.$$

(Hint: By Theorem C.16 it suffices to show that the Laplace transform of $(V(t))_{t\ge 0}$ coincides with the resolvent of $AQ$.)

(ii) Find analogous formulas in the situation of Theorem 3.20.(ii) for the semigroups generated by $(QA)_1$ and $QA$, respectively.

(iii) Let $A$ be a generator on a Banach space $X$. If the range of $B \in \mathcal{L}(X_1, X)$ is contained in the Favard space $F_1$, then $(A + B, D(A))$ is a generator on $X$. (Hint: Use Corollary 3.22.(i) and Corollary 3.6.)

# 4. Trotter–Kato Approximation Theorems

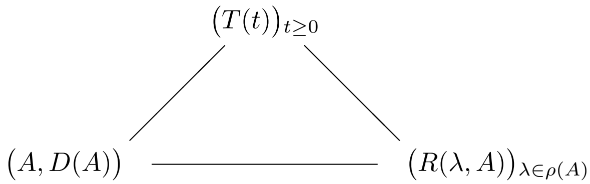
Approximation, besides perturbation, is the other main method used to study a complicated operator and the semigroup it generates. We already encountered an example for such an approximation procedure in our proof of the Generation Theorem II.3.5. For an operator $(A, D(A))$ on $X$ satisfying the Hille–Yosida conditions, we defined the (bounded) *Yosida approximants*[2]

$$A_n := nAR(n, A), \quad n \in \mathbb{N}$$

(see Chapter II, (3.7)) generating the (uniformly continuous) semigroups $(\mathrm{e}^{tA_n})_{t\geq 0}$. Using the fact that $A_n \to A$ pointwise on $D(A)$ as $n \to \infty$ (see Lemma II.3.4.(ii)), we could show that the semigroups converge as well, i.e.,

$$\mathrm{e}^{tA_n} \to T(t) \quad \text{as} \quad n \to \infty.$$

In this section we study this situation systematically and consider the three objects *semigroup*, *generator*, and *resolvent*, visualized by the triangle




from Chapter II. We then try to show that the convergence at one "vertex" implies convergence in the two other "vertices." That the truth is not as simple is shown by the following example.

**4.1 Example.** On the Banach space $X := \mathrm{c}_0$, we take the multiplication operator

$$A(x_k) := (\mathrm{i}kx_k)$$

with domain

$$D(A) := \big\{(x_k) \in \mathrm{c}_0 : (\mathrm{i}kx_k) \in \mathrm{c}_0\big\}.$$

As we know from Example I.4.7.(iii), it generates the strongly continuous semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)(x_k) = (\mathrm{e}^{\mathrm{i}kt}x_k), \quad t \geq 0.$$

Perturbing $A$ by the bounded operators

$$P_n(x_k) := (0, \ldots, nx_n, 0, \ldots),$$

[2] In this context, this notation should not cause any confusion with the operators $A_n$ induced on the abstract Sobolev spaces from Section II.5.

we obtain new operators

$$A_n := A + P_n.$$

Each $A_n$ is the generator of a strongly continuous semigroup $\big(T_n(t)\big)_{t\geq 0}$ (use Theorem 1.3), and for each $x = (x_k) \in D(A)$, we have

$$\|A_n x - Ax\| = \|P_n x\| = n|x_n| \to 0.$$

However, the semigroups $\big(T_n(t)\big)_{t\geq 0}$ do not converge. In fact, one has

$$T_n(t)x = (\mathrm{e}^{\mathrm{i}t}x_1, \mathrm{e}^{2\mathrm{i}t}x_2, \dots, \mathrm{e}^{(\mathrm{i}n+n)t}x_n, \mathrm{e}^{(n+1)\mathrm{i}t}x_{n+1}, \dots)$$

and therefore

$$\|T_n(t)\| \geq \mathrm{e}^{nt} \quad \text{for } n \in \mathbb{N} \text{ and } t \geq 0.$$

By the uniform boundedness principle, this implies that there exists $x \in X$ such that $\big(T_n(t)x\big)_{n\in\mathbb{N}}$ does not converge.

The example shows that the convergence of the generators (pointwise on the domain of the limit operator) does not imply convergence of the corresponding semigroups. Another unpleasant phenomenon may happen for a converging sequence of resolvent operators.

**4.2 Example.** Take $A_n := -n \cdot I$ on any Banach space $X \neq \{0\}$. Then the resolvent operators

$$R(\lambda, A_n) = \frac{1}{\lambda + n} \cdot I$$

and their limit

$$R(\lambda) := \lim_{n\to\infty} R(\lambda, A_n)$$

exist for all $\operatorname{Re}\lambda > 0$. However, the limit $R(\lambda)$ is equal to zero, hence cannot be the resolvent of an operator on $X$.

For our purposes we must exclude such a phenomenon. In order to do so, we need a new concept.

## *a. A Technical Tool: Pseudoresolvents*

In this subsection we consider bounded operators on a Banach space $X$ that depend on a complex parameter and satisfy the resolvent equation (see Chapter IV, (1.2)). Here is the formal definition.

**4.3 Definition.** *Let $\Lambda \subset \mathbb{C}$ and consider operators $\mathfrak{J}(\lambda) \in \mathfrak{L}(X)$ for each $\lambda \in \Lambda$. The family $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ is called a pseudoresolvent if*

$$\mathfrak{J}(\lambda) - \mathfrak{J}(\mu) = (\mu - \lambda)\mathfrak{J}(\lambda)\mathfrak{J}(\mu) \tag{4.1}$$

*holds for all $\lambda, \mu \in \Lambda$.*

The limit operators $R(\lambda)$ from Example 4.2 form a (trivial) pseudoresolvent for $\operatorname{Re}\lambda > 0$. However, they are not injective, and therefore they cannot be the resolvent operators $R(\lambda, A)$ of an operator $A$. It is our goal, and crucial for the proofs in Section 4.b, to find conditions implying that a pseudoresolvent is indeed a resolvent. Before doing so, we discuss the typical situation in which we will encounter pseudoresolvents.

**4.4 Proposition.** *For each $n \in \mathbb{N}$, let $A_n$ be the generator of a contraction semigroup on $X$ and assume that for some $\lambda_0 > 0$,*

$$\lim_{n\to\infty} R(\lambda_0, A_n)x$$

*exists for all $x \in X$. Then, the limit*

$$R(\lambda)x := \lim_{n\to\infty} R(\lambda, A_n)x, \quad x \in X,$$

*exists for all* $\operatorname{Re}\lambda > 0$ *and defines a pseudoresolvent* $\{R(\lambda) : \operatorname{Re}\lambda > 0\}$.

PROOF. Consider the set

$$\Omega := \Big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0,\ \lim_{n\to\infty} R(\lambda, A_n)x \quad \text{exists for all } x \in X\Big\},$$

which is nonempty by assumption. As in Proposition IV.1.3, one shows that for given $\mu \in \Omega$ one has

$$R(\lambda, A_n) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A_n)^{k+1}$$

as long as $|\mu - \lambda| < \operatorname{Re}\mu$ (use (3.6) from Chapter II). The convergence is with respect to the operator norm and uniform in $\{\lambda \in \mathbb{C} : |\mu-\lambda| \le \alpha \operatorname{Re}\mu\}$ for each $0 < \alpha < 1$. Since the series $\sum_{k=0}^{\infty} \alpha^{k+1}$ majorizes all the series $\sum_{k=0}^{\infty} |\mu - \lambda|^k \big\|R(\mu, A_n)^{k+1}\big\|$, we can conclude that $R(\lambda, A_n)x$ converges as $n \to \infty$ for all $\lambda$ satisfying $|\mu-\lambda| \le \alpha \operatorname{Re}\mu$. Therefore, the set $\Omega$ is open.

On the other hand, take an accumulation point $\lambda$ of $\Omega$ with $\operatorname{Re}\lambda > 0$. For $0 < \alpha < 1$, we can find $\mu \in \Omega$ such that $|\mu - \lambda| \le \alpha \operatorname{Re}\mu$; hence, by the above considerations, $\lambda$ must belong to $\Omega$, i.e., $\Omega$ is relatively closed in $S := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. The only set satisfying both properties is $S$ itself; hence we obtain the existence of the operators $R(\lambda)$ for $\operatorname{Re}\lambda > 0$.

Evidently, the resolvent equation (4.1) remains valid for the limit operators. □

In the subsequent lemma, we state the basic properties of pseudoresolvents.

**4.5 Lemma.** *Let $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ be a pseudoresolvent on $X$. Then the following properties hold for all $\lambda, \mu \in \Lambda$.*

(i) $\mathfrak{J}(\lambda)\mathfrak{J}(\mu) = \mathfrak{J}(\mu)\mathfrak{J}(\lambda)$.

(ii) $\ker \mathfrak{J}(\lambda) = \ker \mathfrak{J}(\mu)$.

(iii) $\operatorname{rg} \mathfrak{J}(\lambda) = \operatorname{rg} \mathfrak{J}(\mu)$.

Proof. The commutativity (i) follows from the resolvent equation (4.1). If we rewrite it in the form

$$\mathfrak{J}(\lambda) = \mathfrak{J}(\mu)\bigl[I + (\mu - \lambda)\mathfrak{J}(\lambda)\bigr] = \bigl[I + (\mu - \lambda)\mathfrak{J}(\lambda)\bigr]\mathfrak{J}(\mu),$$

we see that $\operatorname{rg} \mathfrak{J}(\lambda) \subseteq \operatorname{rg} \mathfrak{J}(\mu)$ and $\ker \mathfrak{J}(\mu) \subseteq \ker \mathfrak{J}(\lambda)$. By symmetry, the assertions (ii) and (iii) follow. □

If we now require that $\ker \mathfrak{J}(\lambda) = \{0\}$ and $\operatorname{rg} \mathfrak{J}(\lambda)$ is dense, then the pseudoresolvent $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ becomes the resolvent of a closed, densely defined operator.

**4.6 Proposition.** *For a pseudoresolvent $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ on $X$, the following assertions are equivalent.*

(a) *There exists a densely defined closed operator $\bigl(A, D(A)\bigr)$ such that $\Lambda \subset \rho(A)$ and $\mathfrak{J}(\lambda) = R(\lambda, A)$ for all $\lambda \in \Lambda$.*

(b) $\ker \mathfrak{J}(\lambda) = \{0\}$, *and* $\operatorname{rg} \mathfrak{J}(\lambda)$ *is dense in $X$ for some/all $\lambda \in \Lambda$.*

Proof. We have only to show that (b) implies (a). Since $\mathfrak{J}(\lambda)$ is injective, we can define

$$A := \lambda_0 - \mathfrak{J}(\lambda_0)^{-1}$$

for some $\lambda_0 \in \Lambda$. This yields a closed operator with dense domain $D(A) := \operatorname{rg} \mathfrak{J}(\lambda_0)$. From the definition of $A$, it follows that

$$(\lambda_0 - A)\mathfrak{J}(\lambda_0) = \mathfrak{J}(\lambda_0)(\lambda_0 - A) = I;$$

hence $\mathfrak{J}(\lambda_0) = R(\lambda_0, A)$. For arbitrary $\lambda \in \Lambda$, we have

$$\begin{aligned}
(\lambda - A)\mathfrak{J}(\lambda) &= \bigl[(\lambda - \lambda_0) + (\lambda_0 - A)\bigr]\mathfrak{J}(\lambda)\\
&= \bigl[(\lambda - \lambda_0) + (\lambda_0 - A)\bigr]\mathfrak{J}(\lambda_0)\bigl[I - (\lambda - \lambda_0)\mathfrak{J}(\lambda)\bigr]\\
&= I + (\lambda - \lambda_0)\bigl[\mathfrak{J}(\lambda_0) - \mathfrak{J}(\lambda) - (\lambda - \lambda_0)\mathfrak{J}(\lambda)\mathfrak{J}(\lambda_0)\bigr]\\
&= I,
\end{aligned}$$

and similarly, $\mathfrak{J}(\lambda)(\lambda - A) = I$. This shows that $\mathfrak{J}(\lambda) = R(\lambda, A)$ for all $\lambda \in \Lambda$ and, in particular, that $A$ does not depend on the choice of $\lambda_0$. □

We conclude these considerations with some useful sufficient conditions that make a pseudoresolvent a resolvent.

**4.7 Corollary.** *Let $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ be a pseudoresolvent on $X$ and assume that $\Lambda$ contains an unbounded sequence $(\lambda_n)_{n\in\mathbb{N}}$. If*

$$\lim_{n\to\infty} \lambda_n \mathfrak{J}(\lambda_n)x = x \quad \text{for all } x \in X, \tag{4.2}$$

*then $\{\mathfrak{J}(\lambda) : \lambda \in \Lambda\}$ is the resolvent of a densely defined operator. In particular, (4.2) holds if $\operatorname{rg}\mathfrak{J}(\lambda)$ is dense and*

$$\|\lambda_n \mathfrak{J}(\lambda_n)\| \le M \tag{4.3}$$

*for some constant $M$ and all $n \in \mathbb{N}$.*

Proof. If (4.2) holds, we have $X = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{rg}\mathfrak{J}(\lambda_n)} = \overline{\operatorname{rg}\mathfrak{J}(\lambda)}$, and hence $\mathfrak{J}(\lambda)$ has dense range for each $\lambda \in \Lambda$. If $x \in \ker\mathfrak{J}(\lambda)$, we obtain $x = \lim \lambda_n \mathfrak{J}(\lambda_n)x = 0$; hence $\ker\mathfrak{J}(\lambda) = \{0\}$. The first assertion now follows from Proposition 4.6.(b).

From the estimate $\|\mathfrak{J}(\lambda_n)\| \le \frac{M}{|\lambda_n|}$, $n \in \mathbb{N}$, and the resolvent equation, we obtain

$$\lim_{n\to\infty} \|(\lambda_n \mathfrak{J}(\lambda_n) - I)\mathfrak{J}(\mu)\| = 0$$

for fixed $\mu \in \Lambda$. Therefore, it follows that

$$\lim_{n\to\infty} \lambda_n \mathfrak{J}(\lambda_n)x = x$$

for $x \in \operatorname{rg}\mathfrak{J}(\mu)$. Since this is a dense subspace of $X$, the norm boundedness in (4.3) allows us to conclude that (4.2) holds. □

## *b. The Approximation Theorems*

We now turn our attention to the approximation problem stated above, i.e., we study the relation between convergence of semigroups, generators, and resolvents. The adequate type of convergence for strongly continuous semigroups (and unbounded operators) will be pointwise convergence.

If we *assume* that the limit operator is known to be a generator, we obtain our first main result. However, we need a uniform bound on the semigroups involved.

**4.8 First Trotter–Kato Approximation Theorem.** (Trotter 1958, Kato 1959). *Let $\big(T(t)\big)_{t\ge0}$ and $\big(T_n(t)\big)_{t\ge0}$, $n \in \mathbb{N}$, be strongly continuous semigroups on $X$ with generators $A$ and $A_n$, respectively, and assume that they satisfy the estimate*

$$\|T(t)\|,\ \|T_n(t)\| \le M\mathrm{e}^{wt} \qquad \text{for all } t \ge 0,\ n \in \mathbb{N},$$

*and some constants $M \ge 1$, $w \in \mathbb{R}$. Take $D$ to be a core for $A$ and consider the following assertions.*

(a) $D \subset D(A_n)$ *for all* $n \in \mathbb{N}$ *and* $A_n x \to Ax$ *for all* $x \in D$.

(b) *For each* $x \in D$, *there exists* $x_n \in D(A_n)$ *such that*

$$x_n \to x \qquad \text{and} \qquad A_n x_n \to Ax.$$

(c) $R(\lambda, A_n)x \to R(\lambda, A)x$ *for all* $x \in X$ *and some/all* $\lambda > w$.

(d) $T_n(t)x \to T(t)x$ *for all* $x \in X$, *uniformly for* $t$ *in compact intervals.*

*Then the implications*

$$\text{(a)} \implies \text{(b)} \iff \text{(c)} \iff \text{(d)}$$

*hold, while* (b) *does not imply* (a).

Proof. Before starting, we perform a rescaling and assume without loss of generality that

$$\|T(t)\|,\ \|T_n(t)\| \le M \qquad \text{for all } t \ge 0,\ n \in \mathbb{N}.$$

Since the implication (a) ⇒ (b) is trivial, we start by showing (b) ⇒ (c).

Let $\lambda > 0$. Since $\|R(\lambda, A_n)\| \le {}^{M}\!/\!_{\lambda}$ for all $n \in \mathbb{N}$, it suffices to show that

$$\lim_{n\to\infty} R(\lambda, A_n)y = R(\lambda, A)y$$

for $y$ in the dense subspace $(\lambda - A)D$. Take $x \in D$ and define $y := (\lambda - A)x$. By assumption, there exists $x_n \in D(A_n)$ such that

$$x_n \to x \qquad \text{and} \qquad A_n x_n \to Ax;$$

hence

$$y_n := (\lambda - A_n)x_n \to y.$$

Therefore, the estimate

$$\begin{aligned}\|R(\lambda, A_n)y - R(\lambda, A)y\| \le\ & \|R(\lambda, A_n)y - R(\lambda, A_n)y_n\| \\ & + \|R(\lambda, A_n)y_n - R(\lambda, A)y\| \\ \le\ & \|R(\lambda, A_n)\| \cdot \|y - y_n\| + \|x_n - x\|\end{aligned}$$

implies the assertion.

The implication (c) ⇒ (b) follows if we take $x := R(\lambda, A)y$, and $x_n := R(\lambda, A_n)y$ for fixed $\lambda > 0$ and then observe that

$$A_n x_n = A_n R(\lambda, A_n)y = \lambda R(\lambda, A_n)y - y$$

converges to

$$\lambda R(\lambda, A)y - y = Ax.$$

(d) ⇒ (c). The integral representation of the resolvent yields, for each $\lambda > 0$ and $x \in X$, that

$$\|R(\lambda, A)x - R(\lambda, A_n)x\| \le \int_0^\infty \mathrm{e}^{-\lambda t}\, \|T(t)x - T_n(t)x\|\, dt.$$

The desired convergence is now a consequence of Lebesgue's dominated convergence theorem.

The proof of the final implication (c) ⇒ (d) is not quite so direct. We use an idea of [Kis67], reproduced in [Gol85, Sec. I.7.5], and reduce the convergence problem to a stationary problem to which we can apply Generation Theorem II.3.8.

Observe first that by Proposition 4.4, $\lim_{n\to\infty} R(\lambda, A_n)x$ exists for all $\operatorname{Re}\lambda > 0$. Consider now the Banach space $\mathcal{X} := c(X)$ of all convergent sequences in $X$ endowed with the norm

$$\|\mathfrak{x}\| := \sup_{n\in\mathbb{N}} \|x_n\| \qquad \text{for} \qquad \mathfrak{x} = (x_n)_{n\in\mathbb{N}} \in \mathcal{X}.$$

On this space and for each $\lambda > 0$, define operators $\mathcal{R}(\lambda)$ by

$$\mathcal{R}(\lambda)\mathfrak{x} := \big(R(\lambda, A_n)x_n\big)_{n\in\mathbb{N}}.$$

Since for $(x_n) \in \mathcal{X}$ with $x_n \to x \in X$ we have

$$\begin{aligned}\|R(\lambda, A_n)x_n - R(\lambda, A)x\| \le\ & \|R(\lambda, A_n)\| \cdot \|x_n - x\| \\ & + \|R(\lambda, A_n)x - R(\lambda, A)x\|,\end{aligned}$$

assumption (c) implies $\mathcal{R}(\lambda)\mathfrak{x} \in \mathcal{X}$, i.e., $\mathcal{R}(\lambda)$ is well-defined. In addition, it satisfies

$$\big\|\mathcal{R}(\lambda)^k\big\| \le \frac{M}{\lambda^k} \qquad \text{for all } k \in \mathbb{N}, \tag{4.4}$$

and the following more or less evident properties hold.

(i) $\big\{\mathcal{R}(\lambda) : \lambda > 0\big\}$ is a pseudoresolvent.

(ii) $\operatorname{rg}\mathcal{R}(\lambda)$ is dense in $\mathcal{X}$. This follows, since the eventually constant sequences form a dense subspace in $\mathcal{X}$ and since for each $n \in \mathbb{N}$, the range $\operatorname{rg} R(\lambda, A_n) = D(A_n)$ is dense in $X$.

(iii) $\mathcal{R}(\lambda)$ is injective. Indeed, if $\mathcal{R}(\lambda)\mathfrak{x} = 0$, we must have $R(\lambda, A_n)x_n = 0$ for all $n \in \mathbb{N}$; hence $\mathfrak{x} = 0$.

By Proposition 4.6, there exists a densely defined, closed operator $\mathcal{A}$ such that $\mathcal{R}(\lambda) = R(\lambda, \mathcal{A})$ for all $\lambda > 0$. In addition, this operator satisfies the resolvent estimate (4.4) and therefore, by the Hille–Yosida Generation Theorem II.3.8, it generates a strongly continuous semigroup $\big(\mathcal{T}(t)\big)_{t\ge0}$ on $\mathcal{X}$. For this semigroup we necessarily have

(iv) $\mathcal{T}(t)\mathfrak{x} = \big(T_n(t)x_n\big)_{n\in\mathbb{N}}$ for all $\mathfrak{x} = (x_n)_{n\in\mathbb{N}} \in \mathcal{X}$.

This follows, since the canonical projections $P_n$ onto the $n$th coordinate commute with all the resolvent and semigroup operators; hence

$$P_n\mathcal{T}(t) = T_n(t) \qquad \text{for each } n \in \mathbb{N}.$$

In particular, by taking $\mathfrak{x} = (x, x, \ldots)$, we obtain

$$\big(T_n(t)x\big)_{n\in\mathbb{N}} \in \mathcal{X},$$

and the following definition makes sense.

For every $x \in X$ and $t \geq 0$ we define

$$S(t)x := \lim_{n\to\infty} T_n(t)x. \tag{4.5}$$

We leave it to the reader to verify that $\big(S(t)\big)_{t\geq 0}$ is a strongly continuous semigroup on $X$. We denote its generator by $\big(B, D(B)\big)$ and use the implication (d) $\Rightarrow$ (c) to obtain that $R(\lambda, A_n)$ converges strongly to $R(\lambda, B)$, and hence $R(\lambda, B) = R(\lambda, A)$ and $\big(B, D(B)\big) = \big(A, D(A)\big)$. This shows that the semigroups $\big(S(t)\big)_{t\geq 0}$ and $\big(T(t)\big)_{t\geq 0}$ coincide.

Finally, we show that the convergence in (4.5) is uniform for $t \in [0, t_0]$. Take $\varepsilon > 0$, $x \in X$, and $y := \lim_{n\to\infty} T_n(t)x$ for some $t \in [0, t_0]$. Then there exists $n_t \in \mathbb{N}$ such that $\|T_n(t)x - y\| \leq \varepsilon$ for all $n \geq n_t$ and, by the strong continuity of $\big(\mathcal{T}(t)\big)_{t\geq 0}$, an open neighborhood $\mathcal{U}_t$ of $t$ such that

$$\|\mathcal{T}(s)\mathrm{x} - \mathcal{T}(t)\mathrm{x}\| \leq \varepsilon \qquad \text{for } \mathrm{x} = (x, x, \dots)$$

and all $s \in \mathcal{U}_t$. This implies

$$\|T_n(s)x - y\| \leq \|T_n(s)x - T_n(t)x\| + \|T_n(t)x - y\| \leq 2\varepsilon$$

for all $n \geq n_t$ and $s \in \mathcal{U}_t$. Since $[0, t_0]$ is compact, we obtain the desired uniform convergence.

That (b) does not imply (a) in general can be seen from Counterexample 5.10 below. □

For the above result we had to *assume* that the limit operator $A$ is already known to be a generator. This is a major defect, since in the applications, one wants to approximate the operator $A$ by (simple) operators $A_n$ and then *conclude* that $A$ becomes a generator. Moreover, the semigroup generated by $A$ should be obtained as the limit of the known semigroups generated by the operators $A_n$. In fact, we encountered this problem already in the proof (of the nontrivial implication) of Generation Theorem II.3.5. Therefore, the following result can be viewed as a generalization of the Hille–Yosida theorem.

**4.9 Second Trotter–Kato Approximation Theorem.** (Trotter 1958, Kato 1959). *Let $\big(T_n(t)\big)_{t\geq 0}$, $n \in \mathbb{N}$, be strongly continuous semigroups on $X$ with generators $\big(A_n, D(A_n)\big)$ satisfying the stability condition*

$$\|T_n(t)\| \leq M\mathrm{e}^{wt} \tag{4.6}$$

*for constants $M \geq 1$, $w \in \mathbb{R}$ and all $t \geq 0$, $n \in \mathbb{N}$. For some $\lambda_0 > w$ consider the following assertions.*

(a) *There exists a densely defined operator $\big(A, D(A)\big)$ such that $A_n x \to Ax$ for all $x$ in a core $D$ of $A$ and such that the range $\operatorname{rg}(\lambda_0 - A)$ is dense in $X$.*

(b) *The operators* $R(\lambda_0, A_n)$, $n \in \mathbb{N}$, *converge strongly to an operator* $R \in \mathcal{L}(X)$ *with dense range* $\operatorname{rg} R$.

(c) *The semigroups* $(T_n(t))_{t\geq 0}$, $n \in \mathbb{N}$, *converge strongly (and uniformly for* $t \in [0, t_0]$*) to a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *with generator* $B$ *such that* $R = R(\lambda_0, B)$.

*Then the implications* (a) $\Rightarrow$ (b) $\iff$ (c) *hold. In particular, if* (a) *holds, then* $B = \overline{A}$.

PROOF. Without loss of generality, and after the usual rescaling, it suffices to consider uniformly bounded semigroups only.

(a) $\Rightarrow$ (b). As in the above proof, it suffices to show convergence of the sequence $(R(\lambda_0, A_n)y)_{n\in\mathbb{N}}$ for $y := (\lambda_0 - A)x$, $x \in D$, only. This follows, since

$$\begin{aligned} R(\lambda_0, A_n)y &= R(\lambda_0, A_n)[(\lambda_0 - A_n)x - (\lambda_0 - A_n)x + (\lambda_0 - A)x] \\ &= x + R(\lambda_0, A_n)(A_n x - Ax) \to x = Ry \end{aligned}$$

as $n \to \infty$. Moreover, $\operatorname{rg} R$ contains $D$, hence is dense in $X$.

Since the implication (c) $\Rightarrow$ (b) holds by the above theorem, it remains to prove that (b) $\Rightarrow$ (c). By Proposition 4.4, we obtain a pseudoresolvent $\{R(\lambda) : \lambda > 0\}$ by defining

$$R(\lambda)x := \lim_{n\to\infty} R(\lambda, A_n)x, \quad x \in X.$$

This pseudoresolvent satisfies, for all $\lambda > 0$,

$$\|\lambda R(\lambda)\| \leq M,$$

and, since $R(\lambda)^k = \lim_{n\to\infty} R(\lambda, A_n)^k$,

$$\|\lambda^k R(\lambda)^k\| \leq M \qquad \text{for all } k \in \mathbb{N}.$$

Moreover, it has dense range $\operatorname{rg} R(\lambda) = \operatorname{rg} R$. Therefore, Corollary 4.7 yields the existence of a densely defined operator $(B, D(B))$ such that $R(\lambda) = R(\lambda, B)$ for $\lambda > 0$. Moreover, this operator satisfies the Hille–Yosida estimate

$$\|\lambda^k R(\lambda, B)^k\| \leq M \qquad \text{for all } k \in \mathbb{N},$$

hence generates a bounded strongly continuous semigroup $(T(t))_{t\geq 0}$. We can now apply the implication (c) $\Rightarrow$ (d) from the First Trotter–Kato Approximation Theorem 4.8 in order to conclude that the semigroups $(T_n(t))_{n\geq 0}$ converge—in the desired way—to the semigroup $(T(t))_{t\geq 0}$.

In the final step, we show that (a) implies $\overline{A} = B$. Since $R(\lambda_0, B) = R$, we have

$$R(\lambda_0, B)(\lambda_0 - A)x = x$$

for all $x \in D$. However, $D$ is a core for $\overline{A}$, and therefore

$$R(\lambda_0, B)(\lambda_0 - \overline{A})x = x$$

for all $x \in D(\overline{A})$. From this it follows that $\lambda_0$ is not an approximate eigenvalue of $\overline{A}$. Moreover, $\operatorname{rg}(\lambda_0 - A)$ is dense in $X$ by assumption; hence $\lambda_0$ does not belong to the residual spectrum of $\overline{A}$. Therefore, $\lambda_0 \in \rho(\overline{A})$, and we obtain $R(\lambda_0, \overline{A}) = R(\lambda_0, B)$, i.e., $\overline{A} = B$ as claimed. □

The importance of the above theorems cannot be overestimated. In fact, they yield the theoretical background for many approximation schemes in abstract operator theory and applied numerical analysis. However, we restrict ourselves to rather abstract examples and applications.

## c. *Examples*

The Hille–Yosida Generation Theorem II.3.8 was the main tool in our proof of the Trotter–Kato approximation theorems. Conversely, this theorem was proved using an approximation argument. It is enlightening to start our series of examples by reformulating this part of the proof.

**4.10 Yosida Approximants.** Let $\big(A, D(A)\big)$ be an operator on $X$ satisfying the conditions (in the contractive case, for simplicity) from Generation Theorem II.3.5.(b). For each $n \in \mathbb{N}$, define the *Yosida approximant*

$$A_n := nAR(n, A) \in \mathcal{L}(X).$$

By Lemma II.3.4, the sequence $(A_n)_{n\in\mathbb{N}}$ converges pointwise on $D(A)$ to $A$. Since $\lambda - A$ is already supposed to be surjective, we can apply the Second Trotter–Kato Approximation Theorem 4.9 to conclude the existence of the limit semigroup $\big(T(t)\big)_{t\geq 0}$ with

$$T(t)x := \lim_{n\to\infty} \mathrm{e}^{tA_n}x, \quad x \in X,$$

and generator $\big(A, D(A)\big)$.

Clearly, in a logical sense, these arguments do not re-prove the Hille–Yosida generation theorem, which we already used for the proof of the Trotter–Kato approximation theorem. However, it might be helpful for the beginner to observe that the above approximating sequence enjoys a special feature: The operators $A_n$, $n \in \mathbb{N}$, mutually commute. This property allows a simple and direct proof (as observed by Goldstein [Gol85]) of the essential step in Approximation Theorem 4.8.

**Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ and $\big(T_n(t)\big)_{t\geq 0}$, $n \in \mathbb{N}$, be strongly continuous semigroups on $X$ with generator $\big(A, D(A)\big)$ and bounded generators $A_n$, respectively. In addition, suppose that $\big(T(t)\big)_{t\geq 0}$ and $\big(T_n(t)\big)_{t\geq 0}$ satisfy the stability condition* (4.6) *and that*

$$A_n T(t) = T(t) A_n$$

*for all $n \in \mathbb{N}$ and $t > 0$. If*

$$A_n x \to Ax$$

*for all $x$ in a core $D$ of $A$, then*

$$T_n(t)x \to T(t)x$$

*for all $x \in X$ uniformly for $t \in [0, t_0]$.*

Proof. For $x \in D$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} T_n(t)x - T(t)x &= -\int_0^t \tfrac{d}{ds}[T_n(t-s)T(s)x]\, ds \\ &= \int_0^t T_n(t-s)(A_n - A)T(s)x\, ds \\ &= \int_0^t T_n(t-s)T(s)(A_n x - Ax)\, ds, \end{aligned}$$

hence

$$\|T_n(t)x - T(t)x\| \leq tM^2 \mathrm{e}^{wt}\, \|A_n x - Ax\|.$$

□

We encounter this situation in our next example, by which we re-prove a classical theorem.

**4.11 Weierstrass Approximation Theorem.** Take the function space $X := \mathrm{C}_0(\mathbb{R})$ (or $\mathrm{C}_{\mathrm{ub}}(\mathbb{R})$) and the (left) translation group $\big(T(t)\big)_{t\in\mathbb{R}}$ with

$$T(t)f(s) := f(s+t) \qquad \text{for } s, t \in \mathbb{R}$$

and generator

$$Af := f', \quad D(A) := \{f \in X : f' \in X\}$$

(see Paragraph II.2.10). The bounded operators

$$A_n := \frac{T(1/n) - I}{1/n}, \quad n \in \mathbb{N},$$

- commute with all operators $T(t)$,
- generate contraction semigroups, since

  $$\|\mathrm{e}^{tA_n}\| = \left\|\mathrm{e}^{nt(T(1/n)-I)}\right\| \leq \mathrm{e}^{-nt}\mathrm{e}^{nt\|T(1/n)\|} = 1, \tag{4.7}$$

  and
- satisfy, by definition of the derivative,

  $$A_n f \to Af$$

  for each $f \in D(A)$.

Therefore, the (First) Trotter–Kato Approximation Theorem 4.8 (or the lemma in 4.10) can be applied and yields

$$f(s+t) = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_n^k f)(s) \tag{4.8}$$

for all $f \in X$ and uniformly for $s \in \mathbb{R}$, $t \in [0,1]$. If we now take $s = 0$, choose an appropriate sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers, and observe that $\sum_{k=0}^{m_n} {t^k}/{k!} (A_n^k f)(0)$ is a polynomial, we obtain the Weierstrass approximation theorem as a consequence.

**Proposition.** *For every $f \in X$ there exists a sequence $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that*

$$f(t) = \lim_{n\to\infty} \sum_{k=0}^{m_n} \frac{t^k}{k!} (A_n^k f)(0) \tag{4.9}$$

*uniformly for $t \in [0,1]$.*

It is very instructive to observe how convergence breaks down if we reverse the order of the limit and of the series summation in (4.9). See the illuminating remarks in [Gol85, Sec. I.8.3].

**4.12 A First Approximation Formula.** The idea employed in Paragraph 4.11 is very simple and can be formulated in a general context. Let $\big(T(t)\big)_{t\ge 0}$ be a strongly continuous contraction semigroup on $X$ with generator $\big(A, D(A)\big)$. Then the bounded operators

$$A_n := \frac{T({1}/{n}) - I}{{1}/{n}}, \qquad n \in \mathbb{N},$$

approximate $A$ on $D(A)$ and generate contraction semigroups $\big(\mathrm{e}^{tA_n}\big)_{t\ge 0}$ (see (4.7)). Therefore, we obtain the following approximation formula.

**Proposition.** *With the above definitions, one has*

$$T(t)x = \lim_{n\to\infty} \mathrm{e}^{-nt} \mathrm{e}^{ntT({1}/{n})} x \tag{4.10}$$

*for all $x \in X$ and uniformly in $t \in [0, t_0]$.*

This formula might seem useless, since it assumes that the operators $T(t)$ are already known, at least for small $t > 0$. However, it is the first step towards more interesting approximation formulas to be developed in the next section. Before doing so, we apply our Trotter–Kato approximation theorems to a generalization of the multiplication semigroups from Section I.4.a and Paragraph II.2.9.

**4.13 Operator-Valued Multiplication Semigroups.** For a given Banach space $X$, we consider the $X$-valued function space

$$\mathcal{X} := \mathrm{C}_0(\mathbb{R}, X)$$

of all continuous functions from $\mathbb{R}$ into $X$ vanishing at infinity and endow this space with the sup-norm. We assume that for each $s \in \mathbb{R}$, there are generators $(A(s), D(A(s)))$ of a strongly continuous semigroup $(T_s(t))_{t\ge 0}$ that all satisfy the common estimate

$$\|T_s(t)\| \le M \cdot \mathrm{e}^{wt} \qquad \text{for } t \ge 0 \text{ and } s \in \mathbb{R}.$$

The map $s \mapsto A(s)$ should be *continuous* in the following sense:

For each $s_0 \in \mathbb{R}$, each sequence $s_n \to s_0$, and each $x_0 \in D(A(s_0))$, there exist $x_n \in D(A(s_n))$ such that $x_n \to x_0$ and $A(s_n)x_n \to A(s_0)x_0$.

In the case that there is a common core $D \subset \bigcap_{s\in\mathbb{R}} D(A(s))$, this is implied by

$$\lim_{s\to s_0} A(s)x = A(s_0)x \qquad \text{for all} \qquad s_0 \in \mathbb{R},\ x \in D.$$

In the case that $A(s) \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$ and $\|A(s)\|$ remains bounded on bounded intervals, this means that the function

$$\mathbb{R} \ni s \mapsto A(s) \in \mathcal{L}(X)$$

is continuous for the strong operator topology on $\mathcal{L}(X)$.

The operator-valued function $s \mapsto T_s(t)$, for fixed $t \ge 0$, can now be considered as a single operator on the function space $\mathcal{X}$ acting by pointwise multiplication. This yields an *(operator-valued) multiplication semigroup*.

**Proposition.** *If $s \mapsto A(s)$ is continuous in the above sense, then*

$$\mathcal{M}(t)f(s) := T_s(t)f(s) \qquad \textit{for } s \in \mathbb{R}, t \ge 0 \textit{ and } f \in \mathcal{X}$$

*defines a strongly continuous semigroup $(\mathcal{M}(t))_{t\ge 0}$ on $\mathcal{X}$ satisfying*

$$\|\mathcal{M}(t)\| \le M\mathrm{e}^{wt} \qquad \textit{for } t \ge 0. \tag{4.11}$$

Proof. We start by showing that the map

$$\mathbb{R} \times \mathbb{R}_+ \ni (s,t) \mapsto T_s(t)f(s) \in X \tag{4.12}$$

is continuous for every $f \in \mathcal{X}$. To that purpose we estimate

$$\begin{aligned}\|T_{s_0}(t_0)f(s_0) - T_s(t)f(s)\| \le\ & \|T_{s_0}(t_0)f(s_0) - T_{s_0}(t)f(s_0)\| \\ &+ \|T_{s_0}(t)f(s_0) - T_s(t)f(s_0)\| \\ &+ \|T_s(t)f(s_0) - T_s(t)f(s)\|\end{aligned}$$

and observe that

- $\lim_{t\to t_0} \|(T_{s_0}(t_0) - T_{s_0}(t))f(s_0)\| = 0$, since the semigroup $(T_{s_0}(t))_{t\ge 0}$ is strongly continuous,
- $\lim_{s\to s_0} \|(T_{s_0}(t) - T_s(t))f(s_0)\| = 0$, since, due to the continuity assumption on $s \mapsto A(s)$, we can apply Approximation Theorem 4.8,
- $\lim_{s\to s_0} \|T_s(t)(f(s_0) - f(s))\| = 0$, since $f$ is continuous and $\|T_s(t)\| \le M \cdot \mathrm{e}^{wt}$.

From these considerations, it is now clear that

$$s \mapsto \big(\mathcal{M}(t)f\big)(s) = T_s(t)f(s)$$

is a function in $\mathcal{X}$ and that $(\mathcal{M}(t))_{t\ge 0}$ is a semigroup on $\mathcal{X}$ satisfying the estimate (4.11). Its strong continuity needs to be checked only for functions with compact support. But this follows from the uniform continuity of the map in (4.12) on compact sets of the form $[-n, n] \times [0, 1]$. ☐

As one might expect from the scalar case in Paragraph II.2.9, the generator of the semigroup $(\mathcal{M}(t))_{t\geq 0}$ is the multiplication operator induced by the operators $A(s)$.

**Corollary.** *The generator $(\mathcal{A}, D(\mathcal{A}))$ of the above semigroup $(\mathcal{M}(t))_{t\geq 0}$ is the (operator-valued) multiplication operator*

$$\big(\mathcal{A}f\big)(s) := A(s)f(s), \quad s \in \mathbb{R},$$

*with (maximal) domain*

$$D(\mathcal{A}) := \big\{f \in \mathcal{X} : f(s) \in D\big(A(s)\big) \text{ for } s \in \mathbb{R} \text{ and } A(\cdot)f(\cdot) \in \mathcal{X}\big\}.$$

PROOF. Let $(\mathcal{B}, D(\mathcal{B}))$ be the generator of $(\mathcal{M}(t))_{t\geq 0}$. For $f \in D(\mathcal{A})$, we have

$$\left\| \frac{T_s(t)f(s) - f(s)}{t} - A(s)f(s) \right\| = \left\| \frac{1}{t} \int_0^t \big(T_s(r)A(s)f(s) - A(s)f(s)\big)\, dr \right\| \leq \sup_{0\leq r\leq t} \|T_s(r)A(s)f(s) - A(s)f(s)\|.$$

Since the map $(s,t) \mapsto T_s(t)A(s)f(s)$ is continuous, it follows that

$$\lim_{t\downarrow 0} \sup_{s\in\mathbb{R}} \left\| \frac{T_s(t)f(s) - f(s)}{t} - A(s)f(s) \right\| = 0,$$

and hence $\mathcal{A} \subset \mathcal{B}$. From the definition it follows that the operator $\mu - \mathcal{A}$, for $\mu > w$, has a bounded inverse

$$\big(R(\mu, \mathcal{A})f\big)(s) = R\big(\mu, A(s)\big)f(s)$$

for $f \in \mathcal{X}$, $s \in \mathbb{R}$. On the other hand, $\mu - \mathcal{B}$ is invertible, since $\mathcal{B}$ is the generator of a semigroup satisfying (4.11). This implies $\mu - \mathcal{A} = \mu - \mathcal{B}$; hence $\mathcal{A} = \mathcal{B}$. □

We will use this semigroup in Example 5.9.

**4.14 Exercises.** (1) Discuss the continuity properties of the map $s \mapsto A(s)$ stated at the beginning of Paragraph 4.13. Find an example satisfying the first, but not the second, property.

(2) Consider the operator $Af := f''$ with maximal domain on $X := \mathrm{C}_0(\mathbb{R})$. For each $n \in \mathbb{N}$, we define bounded difference operators

$$A_n f(s) := n^2\big[f(s + {}^1\!/\!_n) - 2f(s) + f(s - {}^1\!/\!_n)\big], \quad s \in \mathbb{R}, f \in X.$$

Prove the following statements.

(i) $(A, D(A))$ is a closed, densely defined operator.

(ii) $\big\|\mathrm{e}^{tA_n}\big\| \leq 1$ for each $n \in \mathbb{N}$, and $A_n f \to Af$ for $f \in D(A)$.

(iii) For each $g \in X$, there exists a unique $f \in D(A)$ such that $f - f'' = g$. (Hint: Use the formal identity $2(I - ({}^d\!/\!_{ds})^2)^{-1} = (I - {}^d\!/\!_{ds})^{-1}(I + {}^d\!/\!_{ds})^{-1}$ and the resolvent formula for ${}^d\!/\!_{ds}$ from Paragraph IV.1.2. Check that this yields the correct solution.)

(iv) $(A, D(A))$ generates the strongly continuous semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)f(s) = \lim_{n\to\infty} \mathrm{e}^{-2n^2 t} \sum_{k=0}^{\infty} \frac{(n^2 t)^k}{k!} \sum_{l=0}^{k} \binom{k}{l} f\big(s + {}^{(k-2l)}\!/\!_n\big)$$

for $s \in \mathbb{R}$, $f \in X$.

# 5. Approximation Formulas

As announced in the previous section, it is now our aim to obtain more or less explicit formulas for the semigroup operators $T(t)$. These formulas are based on some knowledge of the generator (and its resolvent) and the Trotter–Kato approximation theorem.

## a. Chernoff Product Formula

Our first approach is via the Chernoff product formula, from which many approximation formulas can be derived. For its proof the following estimate will be essential.

**5.1 Lemma.** *Let $S \in \mathcal{L}(X)$ satisfy $\|S^m\| \le M$ for some $M \ge 1$ and all $m \in \mathbb{N}$. Then we have*

$$\left\| \mathrm{e}^{n(S-I)}x - S^n x \right\| \le \sqrt{n} M \, \|Sx - x\| \tag{5.1}$$

*for every $n \in \mathbb{N}$ and $x \in X$.*

PROOF. Let $n \in \mathbb{N}$ be fixed and observe that

$$\mathrm{e}^{n(S-I)} - S^n = \mathrm{e}^{-n}\left(\mathrm{e}^{nS} - \mathrm{e}^n S^n\right) = \mathrm{e}^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!}\left(S^k - S^n\right).$$

For $k > n$, we write

$$S^k - S^n = \sum_{j=n}^{k-1}\left(S^{j+1} - S^j\right) = \sum_{j=n}^{k-1} S^j (S - I),$$

and similarly for $k < n$. Therefore, and since $\|S^m\| \le M$, we obtain

$$\left\|S^k x - S^n x\right\| \le |n-k| \cdot M \, \|Sx - x\|$$

for all $k \in \mathbb{N}$, $x \in X$. This allows the estimate

$$\begin{aligned}
\left\| \mathrm{e}^{n(S-I)}x - S^n x \right\| &\le \mathrm{e}^{-n} M \, \|Sx - x\| \cdot \sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right)^{1/2} \left(\frac{n^k}{k!}\right)^{1/2} |n-k| \\
&\le \mathrm{e}^{-n} M \, \|Sx - x\| \cdot \left(\sum_{k=0}^{\infty} \frac{n^k}{k!}\right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2\right)^{1/2} \\
&= \mathrm{e}^{-n} M \, \|Sx - x\| \cdot \left(\mathrm{e}^n\right)^{1/2} \left(n\mathrm{e}^n\right)^{1/2} \\
&= \sqrt{n} M \, \|Sx - x\|,
\end{aligned}$$

where we used the Cauchy–Schwarz inequality and the identity

$$\sum_{k=0}^{\infty} \frac{n^k}{k!}(n-k)^2 = n\mathrm{e}^n.$$

$\square$

This lemma, combined with Approximation Theorem 4.9, yields the main result of this section.

**5.2 Theorem.** (*Chernoff Product Formula*). *Consider a function*

$$V : \mathbb{R}_+ \to \mathcal{L}(X)$$

*satisfying* $V(0) = I$ *and* $\big\|\big[V(t)\big]^m\big\| \le M$ *for all* $t \ge 0$, $m \in \mathbb{N}$, *and some* $M \ge 1$. *Assume that*

$$Ax := \lim_{h \downarrow 0} \frac{V(h)x - x}{h}$$

*exists for all* $x \in D \subset X$, *where* $D$ *and* $(\lambda_0 - A)D$ *are dense subspaces in* $X$ *for some* $\lambda_0 > 0$. *Then the closure* $\overline{A}$ *of* $A$ *generates a bounded strongly continuous semigroup* $\big(T(t)\big)_{t\ge 0}$, *which is given by*

$$T(t)x = \lim_{n\to\infty} \left[V({}^t\!/\!_n)\right]^n x \tag{5.2}$$

*for all* $x \in X$ *and uniformly for* $t \in [0, t_0]$.

Proof. For $s > 0$, define

$$A_s := \frac{V(s) - I}{s} \in \mathcal{L}(X),$$

and observe that $A_s x \to Ax$ for all $x \in D$ as $s \downarrow 0$. The semigroups $(\mathrm{e}^{tA_s})_{t\ge 0}$ all satisfy

$$\big\|\mathrm{e}^{tA_s}\big\| \le \mathrm{e}^{-t/s} \Big\|\mathrm{e}^{tV(s)/s}\Big\| \le \mathrm{e}^{-t/s} \sum_{m=0}^{\infty} \frac{t^m \,\big\|\big[V(s)\big]^m\big\|}{s^m m!} \le M \quad \text{for every } t \ge 0.$$

This shows that the assumptions of the Second Trotter–Kato Approximation Theorem 4.9 are fulfilled (with the discrete parameter $n \in \mathbb{N}$ replaced by the continuous parameter $s > 0$). Hence, the closure $\overline{A}$ of $A$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ satisfying

$$\big\|T(t)x - \mathrm{e}^{tA_s}x\big\| \to 0 \qquad \text{for all } x \in X \text{ as } s \downarrow 0$$

uniformly for $t \in [0, t_0]$, and therefore

$$\Big\|T(t)x - \mathrm{e}^{tA_{t/n}}x\Big\| \to 0 \qquad \text{for all } x \in X \text{ as } n \to \infty \tag{5.3}$$

uniformly for $t \in [0, t_0]$.

On the other hand, we have by Lemma 5.1 that

$$(5.4)\quad \begin{aligned}\left\| \mathrm{e}^{tA_{t/n}}x - [V({}^t\!/\!_n)]^n\, x\right\| &= \left\| \mathrm{e}^{n(V({}^t\!/\!_n)-I)}x - [V({}^t\!/\!_n)]^n\, x\right\| \\ &\le \sqrt{n}M \left\| V({}^t\!/\!_n)x - x\right\| \\ &= \frac{tM}{\sqrt{n}} \left\| A_{t/n}x\right\| \to 0\end{aligned}$$

for all $x \in D$ as $n \to \infty$, uniformly on $(0, t_0]$. Since $\left\| \mathrm{e}^{tA_{t/n}} - [V({}^t\!/\!_n)]^n \right\| \le 2M$, the combination of (5.3), (5.4), and Proposition A.3 yields (5.2). □

As before, we pass to the unbounded case by a rescaling procedure.

**5.3 Corollary.** *Consider a function*

$$V : \mathbb{R}_+ \to \mathcal{L}(X)$$

*satisfying* $V(0) = I$ *and*

$$\left\| [V(t)]^k \right\| \le M\mathrm{e}^{kwt} \qquad \text{for all } t \ge 0, k \in \mathbb{N}$$

*and some constants* $M \ge 1$, $w \in \mathbb{R}$. *Assume that*

$$Ax := \lim_{t \downarrow 0} \frac{V(t)x - x}{t}$$

*exists for all* $x \in D \subset X$, *where* $D$ *and* $(\lambda_0 - A)D$ *are dense subspaces in* $X$ *for some* $\lambda_0 > w$. *Then the closure* $\overline{A}$ *of* $A$ *generates a strongly continuous semigroup* $\big(T(t)\big)_{t\ge0}$ *given by*

$$(5.5)\qquad T(t)x = \lim_{n\to\infty} [V({}^t\!/\!_n)]^n x$$

*for all* $x \in X$ *and uniformly for* $t \in [0, t_0]$. *Moreover,* $\big(T(t)\big)_{t\ge0}$ *satisfies the estimate*

$$\|T(t)\| \le M\mathrm{e}^{wt} \qquad \text{for all } t \ge 0.$$

Proof. From the function $V(\cdot)$ we pass to

$$\widetilde{V}(t) := \mathrm{e}^{-wt}V(t),$$

which then satisfies

$$\left\| \widetilde{V}(t)^k \right\| \le M \qquad \text{for all } k \in \mathbb{N} \text{ and } t \ge 0$$

and whose derivative in zero is the operator $A - w$. The assertions then follow from Theorem 5.2. □

Next, we substitute the "time steps" of size "${}^t\!/\!_n$" in the definition of the approximating operators $V({}^t\!/\!_n)$ by an arbitrary null sequence $(t_n)_{n\in\mathbb{N}}$.

**5.4 Corollary.** *Let $V : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfy the assumptions in Corollary 5.3. If for fixed $t > 0$ we take a positive null sequence $(t_n)_{n\in\mathbb{N}}$ and a strictly increasing sequence of integers $k_n$ such that*

$$k_n t_n \to t,$$

*then*

$$T(t)x = \lim_{n\to\infty} [V(t_n)]^{k_n} x \tag{5.6}$$

*for all $x \in X$.*

PROOF. Using the function

$$\xi(s) := \begin{cases} s \cdot {}^{(t_n k_n)}/_t & \text{for } s \in ({}^t/_{k_{n+1}}, {}^t/_{k_n}], \\ 0 & \text{for } s = 0 \quad \text{or } s > {}^t/_{k_1}, \end{cases}$$

we introduce a new operator-valued function $W : \mathbb{R}_+ \to \mathcal{L}(X)$ by

$$W(t) := V\big(\xi(t)\big), \quad t \geq 0.$$

This function still satisfies $W(0) = I$ and $\|W(t)^k\| \leq M\mathrm{e}^{kwt}$ for all $t \geq 0$, $k \in \mathbb{N}$. For $x \in D$, we show that

$$\lim_{t\downarrow 0} \frac{W(t)x - x}{t} = Ax.$$

Let $(t_n)_{n\in\mathbb{N}}$ be an arbitrary null sequence and for each $t_m$ choose $n_m \in \mathbb{N}$ such that $t_m \in ({}^t/_{k_{n_m+1}}, {}^t/_{k_{n_m}}]$. Then

$$\begin{aligned} \frac{W(t_m)x - x}{t_m} &= \frac{V\big(\xi(t_m)\big)x - x}{\xi(t_m)} \cdot \frac{\xi(t_m)}{t_m} \\ &= \frac{V\big(\xi(t_m)\big)x - x}{\xi(t_m)} \cdot \frac{t_{n_m} k_{n_m} \cdot t_m}{t \cdot t_m}, \end{aligned}$$

hence

$$\lim_{m\to\infty} \frac{W(t_m)x - x}{t_m} = Ax \cdot \lim_{m\to\infty} \frac{t_{n_m} k_{n_m}}{t} = Ax.$$

By Corollary 5.3, we conclude that $\overline{A}$ generates the semigroup $\big(T(t)\big)_{t\geq 0}$ given by

$$T(t)x = \lim_{n\to\infty} [W({}^t/_n)]^n x$$

uniformly for $t \in [0, t_0]$. In particular, we obtain for the subsequence $({}^t/_{k_n})_{n\in\mathbb{N}}$ that

$$\begin{aligned} T(t)x &= \lim_{n\to\infty} [W({}^t/_{k_n})]^{k_n} x \\ &= \lim_{n\to\infty} \big[V\big(\xi({}^t/_{k_n})\big)\big]^{k_n} x \\ &= \lim_{n\to\infty} \big[V(t_n)\big]^{k_n} x \qquad \text{for all } x \in X. \end{aligned}$$

□

The following application of the Chernoff Product Formula Theorem 5.2 (or of Corollary 5.3) finally gives us an explicit formula, called the *Post–Widder Inversion Formula*, for the semigroup in terms of the resolvent of its generator. This adds a missing arrow to the "triangle" from Diagram II.1.14, and, at the same time, corresponds to Hille's original proof of Generation Theorem II.3.5.

**5.5 Corollary.** *For every strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ on $X$ with generator $\big(A, D(A)\big)$, one has*

$$T(t)x = \lim_{n\to\infty} \left[{}^{n}\!/\!{}_{t} R({}^{n}\!/\!{}_{t}, A)\right]^n x = \lim_{n\to\infty} \left[I - {}^{t}\!/\!{}_{n} A\right]^{-n} x, \quad x \in X, \tag{5.7}$$

*uniformly for $t$ in compact intervals.*

Proof. Assume that $\|T(t)\| \le M e^{wt}$ for constants $M \ge 1$, $w > 0$ and define

$$V(t) := \begin{cases} I & \text{for } t = 0, \\ {}^{1}\!/\!{}_{t}\, R({}^{1}\!/\!{}_{t}, A) & \text{for } t \in (0, \delta), \\ 0 & \text{for } t \ge \delta, \end{cases}$$

for some $\delta \in (0, {}^{1}\!/\!{}_{w})$. In this way we obtain a function $V : \mathbb{R}_+ \to \mathcal{L}(X)$ satisfying

$$\left\|V(t)^k\right\| \le {}^{1}\!/\!{}_{t^k} \left\|R({}^{1}\!/\!{}_{t}, A)^k\right\| \le \frac{M}{t^k({}^{1}\!/\!{}_{t} - w)^k} = \frac{M}{(1 - wt)^k} \le M e^{k(w+1)t}$$

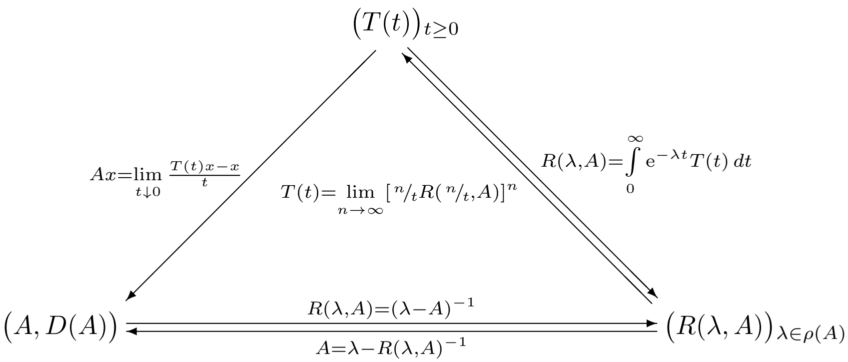
for all $t \in (0, \delta)$, provided that we choose $\delta > 0$ sufficiently small. Moreover, by Lemma II.3.4, we have

$$\lim_{t\downarrow 0} \frac{V(t)x - x}{t} = \lim_{t\downarrow 0} {}^{1}\!/\!{}_{t} R({}^{1}\!/\!{}_{t}, A)Ax = Ax \qquad \text{if } x \in D(A).$$

Therefore, the Chernoff product formula as stated in Corollary 5.3 applies, and (5.5) becomes the above formula. □

For the sake of completeness, we add this new relation to the diagram relating the semigroup, its generator, and its resolvent operators.

**5.6 Diagram.**



We now apply the Trotter–Kato approximation theorems and, in particular, the Chernoff product formula from Theorem 5.2 to more concrete situations. First, we relate it to a classical approximation process via Bernstein polynomials for continuous functions on $[0,1]$.

**5.7 Bernstein Approximation of a Diffusion Semigroup.** We recall that for each function $f \in \mathrm{C}[0,1]$ the corresponding $n$th *Bernstein polynomial* is defined by

$$B_n f(s) := \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} f(k/n), \quad s \in [0,1]. \tag{5.8}$$

In this way, we obtain the sequence of *Bernstein operators*

$$B_n : f \mapsto B_n f,$$

which are positive contractions on the Banach space $\mathrm{C}[0,1]$ and converge strongly to the identity operator.

**Lemma 1.** *For every $f \in \mathrm{C}[0,1]$, we have*

$$\lim_{n\to\infty} B_n f = f. \tag{5.9}$$

This lemma, which also re-proves the Weierstrass Approximation Theorem 4.11, is one of the fundamental results of classical approximation theory and can be proved in many different ways. A very elegant proof uses Korovkin's theorem, which assures that

$$\lim_{n\to\infty} B_n f = f \qquad \text{for } all \ f \in \mathrm{C}[0,1],$$

if this convergence holds for the three functions

$$\mathbb{1}(s) := 1, \quad id(s) := s, \qquad \text{and} \qquad id^2(s) := s^2 \qquad \text{for } s \in [0,1].$$

This, however, is straightforward, since for $s \in [0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned} B_n \mathbb{1}(s) &= 1, \\ B_n id(s) &= s, \\ B_n id^2(s) &= s^2 + \frac{s(1-s)}{n}. \end{aligned} \tag{5.10}$$

We refer to [AC94, Thm. 4.2.7] for details or to [Lor53, Thm. 1.1.1] for a direct proof.

For our purpose, it is more important to prove a formula due to E. Voronovskaja [Vor32] that relates the Bernstein operators to a certain differential operator.

**Lemma 2.** *For every $f \in \mathrm{C}^2[0,1]$, we have*

$$\lim_{n\to\infty} \frac{B_n f(s) - f(s)}{1/n} = \frac{1}{2} s(1-s) f''(s) \qquad \textit{uniformly for } s \in [0,1]. \tag{5.11}$$

Proof. Since $f$ is twice continuously differentiable, we can write

$$f(t) - f(s) = (t-s) f'(s) + (t-s)^2 \left(\tfrac{1}{2} f''(s) + \eta(s,t)\right)$$

for $s, t \in [0,1]$ and a bounded function $\eta : [0,1]^2 \to \mathbb{C}$ satisfying

$$\lim_{t \to s} \eta(s,t) = 0$$

uniformly for $s \in [0,1]$. We use this identity for $t = {}^{k}/_{n}$ and obtain

$$\begin{aligned}
B_n f(s) - f(s) &= \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \left( f\left(\frac{k}{n}\right) - f(s) \right) \\
&= f'(s)(B_n id(s) - s) + \frac{1}{2} f''(s) \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \left(\frac{k}{n} - s\right)^2 \\
&\quad + \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 \\
&= \frac{1}{2} f''(s) \left( B_n id^2(s) - 2s B_n id(s) + s^2 B_n \mathbb{1}(s) \right) \qquad (5.12) \\
&\quad + \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 \\
&= \frac{s(1-s)}{2n} f''(s) + \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 .
\end{aligned}$$

Take $M > 0$ such that $|\eta(s,t)| \leq M$ for every $s, t \in [0,1]$, and for $\varepsilon > 0$, choose $\delta > 0$ such that $|\eta(s,t)| < \varepsilon$ whenever $|s - t| < \delta$.

Moreover, we observe that with a simple calculation based on the formulas in (5.10), we can evaluate $B_n id^3$ and $B_n id^4$ and obtain

$$\sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \left(\frac{k}{n} - s\right)^4 = \frac{3s^2(1-s)^2}{n^2} + \frac{s(1-s)(1-6s(1-s))}{n^3} \leq \frac{C}{n^2}$$

for a suitable constant $C > 0$ and hence

$$\begin{aligned}
&\left| \sum_{k=0}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 \right| \\
&\qquad \leq \left| \sum_{\substack{k=0 \\ |{}^{k}/_{n} - s| < \delta}}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 \right| \\
&\qquad\quad + \left| \sum_{\substack{k=0 \\ |{}^{k}/_{n} - s| \geq \delta}}^{n} \binom{n}{k} s^k (1-s)^{n-k} \eta\left(s, \frac{k}{n}\right) \left(\frac{k}{n} - s\right)^2 \right| \\
&\qquad < \varepsilon \frac{s(1-s)}{2n} + \frac{M}{\delta^2} \sum_{\substack{k=0 \\ |{}^{k}/_{n} - s| \geq \delta}}^{n} \binom{n}{k} s^k (1-s)^{n-k} \left(\frac{k}{n} - s\right)^4 \\
&\qquad \leq \frac{\varepsilon}{8n} + \frac{M}{\delta^2} \cdot \frac{C}{n^2}.
\end{aligned}$$

By (5.12), we obtain

$$\left| n\big(B_n f(s) - f(s)\big) - \frac{s(1-s)}{2} f''(s) \right| \le \frac{\varepsilon}{8} + \frac{M}{\delta^2} \cdot \frac{C}{n}$$

and therefore

$$\overline{\lim_{n\to+\infty}} \left| n\big(B_n f(s) - f(s)\big) - \frac{s(1-s)}{2} f''(s) \right| \le \frac{\varepsilon}{8}.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

The differential operator appearing in (5.11) is already familiar to us. In fact, in Paragraph II.3.30.(iii) we have seen that the operator

$$Af(s) := \frac{1}{2} s(1-s) f''(s), \quad s \in [0,1],$$

with domain

$$D(A) := \left\{ f \in \mathrm{C}[0,1] : f \in \mathrm{C}^2(0,1) \text{ and } \lim_{s\to 0,1} s(1-s) f''(s) = 0 \right\}$$

generates a strongly continuous contraction semigroup $(T(t))_{t\ge 0}$ on $\mathrm{C}[0,1]$. The space $\mathrm{C}^2[0,1]$ is a core for $(A, D(A))$; hence the limit in (5.11) coincides on a core with a generator. This allows us to apply Theorem 5.2.

**Proposition.** *The semigroup $(T(t))_{t\ge 0}$ generated by the differential operator*

$$\begin{aligned} Af(s) &:= \frac{1}{2} s(1-s) f''(s), \quad s \in [0,1], \\ D(A) &:= \left\{ f \in \mathrm{C}[0,1] : f \in \mathrm{C}^2(0,1) \text{ and } \lim_{s\to 0,1} s(1-s) f''(s) = 0 \right\} \end{aligned}$$

*can be obtained as*

$$T(t)f = \lim_{n\to\infty} B_n^{k_n} f \qquad \text{for all } f \in \mathrm{C}[0,1].$$

*Here, the sequence of natural numbers $k_n$ depends on $t > 0$ and has to satisfy*

$$\lim_{n\to\infty} \frac{k_n}{n} = t.$$

PROOF. We use (the second part of) Corollary 5.3 and define

$$V(t) := B_n \qquad \text{for } \frac{1}{n} \le t < \frac{1}{n-1}, \quad n \ge 2.$$

Then the function $V(\cdot)$ consists of contractions and, due to Lemma 2, satisfies all the other assumptions of Corollary 5.3. Now take

$$t_n := \frac{1}{n} \qquad \text{and } k_n \in \mathbb{N} \qquad \text{such that } k_n t_n \to t.$$

Then (5.6) becomes the above assertion, since $V(t_n) = B_n$. □

Our next application of the Chernoff product formula from Theorem 5.2 is to perturbation theory and yields another important formula, called the *Trotter product formula*, for the perturbed semigroup. In contrast to the situation studied in Sections 1, 2, and 3, we obtain a result that is symmetric in the operators $A$ and $B$.

**5.8 Corollary.** *Let $\big(T(t)\big)_{t\ge0}$ and $\big(S(t)\big)_{t\ge0}$ be strongly continuous semigroups on $X$ satisfying the stability condition*

$$\|[T(t/n)S(t/n)]^n\| \le M\mathrm{e}^{wt} \qquad \text{for all } t\ge 0, n\in\mathbb{N}, \tag{5.13}$$

*and for constants $M\ge 1$, $w\in\mathbb{R}$. Consider the "sum" $A+B$ on $D := D(A)\cap D(B)$ of the generators $\big(A,D(A)\big)$ of $\big(T(t)\big)_{t\ge0}$ and $\big(B,D(B)\big)$ of $\big(S(t)\big)_{t\ge0}$, and assume that $D$ and $(\lambda_0 - A - B)D$ are dense in $X$ for some $\lambda_0 > w$. Then $C := \overline{A+B}$ generates a strongly continuous semigroup $\big(U(t)\big)_{t\ge0}$ given by the Trotter product formula*

$$U(t)x = \lim_{n\to\infty} [T(t/n)S(t/n)]^n\, x, \quad x\in X, \tag{5.14}$$

*with uniform convergence for $t$ in compact intervals.*

Proof. In order to apply the Chernoff product formula from Corollary 5.3, it suffices to define

$$V(t) := T(t)S(t), \quad t\ge 0,$$

and observe that

$$\begin{aligned}\lim_{t\downarrow0}\frac{T(t)S(t)y-y}{t} &= \lim_{t\downarrow0} T(t)\,\frac{S(t)y-y}{t} + \lim_{t\downarrow0}\frac{T(t)y-y}{t}\\ &= By + Ay\end{aligned}$$

for all $y\in D$. □

The following application of the Trotter product formula is to *nonautonomous Cauchy problems*. A systematic treatment of these problems via semigroups will be given in Section VI.9.

**5.9 Example.** For bounded operators $A(t)$, $t\in\mathbb{R}$, on a Banach space $X$ we consider the nonautonomous abstract Cauchy problem

$$\begin{cases}\dot u(t) = A(t)u(t), & t\ge s,\\ u(s) = x\in X,\end{cases} \tag{nACP}$$

cf. Section VI.9. If we assume that $t\mapsto A(t)$ is strongly continuous, it is well known that there is a unique family of bounded operators $U(t,s)$, $t,s\in\mathbb{R}$, $t\ge s$, on $X$ such that

$$\begin{aligned}&U(s,s)=I, \qquad U(t,s) = U(t,r)U(r,s),\\ &\big\{(\tau,\sigma)\in\mathbb{R}^2 : \tau\ge\sigma\big\} \ni (t,s)\mapsto U(t,s)\in\mathcal{L}(X) \quad \text{is differentiable, and}\\ &\tfrac{d}{dt}U(t,s) = A(t)U(t,s), \qquad \tfrac{d}{ds}U(t,s) = -U(t,s)A(s),\end{aligned}$$

see [Fat83, Thm. 7.1.1 and Expl. 7.1.6] (see also [DK74, § III.1]). In particular, the map $t\mapsto U(t,s)x$ belongs to $\mathrm{C}^1([s,\infty),X)$ and is the unique solution of (nACP).

We now want to use the Trotter product formula to approximate the operator $U(t,s)$, i.e., the solutions of (nACP). As approximating operators we choose the product

$$\prod_{k=1}^{n} \mathrm{e}^{t/n A(s+kt/n)} = \mathrm{e}^{t/n A(s+t)} \cdot \mathrm{e}^{t/n A(s+t-t/n)} \cdots \mathrm{e}^{t/n A(s+t/n)},$$

which correspond to the problem (nACP) on the interval $[s, s+t]$ with $A(t)$ replaced by the piecewise constant operators $A(s+kt/n)$.

**Proposition.** *Let* $A(\cdot) \in \mathrm{C}(\mathbb{R}, \mathcal{L}_s(X))$. *Then*

$$\lim_{n\to\infty} \prod_{k=1}^{n} \mathrm{e}^{t/n A(s+kt/n)} x = U(s+t,s)x$$

*for all* $x \in X$ *and uniformly for* $s$ *and* $t$ *in compact intervals of* $\mathbb{R}$ *and* $\mathbb{R}_+$, *respectively.*

Proof. Let $a < b$ and notice that

$$\left\| \prod_{k=1}^{n} \mathrm{e}^{t/n A(s+kt/n)} \right\| \le \prod_{k=1}^{n} \mathrm{e}^{tc/n} \le \mathrm{e}^{(b-a)c} =: M \tag{5.15}$$

for $c := \sup_{a\le s\le b} \|A(s)\| < \infty$ and $a \le s \le s+t \le b$. On the space $\mathcal{X} := \mathrm{C}_0((a,b],X)$ we define the bounded multiplication operator

$$(\mathcal{A}f)(s) := A(s)f(s) \qquad \text{for } a < s \le b \text{ and } f \in \mathcal{X},$$

which generates the multiplication semigroup $(\mathcal{M}(t))_{t\ge0}$ on $\mathcal{X}$ given by

$$(\mathcal{M}(t)f)(s) = \mathrm{e}^{tA(s)} f(s);$$

see Paragraph 4.13. Moreover, the left translation semigroup $(S(t))_{t\ge0}$ on $\mathcal{X}$ defined by

$$\big(S(t)f\big)(s) = \begin{cases} f(s-t) & \text{if } a < s-t \le s \le b, \\ 0 & \text{if } s-t \le a < s \le b, \end{cases}$$

is generated by the operator

$$Bf := -f' \qquad \text{for } f \in D(B) := \big\{ f \in \mathcal{X} \cap \mathrm{C}^1\big((a,b],X\big) : Bf \in \mathcal{X} \big\};$$

see Exercise I.4.19.(5). Since $\mathcal{A}$ is bounded, we have

$$\big(\lambda - (\mathcal{A}+B)\big)D(B) = \mathcal{X} \qquad \text{for all } \lambda > \|\mathcal{A}\|,$$

and $(\mathcal{A}+B, D(B))$ generates a semigroup $(T(t))_{t\ge0}$ on $\mathcal{X}$ (use Theorem 1.3). Moreover, for $f \in \mathcal{X}$ and $a < s \le s+t \le b$, we compute

$$\begin{aligned} \Big[\big(\mathcal{M}(t/n)S(t/n)\big)^n f\Big](s+t) &= \mathrm{e}^{t/n A(s+t)} \Big[\big(\mathcal{M}(t/n)S(t/n)\big)^{n-1} f\Big](s+t-t/n) \\ &= \cdots \\ &= \mathrm{e}^{t/n A(s+t)} \mathrm{e}^{t/n A(s+t-t/n)} \cdots \mathrm{e}^{t/n A(s+t/n)} f(s). \end{aligned} \tag{5.16}$$

Hence, $\|(\mathcal{M}(t/n)S(t/n))^n\| \le M$ by (5.15). The Trotter product formula from Corollary 5.8 now yields

$$\lim_{n\to\infty} \big(\mathcal{M}(t/n)S(t/n)\big)^n f = T(t)f$$

uniformly for $0 \le t \le b-a$.

For given $x \in X$ and $c \in (a,b)$ now take $f \in D(B)$ such that $f(s) = x$ for $c \le s \le b$. Then (5.16) implies

$$\lim_{n\to\infty} \left\| \mathrm{e}^{t/_n A(s+t)} \cdots \mathrm{e}^{t/_n A(s+t/_n)} x - \big(T(t)f\big)(s+t) \right\| \\ \le \lim_{n\to\infty} \left\| \big(\mathcal{M}(t/_n)S(t/_n)\big)^n f - T(t)f \right\| = 0$$

uniformly for $c \le s \le s+t \le b$. It remains to show that

$$\big(T(t)f\big)(s+t) = U(s+t,s)x$$

for $c \le s < b$. In fact, let

$$v(s+t) := \big(T(t)f\big)(s+t) \qquad \text{for } 0 \le t \le b-s.$$

Since $T(t)f \in D(B) = D(\mathcal{A}+B)$, we have $v(s) = x$ and

$$\begin{aligned} \tfrac{d}{dt} v(s+t) &= \big((\mathcal{A}+B)T(t)f\big)(s+t) - \big(BT(t)f\big)(s+t) \\ &= A(s+t)v(s+t). \end{aligned}$$

Hence, $v$ solves (nACP) on $[s,b]$, and thus $v(s+t) = U(s+t,s)x$. □

We now show, by essentially the same example, first that the density of $D(A) \cap D(B)$ is not necessary for the convergence (to a strongly continuous semigroup) of the Trotter Product Formula (5.14) and second that the converse of the implication (a) ⇒ (b) in the First Trotter–Kato Approximation Theorem 4.8 does not hold.

**5.10 Counterexample.** On $X := \mathrm{L}^2(\mathbb{R})$ we take the right translation semigroup $(T(t))_{t\ge0}$ with generator $A$ (see Section I.4.c and Paragraph II.2.10) and the multiplication semigroup $(S(t))_{t\ge0}$ generated by $B = M_{\mathrm{i}q}$ for $q : \mathbb{R} \to \mathbb{R}$ a measurable and locally integrable function. For $f \in X$ and as in (5.16), we can compute the products

$$[T(t/_n)S(t/_n)]^n f(s) = \exp\left( \mathrm{i} \sum_{k=1}^{n} q\,(s - {}^{kt}/_n)\, {}^t/_n \right) \cdot f(s-t) \qquad \text{for } t \ge 0,\ s \in \mathbb{R}.$$

They converge in $\mathrm{L}^2$-norm to $U(t)f$ with

$$U(t)f(s) := \mathrm{e}^{\mathrm{i}\int_{s-t}^{s} q(\tau)\,d\tau} \cdot f(s-t).$$

These operators $U(t)$ form a strongly continuous semigroup (of isometries) on $X$. Observe that no assumption on $D(A) \cap D(B)$ was made.

In fact, this intersection can be $\{0\}$. Take $\mathbb{Q} = \{\alpha_k : k \in \mathbb{N}\}$ and define

$$q(s) := \sum_{k=1}^{\infty} \frac{1}{k!} |s - \alpha_k|^{-1/2} \qquad \text{for } s \in \mathbb{R}.$$

Then $q \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. However, $q \notin \mathrm{L}^2[a,b]$ for any $a < b$. Therefore, no continuous function belongs to $D(B)$; hence $D(A) \cap D(B) = 0$. However, at least formally, the generator $C$ of $(U(t))_{t\ge0}$ is the "sum" $A + B$.

In fact, one can show that the domain of $C$ is

$$D(C) = \left\{f \in \mathrm{L}^2(\mathbb{R}) : f \text{ is absolutely continuous and } -f' + qf \in \mathrm{L}^2(\mathbb{R})\right\}$$

and

$$Cf = -f' + qf \quad \text{for } f \in D(C).$$

Using the same $q$, we now define semigroups on $X$ by

$$U_n(t)f(s) := \mathrm{e}^{{}^{\mathrm{i}}\!/_{n} \int_{s-{}^{t(n+1)}\!/_{n}}^{s} q(\tau)\,d\tau} \cdot f\left(s - {}^{t(n+1)}\!/_{n}\right)$$

for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} U_n(t)f = T(t)f$ for every $f \in X$, and the semigroups $(U_n(t))_{t\geq 0}$ and the right translation semigroup $(T(t))_{t\geq 0}$ satisfy the equivalent conditions (b), (c), and (d) in the First Trotter–Kato Approximation Theorem 4.8. However, the intersections of the respective domains are trivial; hence condition (a) does not hold.

**5.11 Exercises.** (1) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ on a Banach space $X$. If $B \in \mathcal{L}(X)$, then the semigroup $(U(t))_{t\geq 0}$ generated by $A + B$ is given by the Trotter product formula

$$U(t)x = \lim_{n\to\infty} \left[T({}^{t}\!/_{n})\mathrm{e}^{{}^{tB}\!/_{n}}\right]^n x$$

for all $t \geq 0$ and $x \in X$. (Hint: By renorming $X$ as in Chapter II, (3.18) (or by Lemma II.3.10) one may assume that $(T(t))_{t\geq 0}$ is a contraction semigroup. To verify the stability condition (5.13) observe that $\|\mathrm{e}^{tB}\| \leq \mathrm{e}^{t\|B\|}$.)

(2) Let $A(\cdot) \in \mathrm{C}(\mathbb{R}, \mathcal{L}_s(X))$ and assume that

$$\left\| \prod_{k=1}^{n} \mathrm{e}^{{}^{t}\!/_{n} A(s + {}^{kt}\!/_{n})} \right\| \leq M\mathrm{e}^{wt}$$

for some constants $M \geq 0$, $w \in \mathbb{R}$, and all $n \in \mathbb{N}$, $s \in \mathbb{R}$, and $t \geq 0$. Moreover, let $(U(t,s))_{t\geq s}$ be the evolution family as given in Example 5.9. Show that the following assertions are true.

(i) $\|U(t,s)\| \leq M\mathrm{e}^{w(t-s)}$ for $t \geq s$.

(ii) $(T(t)f)(s) := U(s, s-t)f(s-t)$ for $s \in \mathbb{R}$, $t \geq 0$ and $f \in \mathfrak{X} := \mathrm{C}_0(\mathbb{R}, X)$ defines a strongly continuous semigroup on $\mathfrak{X}$, cf. Lemma VI.9.10.

(iii*) The generator $G$ of $(T(t))_{t\geq 0}$ is given by

$$Gf = -f' + A(\cdot)f, \quad f \in D(G) = \left\{f \in \mathfrak{X} \cap \mathrm{C}^1(\mathbb{R}, X) : -f' + A(\cdot)f \in \mathfrak{X}\right\},$$

compare Lemma VI.9.28. (Hint: Consider first functions having compact support.)

## b. Inversion Formulas

Using the Trotter–Kato approximation theorems, we obtained in the previous section formulas for the semigroup operators $\big(T(t)\big)_{t\geq 0}$ based on the resolvent operators $R(\lambda, A)$. A typical example is the Post–Widder inversion formula

$$T(t) = \lim_{n\to\infty} \big[{}^n\!/\!_t R({}^n\!/\!_t, A)\big]^n \tag{5.17}$$

from Corollary 5.5. It has, however, the drawback that we need to compute not only the resolvent $R(\lambda, A)$ but also its powers $R(\lambda, A)^n$ for large $\lambda$. Recalling the opposite arrow from Diagram 5.6, which expresses the resolvent by the semigroup operators as

$$R(\lambda, A) = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)\, dt, \quad \operatorname{Re}\lambda > \omega_0, \tag{5.18}$$

we are led to a different approach. In fact, (5.18) states that the resolvent $R(\lambda, A)$ is the Laplace transform of the semigroup $\big(T(t)\big)_{t\geq 0}$. Therefore, we try to obtain $T(t)$ as the *inverse* Laplace transform of the resolvent $R(\lambda, A)$.

In order to carry out such an inversion, we need some preparation. We start by defining a *functional calculus*, which will also be useful later; see Section IV.3.c.

**5.12 Proposition.** *Let $\big(T(t)\big)_{t\geq 0}$ be a bounded strongly continuous semigroup on a Banach space $X$. Then the map*

$$\mathcal{T} : \mathrm{L}^1(\mathbb{R}_+) \to \mathcal{L}(X), \quad [\mathcal{T} f]x := \int_0^\infty f(s) T(s)x\, ds,$$

*defines a bounded operator.*

The simple proof is left to the reader.

This functional calculus yields the formula

$$S(t)x := \int_0^t T(s)x\, ds = \big[\mathcal{T}\mathbb{1}_{[0,t]}\big]\, x, \quad x \in X, \tag{5.19}$$

where $\mathbb{1}_{[0,t]} \in \mathrm{L}^1(\mathbb{R}_+)$ denotes the characteristic function of the interval $[0, t]$. Moreover, (5.18) can be restated as

$$R(\lambda, A) = [\mathcal{T}\varepsilon_{-\lambda}(\cdot)] \tag{5.20}$$

for the exponential functions $\varepsilon_{-\lambda}(s) := \mathrm{e}^{-\lambda s}$. The idea is now to approximate $\mathbb{1}_{[0,t]}$ in terms of these exponential functions $\varepsilon_{-\lambda}(\cdot)$. This will yield an inversion formula for the "integrated semigroup" $\big(S(t)\big)_{t\geq 0}$.

The approximation of $\mathbb{1}_{[0,t]}$ given in the following lemma is our key tool.

**5.13 Lemma.** *For all $\varepsilon > 0$, $t \geq 0$, and $n \in \mathbb{N}$ consider*

$$H_{n,t}(\cdot) := \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{\lambda t}}{\lambda}\, \varepsilon_{-\lambda}(\cdot)\, d\lambda.$$

*Then $H_{n,t}(\cdot) \in \mathrm{L}^1(\mathbb{R}_+)$, and $\lim_{n\to\infty} H_{n,t}(\cdot) = \mathbb{1}_{[0,t]}$ in $\mathrm{L}^1(\mathbb{R}_+)$ holds uniformly for $t$ in bounded intervals.*

Proof. We start by decomposing

$$\begin{aligned}
&\|H_{n,t}(\cdot) - \mathbb{1}_{[0,t]}\|_1 \\
&\quad = \int_0^t \left| \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{\lambda(t-s)}}{\lambda}\, d\lambda - 1 \right| ds + \int_t^\infty \left| \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{\lambda(t-s)}}{\lambda}\, d\lambda \right| ds \\
&\quad = \int_0^t \left| \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{u\lambda}}{\lambda}\, d\lambda - 1 \right| du + \int_0^\infty \left| \frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{-u\lambda}}{\lambda}\, d\lambda \right| du \\
&\quad := J_1(n,t) + J_2(n).
\end{aligned}$$

In order to estimate $J_1(n,t)$, we observe that by the residue theorem

$$\frac{1}{2\pi \mathrm{i}} \int_{\varepsilon - \mathrm{i}n}^{\varepsilon + \mathrm{i}n} \frac{\mathrm{e}^{u\lambda}}{\lambda}\, d\lambda - 1 = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_n} \frac{\mathrm{e}^{u\lambda}}{\lambda}\, d\lambda \qquad \text{for } n > \varepsilon \tag{5.21}$$

and $\gamma_n : [\pi/2, 3\pi/2] \to \mathbb{C}$, $\gamma_n(r) := \varepsilon + n\mathrm{e}^{\mathrm{i}r}$. Using the estimate $\cos r \geq 1 - 2r/\pi$, valid for all $r \in [0, \pi/2]$, we obtain

$$\begin{aligned}
\left| \frac{1}{2\pi \mathrm{i}} \int_{\gamma_n} \frac{\mathrm{e}^{u\lambda}}{\lambda}\, d\lambda \right| &\leq \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} \frac{n}{n-\varepsilon}\, \mathrm{e}^{u(\varepsilon + n\cos r)}\, dr \\
&\leq \frac{1}{\pi} \frac{n}{n-\varepsilon} \int_0^{\pi/2} \mathrm{e}^{u[\varepsilon - n(1 - \frac{2}{\pi} r)]}\, dr \\
&= \frac{\mathrm{e}^{u\varepsilon}}{2} \frac{n}{n-\varepsilon} \left[ \frac{1 - \mathrm{e}^{-nu}}{nu} \right]
\end{aligned} \tag{5.22}$$

for $n > \varepsilon$ and $u > 0$. Since

$$0 \leq \frac{1 - \mathrm{e}^{-x}}{x} \leq 1 \quad \text{for all } x > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1 - \mathrm{e}^{-x}}{x} = 0,$$

the dominated convergence theorem implies that

$$\lim_{n\to\infty} J_1(n,t) = 0$$

uniformly for $t$ in bounded intervals of $\mathbb{R}_+$.

As our next step, we estimate $J_2(n)$. To this end, we first note that by Cauchy's integral theorem, we have

$$\int_{\varepsilon-\mathrm{i}n}^{\varepsilon+\mathrm{i}n} \frac{\mathrm{e}^{-u\lambda}}{\lambda}\, d\lambda = \int_{\tau_n} \frac{\mathrm{e}^{-u\lambda}}{\lambda}\, d\lambda,$$

where $\tau_n : [-\pi/2, \pi/2] \to \mathbb{C}$, $\tau_n(r) := \varepsilon + n\mathrm{e}^{\mathrm{i}r}$. Proceeding as above, we obtain

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\int_{\tau_n} \frac{\mathrm{e}^{-u\lambda}}{\lambda}\, d\lambda\right| &\le \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{-u(\varepsilon+n\cos r)}\, dr\\ &\le \frac{1}{\pi}\int_0^{\pi/2} \mathrm{e}^{-u[\varepsilon+n(1-\frac{2}{\pi}r)]}\, dr\\ &= \frac{\mathrm{e}^{-u\varepsilon}}{2}\left[\frac{1-\mathrm{e}^{-nu}}{nu}\right] =: g_n(u).\end{aligned}$$

Since the function $x \mapsto \frac{1-\mathrm{e}^{-x}}{x}$ is positive and decreasing on $(0,\infty)$, we have

$$0 \le g_n \le g_1 \in \mathrm{L}^1(\mathbb{R}_+)$$

for all $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} g_n(u) = 0$ for all $u > 0$; hence by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} J_2(n) = 0,$$

and the proof is complete. □

Putting together Proposition 5.12 and Lemma 5.13, we arrive at our first inversion formula.

**5.14 Theorem.** (*Complex Inversion Formula*). *Let $\big(T(t)\big)_{t\ge0}$ be a bounded strongly continuous semigroup on a Banach space $X$. Then*

$$\int_0^t T(s)x\, ds = \lim_{n\to\infty} \frac{1}{2\pi\mathrm{i}}\int_{\varepsilon-\mathrm{i}n}^{\varepsilon+\mathrm{i}n} \frac{\mathrm{e}^{\lambda t}}{\lambda}\, R(\lambda,A)x\, d\lambda \tag{5.23}$$

*for every $\varepsilon > 0$ and all $x \in X$, the convergence being uniform for $t$ in bounded intervals.*

Proof. If $\operatorname{Re}\lambda = \varepsilon > 0$, then $\varepsilon_{-\lambda} \in \mathrm{L}^1(\mathbb{R}_+)$ and $\mathcal{T}\varepsilon_{-\lambda}(\cdot) = R(\lambda,A)$. Hence, from Proposition 5.12 and Lemma 5.13, we obtain

$$\begin{aligned}\int_0^t T(s)x\, ds &= \big[\mathcal{T}\mathbb{1}_{[0,t]}\big]\, x = \Big[\mathcal{T}\lim_{n\to\infty} H_{n,t}(\cdot)\Big]\, x\\ &= \lim_{n\to\infty}\,[\mathcal{T}H_{n,t}(\cdot)]\, x = \lim_{n\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\varepsilon-\mathrm{i}n}^{\varepsilon+\mathrm{i}n} \frac{\mathrm{e}^{\lambda t}}{\lambda}\,[\mathcal{T}\varepsilon_{-\lambda}(\cdot)]\, x\, d\lambda\\ &= \lim_{n\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\varepsilon-\mathrm{i}n}^{\varepsilon+\mathrm{i}n} \frac{\mathrm{e}^{\lambda t}}{\lambda}\, R(\lambda,A)x\, d\lambda\end{aligned}$$

for all $x \in X$. □

If we take $x \in D(A)$, we can derive the following formula for $T(t)x$. Observe that we no longer require the semigroup to be bounded.

**5.15 Corollary.** *Let $\mathcal{T} = \big(T(t)\big)_{t\ge 0}$ be a strongly continuous semigroup on a Banach space $X$. Then*

$$T(t)x = \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{w-in}^{w+in} \mathrm{e}^{\lambda t} R(\lambda, A)x \, d\lambda \quad \text{for all } x \in D(A) \text{ and } w > \omega_0(\mathcal{T})$$

*with uniform convergence for $t$ in compact intervals of $(0, \infty)$.*

PROOF. By rescaling, we may assume that $\big(T(t)\big)_{t\ge 0}$ is bounded and that $w > 0$. Then, by the previous result and Lemma II.1.3.(iv), we have

$$\begin{aligned} T(t)x - x &= \int_0^t T(s)Ax \, ds \\ &= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{w-in}^{w+in} \frac{\mathrm{e}^{\lambda t}}{\lambda} R(\lambda, A)Ax \, d\lambda \\ &= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{w-in}^{w+in} \left[ \mathrm{e}^{\lambda t} R(\lambda, A)x - \frac{\mathrm{e}^{\lambda t}}{\lambda} x \right] d\lambda \end{aligned}$$

for $x \in D(A)$. However, from (5.21) and (5.22) and for $u = t$ we obtain

$$\lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{w-in}^{w+in} \frac{\mathrm{e}^{\lambda t}}{\lambda} \, d\lambda = 1$$

uniformly for $t$ in compact subsets of $(0, \infty)$. This completes the proof. □

Since in the previous inversion formula the integral does not converge absolutely, it will be difficult to derive from it estimates on the semigroup operators $T(t)$. The following representation of $T(t)x$ will converge absolutely. It requires, however, more regularity on the data $x$. It was already used in Paragraph II.4.20, and another application will be made in Theorem V.1.11.

**5.16 Corollary.** *Let $A$ generate a strongly continuous semigroup $\mathcal{T} = \big(T(t)\big)_{t\ge 0}$ on $X$. Then*

$$T(t)x = \frac{(k-1)!}{t^{k-1}} \frac{1}{2\pi \mathrm{i}} \lim_{n\to\infty} \int_{w-in}^{w+in} \mathrm{e}^{\lambda t} R(\lambda, A)^k x \, d\lambda$$

*for all $w > \omega_0(\mathcal{T})$, $k \in \mathbb{N}$, $t > 0$, and $x \in D(A^2)$. Moreover, if $k \ge 2$, then the integral converges absolutely and uniformly for $t > 0$.*

Proof. For $k = 1$ this is just Corollary 5.15. Note that for $x \in D(A)$ we have

$$R(\lambda, A)x = \frac{1}{\lambda}\big(R(\lambda, A)Ax + x\big).$$

In particular $\lim_{s\to\pm\infty} R(w + is, A)x = 0$. Using this and integration by parts, the formula follows easily by induction. Moreover, for $x \in D(A^2)$ we have

$$R(\lambda, A)^2x = \frac{1}{\lambda^2}\left(R(\lambda, A)^2A^2x + 2R(\lambda, A)Ax + x\right),$$

and hence the integral converges absolutely and uniformly for $t > 0$. □

From the proof of Theorem 5.14 it is clear that every approximation of the characteristic function $\mathbb{1}_{[0,t]}$ in terms of exponential functions $\varepsilon_{-\lambda}$ will yield an inversion formula for the Laplace transform. The following is a particularly nice example.

**5.17 Lemma.** *The sequence*

$$B_{n,t}(\cdot) := \mathbb{1} - \exp\big(-e^{nt}\varepsilon_{-n}(\cdot)\big)$$

*converges in $L^1(\mathbb{R}_+)$ to the characteristic function $\mathbb{1}_{[0,t]}$ as $n \to \infty$ uniformly for $t$ in bounded intervals of $\mathbb{R}_+$.*

Proof. As in the proof of Theorem 5.14, we start by decomposing

$$\begin{aligned}
\|B_{n,t}(\cdot) - \mathbb{1}_{[0,t]}\|_1 &= \int_0^t |B_{n,t}(s) - 1|\, ds + \int_t^\infty |B_{n,t}(s)|\, ds\\
&= \int_0^t \exp\big(-e^{nu}\big)\, du + \int_0^\infty \big(1 - \exp(-e^{-nu})\big)\, du\\
&= I_1(n,t) + I_2(n).
\end{aligned}$$

Since the functions $u \mapsto \exp\big(-e^{nu}\big)$ for $u \in [0,t]$ are bounded by 1 and converge to zero for all $u \in (0,t]$, $I_1(n,t)$ converges to zero by the dominated convergence theorem uniformly for $t$ in bounded intervals of $\mathbb{R}_+$.

In order to estimate $I_2(n,t)$, we observe that by the mean value theorem for each $r > 0$ there exists $\delta \in (0,1)$ such that

$$\frac{e^{-r} - 1}{r} = -e^{-\delta r};$$

hence $1 - e^{-r} = re^{-\delta r} \le r$ for all $r > 0$. Putting $r = e^{-nu}$, we therefore obtain

$$\int_0^\infty \big(1 - \exp(-e^{-nu})\big)\, du \le \int_0^\infty e^{-nu} = \frac{1}{n},$$

and the assertion follows. □

**5.18 Theorem.** (*Phragmén Inversion Formula*). *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $X$. Then*

$$\int_0^t T(s)x\,ds = \lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\,\mathrm{e}^{tkn} R(kn,A)x \qquad \text{for all } x \in X,$$

*where the convergence is uniform for $t$ in bounded intervals of $\mathbb{R}_+$.*

PROOF. Note that the identity

$$B_{n,t}(\cdot) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\,\mathrm{e}^{tkn} \varepsilon_{-kn}(\cdot)$$

holds. Hence, the assertion follows from the previous lemma as in the proof of Theorem 5.14 by replacing $H_{n,t}(\cdot)$ by $B_{n,t}(\cdot)$. □

As a remarkable consequence of this formula we note that the values of the resolvent in $m_0 + \mathbb{N}$, i.e., $R(m, A)$ for $m \geq m_0$, already determine uniquely the associated semigroup $\big(T(t)\big)_{t\geq 0}$.

**5.19 Exercise.** (i) Show that the sequence $(G_{n,t})_{n\in\mathbb{N}} \subset \mathrm{L}^1(\mathbb{R}_+)$ defined by

$$G_{n,t}(s) := \frac{s^{n+1}}{n!} \int_{n/t}^{\infty} \mathrm{e}^{-sr} r^n\, dr$$

converges in $\mathrm{L}^1(\mathbb{R}_+)$ to the characteristic function $\mathbb{1}_{[0,1]}$ as $n \to \infty$.
(ii) Show that the sequence $(S_{n,t})_{n\in\mathbb{N}} \subset \mathrm{L}^1(\mathbb{R}_+)$ defined by

$$S_{n,t}(s) := \mathrm{e}^{-sn/t} \sum_{k=0}^{n} \frac{1}{k!} \Big(\frac{sn}{t}\Big)^k$$

converges in $\mathrm{L}^1(\mathbb{R}_+)$ to the characteristic function $\mathbb{1}_{[0,1]}$ as $n \to \infty$.
(iii) Re-prove the Post–Widder inversion formula in (5.17) using either (i) or (ii) and the ideas in the proof of Theorems 5.14 or 5.18.

## Notes to Chapter III

Many of the results of this chapter are already contained in the two classic monographs by Hille–Phillips [HP57] and Kato [Kat80] as well as in the other books on semigroups (e.g. [Dav80], [Gol85], [Paz83]).

**Section 1.** Corollary 1.5 is due to Desch–Schappacher [DS84], and we return in Section 3.a to their main perturbation result. Phillips [Phi53] started the investigation of permanence properties of semigroups under bounded perturbations. He proved most of Proposition 1.12, while Pazy [Paz68] continued this research. More recent contributions to this subject are [Ren95], [DHW97], and [NP98].

**Section** 2. The theory of unbounded perturbations is quite old and started in Hilbert spaces. We refer to Kato [Kat80] and Reed–Simon [RS75, Sec. X.2] for this aspect. Our Theorem 2.7 is due to Gustafson [Gus66], Corollary 2.8 and Corollary 2.9 are in [Che72], while Theorem 2.10 already appears in [Hil42].

**Section** 3. Our simultaneous treatment of Desch–Schappacher and Miyadera–Voigt perturbations via an abstract Volterra operator is new.

**Section** 3.a. The perturbation results in this subsection are slight generalizations of results due to Desch and Schappacher; cf. [DS89] and Section 3.d. Concrete applications of these results, in particular of Corollary 3.6, are made, e.g., in [NR95] and [Rha98].

**Section** 3.b. The comparison result Theorem 3.9 (see also Exercise 3.13.(1)) has been studied by Robinson [Rob77] (see also Bratteli–Robinson [BR79, 3.1.5]), Desch–Schappacher [DS87] and Diekmann–Gyllenberg–Heijmans [DGH89].

**Section** 3.c. Perturbations satisfying the estimate (3.26) where introduced by Miyadera in [Miy66]. In particular, he proved the first part of Corollary 3.16. The extension of this result treated in Exercises 2.18.(2) and 3.17.(2) is due to Voigt; see [Voi77], who used it extensively in the study of Schrödinger and transport operators (e.g., [Voi85], [Voi88]).

**Section** 3.d. The research on multiplicative perturbations started with the work of Dorroh [Dor66] and Gustafson [Gus68] on perturbations of contraction semigroups. For further references and recent developments we refer to [CDG$^+$87], [DGT93], [DH93], [DLS85], [Gre87], [Lum89a], [Lum89b], and [PS95].

Based on Exercise 3.23.(2.i) and (2.ii) one can show that Proposition 3.18 prevails without the assumption $\rho(A) \neq \emptyset$. However, in this case the proof is more involved, see [DS89, Thm. 1].

**Section** 4. The Trotter–Kato theorems got their name from the papers [Tro58] and [Kat59], but the contraction case has also been proved independently by Neveu [Nev58]. The monograph [BB67] is devoted to the approximation of semigroups. The idea of using a sequence space to convert the approximation to a stationary problem is due to Kisyński [Kis67] and has been propagated by Goldstein (see [Gol85, Secs. I.6&8]). From there we also took some of our examples in Section 4.c.

**Section** 5.a. The product formula

$$\mathrm{e}^{(A+B)} = \lim_{n\to\infty} \left(\mathrm{e}^{A/n}\mathrm{e}^{B/n}\right)^n$$

for matrices $A$ and $B$ goes back to Lie. It was extended to unbounded operators by Trotter [Tro59], and Chernoff [Che68] deduced it from his product formula. Goldstein [Gol70] and others used it to define a generalized sum of two generators. Surprisingly, Kühnemund–Wacker [KW99] constructed an example showing that the "sum" semigroup is not necessarily given by the Trotter product formula.

The discrete approximation in Corollary 5.4 is particularly useful for numerical applications (see [Paz83, Sec. 3.6]), while the Bernstein approximation in Paragraph 5.7 is in the spirit of Altomare–Campiti, who in their monograph [AC94] treat much more general approximation schemes.

Finally, the application of product formulas to nonautonomous Cauchy problems as in Example 5.9 is due to Nickel [Nic99]. Further applications can be made to the central limit theorem or the Feynman–Kac formula (see [Gol76], [Gol85, Sec. I.8] or [Cas85, App.]).

**Section** 5.b. This approach to inversion formulas is taken from [HN93]. For more information on the Laplace transform and its inversion we refer to the monograph [ABHN99].

# Chapter IV

# Spectral Theory for Semigroups and Generators

Up to now, our main concern was to show that strongly continuous semigroups have generators and, conversely, that certain operators generate strongly continuous semigroups. In the perspective of Section II.6 this means that certain evolution equations have unique solutions, hence are well-posed.

Having established this kind of well-posedness, that is, the existence of a strongly continuous semigroup, we now turn our attention to the qualitative behavior of these solutions, i.e., of these semigroups. Our main tool for this investigation is provided by *spectral theory*.

This is already evident from the Hille–Yosida theorem (and its variants), where generators were characterized by the location of their spectrum and by norm estimates of the resolvent. Moreover, in the Liapunov Stability Theorem I.3.14 we could, at least in the uniformly continuous case, characterize stability of the semigroup by a spectral property.

In order to continue in this direction, we now develop a spectral theory for semigroups and their generators. The importance of these techniques will become evident in Chapter V, where we will apply it to the study of the asymptotic behavior of strongly continuous semigroups.

We start with an introductory section, in which we explain the basic spectral-theoretic notions and results for general closed operators. Since many of these notions have already been used in the preceding chapters, the reader may skip (most of) this section.

# 1. Spectral Theory for Closed Operators

In this section our object of interest is a closed, linear operator

$$A : D(A) \subset X \to X$$

on some Banach space $X$. Note that we do not assume a dense domain, while the closedness is necessary for a reasonable spectral theory.

**1.1 Definition.** *We call*

$$\rho(A) := \big\{\lambda \in \mathbb{C} : \lambda - A : D(A) \to X \text{ is bijective}\big\}$$

*the resolvent set and its complement* $\sigma(A) := \mathbb{C} \setminus \rho(A)$ *the spectrum of* $A$. *For* $\lambda \in \rho(A)$, *the inverse*

$$R(\lambda, A) := (\lambda - A)^{-1}$$

*is, by the closed graph theorem, a bounded operator on* $X$ *and will be called the resolvent (of* $A$ *at the point* $\lambda$*).*

It follows immediately from the definition that the identity

$$AR(\lambda, A) = \lambda R(\lambda, A) - I \tag{1.1}$$

holds for every $\lambda \in \rho(A)$. The next identity is the reason for many of the nice properties of the resolvent set $\rho(A)$ and the *resolvent map*

$$\rho(A) \ni \lambda \mapsto R(\lambda, A) \in \mathcal{L}(X).$$

**1.2 Resolvent Equation.** *For* $\lambda, \mu \in \rho(A)$, *one has*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \tag{1.2}$$

PROOF. The definition of the resolvent implies

$$[\lambda R(\lambda, A) - AR(\lambda, A)]R(\mu, A) = R(\mu, A)$$

and

$$R(\lambda, A)[\mu R(\mu, A) - AR(\mu, A)] = R(\lambda, A).$$

If we subtract these equations and use the fact that $R(\lambda, A)$ and $R(\mu, A)$ commute, we obtain (1.2). □

The basic properties of the resolvent set and the resolvent map are now collected in the following proposition.

**1.3 Proposition.** *For a closed operator $A : D(A) \subset X \to X$, the following properties hold.*

(i) *The resolvent set $\rho(A)$ is open in $\mathbb{C}$, and for $\mu \in \rho(A)$ one has*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1} \tag{1.3}$$

*for all $\lambda \in \mathbb{C}$ satisfying $|\mu - \lambda| < {}^1/_{\|R(\mu,A)\|}$.*

(ii) *The resolvent map $\lambda \mapsto R(\lambda, A)$ is locally analytic with*

$$\tfrac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n!\, R(\lambda, A)^{n+1} \qquad \text{for all } n \in \mathbb{N}. \tag{1.4}$$

(iii) *Let $\lambda_n \in \rho(A)$ with $\lim_{n\to\infty} \lambda_n = \lambda_0$. Then $\lambda_0 \in \sigma(A)$ if and only if $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$.*

Proof. (i) For $\lambda \in \mathbb{C}$ write

$$\lambda - A = \mu - A + \lambda - \mu = [I - (\mu - \lambda)R(\mu, A)](\mu - A).$$

This operator is bijective if $[I - (\mu - \lambda)R(\mu, A)]$ is invertible, which is the case for $|\mu - \lambda| < {}^1/_{\|R(\mu,A)\|}$. The inverse is then obtained as

$$R(\lambda, A) = R(\mu, A)[I - (\mu - \lambda)R(\mu, A)]^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\mu, A)^{n+1}.$$

Assertion (ii) follows immediately from the series representation (1.3) for the resolvent.

To show (iii) we use (i), which implies $\|R(\mu, A)\| \geq \frac{1}{\operatorname{dist}(\mu,\sigma(A))}$ for all $\mu \in \rho(A)$. This already proves one implication. For the converse, assume that $\lambda_0 \in \rho(A)$. Then the continuous resolvent map remains bounded on the compact set $\{\lambda_n : n \geq 0\}$. This contradicts the assumption that $\lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty$; hence $\lambda_0 \in \sigma(A)$. □

As an immediate consequence, we have that the spectrum $\sigma(A)$ is a closed subset of $\mathbb{C}$. Nothing more can be said in general (see the examples below). However, if $A$ is bounded, it follows that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\},$$

since

$$R(\lambda, A) = \frac{1}{\lambda}\left(1 - \frac{A}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{\lambda^{n+1}}$$

exists for all $|\lambda| > \|A\|$. In addition, an application of Liouville's theorem to the resolvent map implies $\sigma(A) \neq \emptyset$ (see [TL80, Chap. V, Thm. 3.2]).

**1.4 Corollary.** *For a bounded operator $A$ on a Banach space $X$, the spectrum $\sigma(A)$ is always compact and nonempty; hence its spectral radius*

$$\mathrm{r}(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

*is finite and satisfies* $\mathrm{r}(A) \le \|A\|$.

Before proceeding with a more detailed analysis of $\sigma(A)$, we show by some simple examples that $\sigma(A)$ can be any closed subset of $\mathbb{C}$.

**1.5 Examples.** (i) On $X := \mathrm{C}[0,1]$ take the differential operators

$$A_i f := f' \qquad \text{for } i = 1, 2$$

with domain

$$D(A_1) := \mathrm{C}^1[0,1] \quad \text{and}$$
$$D(A_2) := \{f \in \mathrm{C}^1[0,1] : f(1) = 0\}.$$

Then

$$\sigma(A_1) = \mathbb{C},$$

since for each $\lambda \in \mathbb{C}$ one has $(\lambda - A_1)\varepsilon_\lambda = 0$ for $\varepsilon_\lambda := \mathrm{e}^{\lambda s}$, $0 \le s \le 1$. On the other hand,

$$\sigma(A_2) = \emptyset,$$

since

$$R_\lambda f(s) := \int_s^1 \mathrm{e}^{\lambda(s-t)} f(t)\, dt, \quad 0 \le s \le 1,\ f \in X,$$

yields the inverse of $(\lambda - A_2)$ for every $\lambda \in \mathbb{C}$.

(ii) Take any nonempty, closed subset $\Omega \subset \mathbb{C}$. On the space $X := \mathrm{C}_0(\Omega)$ consider the multiplication operator

$$Mf(\lambda) := \lambda \cdot f(\lambda)$$

for $\lambda \in \Omega$, $f \in X$. From Proposition I.4.2 we obtain that

$$\sigma(M) = \Omega.$$

As a next step, we look at the fine structure of the spectrum. We start with a particularly important subset of $\sigma(A)$.

**1.6 Definition.** *For a closed operator $A : D(A) \subseteq X \to X$, we call*

$$P\sigma(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective}\}$$

*the point spectrum of $A$. Moreover, each $\lambda \in P\sigma(A)$ is called an eigenvalue, and each $0 \ne x \in D(A)$ satisfying $(\lambda - A)x = 0$ is an eigenvector of $A$ (corresponding to $\lambda$).*

In most cases, the eigenvalues are simpler to determine than arbitrary spectral values. However, they do not, in general, exhaust the entire spectrum.

**1.7 Examples.** (i) For the operator $A_1$ in Example 1.5.(i), one has

$$\sigma(A_1) = P\sigma(A_1) = \mathbb{C}.$$

(ii) In contrast, for the multiplication operator $M$ in Example 1.5.(ii) one has

$$\sigma(M) = \Omega, \quad \text{but} \quad P\sigma(M) = \{\lambda \in \mathbb{C} : \lambda \text{ is isolated in } \Omega\}.$$

As a variant of the point spectrum, we introduce the following larger subset of $\sigma(A)$.

**1.8 Definition.** *For a closed operator $A : D(A) \subseteq X \to X$, we call*

$$A\sigma(A) := \left\{\lambda \in \mathbb{C} : \begin{array}{l} \lambda - A \text{ is not injective or} \\ \operatorname{rg}(\lambda - A) \text{ is not closed in } X \end{array}\right\}$$

*the approximate point spectrum of $A$.*

The inclusion $P\sigma(A) \subset A\sigma(A)$ is evident from the definition, but the reason for calling it "approximate point spectrum" is not. This is explained by the next lemma.

**1.9 Lemma.** *For a closed operator $A : D(A) \subset X \to X$ and a number $\lambda \in \mathbb{C}$ one has $\lambda \in A\sigma(A)$, i.e., $\lambda$ is an approximate eigenvalue, if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$, called an approximate eigenvector, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0$.*

PROOF. We only have to consider the case in which $\lambda - A$ is injective. As usual, we denote by $X_1 := \big(D(A), \|\cdot\|_A\big)$ the first Sobolev space for $A$, cf. Exercise II.5.9.(1). Then the inverse $(\lambda - A)^{-1} : \operatorname{rg}(\lambda - A) \to X_1$ exists and, by the closed graph theorem, is unbounded if and only if $\operatorname{rg}(\lambda - A)$ is not closed. On the other hand, if $(\lambda - A)^{-1} : \operatorname{rg}(\lambda - A) \to X$ is bounded, the closedness of $A$ implies the closedness of $\operatorname{rg}(\lambda - A)$. Hence $(\lambda - A)^{-1} : X \to X_1$ is unbounded if and only if $(\lambda - A)^{-1} : X \to X$ is unbounded, and this property can be expressed by the condition above. □

The approximate point spectrum generalizes the point spectrum. However, as we show in the following corollary, it has the advantage that it is never empty unless $\sigma(A) = \emptyset$ or $\sigma(A) = \mathbb{C}$.

**1.10 Proposition.** *For a closed operator $A : D(A) \subset X \to X$, the topological boundary $\partial\sigma(A)$ of the spectrum $\sigma(A)$ is contained in the approximate point spectrum $A\sigma(A)$.*

Proof. For each $\lambda_0 \in \partial\sigma(A) \subseteq \sigma(A)$ we can find a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \rho(A)$ such that $\lambda_n \to \lambda_0$. By Proposition 1.3.(iii), using the uniform boundedness principle and passing to a subsequence, we find $x \in X$ such that $\lim_{n\to\infty} \|R(\lambda_n, A)x\| = \infty$. Define $y_n \in D(A)$ by

$$y_n := \frac{R(\lambda_n, A)x}{\|R(\lambda_n, A)x\|}.$$

The identity

$$(\lambda_0 - A)y_n = (\lambda_0 - \lambda_n)y_n + (\lambda_n - A)y_n$$

shows that $(y_n)$ is an approximate eigenvector corresponding to $\lambda_0$. □

The remaining part of the spectrum is now taken care of by the following definition.

**1.11 Definition.** *For a closed operator $A : D(A) \subseteq X \to X$, we call*

$$R\sigma(A) := \{\lambda \in \mathbb{C} : \text{rg}(\lambda - A) \text{ is not dense in } X\}$$

*the residual spectrum of $A$.*

All possibilities for $\lambda - A$ *not* being bijective are now covered by Definitions 1.8 and 1.11, and hence

$$\sigma(A) = A\sigma(A) \cup R\sigma(A).$$

However, there is no reason for the union to be disjoint. It is easy to find examples by applying the following very useful dual characterization of $R\sigma(A)$. Note that we now need a dense domain in order to define the adjoint operator (see Definition B.8).

**1.12 Proposition.** *For a closed, densely defined operator $A$, the residual spectrum $R\sigma(A)$ coincides with the point spectrum $P\sigma(A')$ of $A'$.*

Proof. The closure of $\text{rg}(\lambda - A)$ is different from $X$ if and only if there exists a linear form $0 \neq x' \in X'$ vanishing on $\text{rg}(\lambda - A)$. By the definition of $A'$, this means $x' \in D(A')$ and $(\lambda - A')x' = 0$. □

In the next theorem we show that for each $\lambda_0 \in \rho(A)$ there is a canonical relation, called the *spectral mapping theorem*, between the spectrum of the unbounded operator $A$ and the spectrum of the bounded operator $R(\lambda_0, A)$. This will allow us to transfer results from the spectral theory of bounded operators to the unbounded case.

**1.13 Spectral Mapping Theorem for the Resolvent.** *Let $A : D(A) \subseteq X \to X$ be a closed operator with nonempty resolvent set $\rho(A)$.*

(i) $\sigma\big(R(\lambda_0, A)\big) \setminus \{0\} = \big(\lambda_0 - \sigma(A)\big)^{-1} := \big\{\frac{1}{\lambda_0 - \mu} : \mu \in \sigma(A)\big\}$ *for each* $\lambda_0 \in \rho(A)$.

(ii) *Analogous statements hold for the point, approximate point, and residual spectra of $A$ and $R(\lambda_0, A)$.*

Proof. For $0 \neq \mu \in \mathbb{C}$ and $\lambda_0 \in \rho(A)$ we have

$$\begin{aligned}\big(\mu - R(\lambda_0, A)\big)x &= \mu\big[(\lambda_0 - \tfrac{1}{\mu}) - A\big]R(\lambda_0, A)x && \text{for } x \in X,\\ &= \mu R(\lambda_0, A)\big[(\lambda_0 - \tfrac{1}{\mu}) - A\big]x && \text{for } x \in D(A).\end{aligned}$$

This identity shows that

$$\ker\big(\mu - R(\lambda_0, A)\big) = \ker\big[(\lambda_0 - \tfrac{1}{\mu}) - A\big]$$

and

$$\operatorname{rg}\big(\mu - R(\lambda_0, A)\big) = \operatorname{rg}\big[(\lambda_0 - \tfrac{1}{\mu}) - A\big].$$

Recalling Definitions 1.6, 1.8, and 1.11 for the various parts of the spectrum, we see that $\mu \in P\sigma\big(R(\lambda_0, A)\big)$ if and only if $(\lambda_0 - {}^1\!/\!_\mu) \in P\sigma(A)$ and similarly for the approximate point spectrum and the residual spectrum. This proves assertion (ii), and hence (i). □

This relation between $\sigma(A)$ and $\sigma\big(R(\lambda_0, A)\big)$ determines the spectral radius of $R(\lambda_0, A)$.

**1.14 Corollary.** *For each $\lambda_0 \in \rho(A)$ one has*

$$\operatorname{dist}(\lambda_0, \sigma(A)) = \frac{1}{\mathrm{r}\big(R(\lambda_0, A)\big)} \geq \frac{1}{\|R(\lambda_0, A)\|}. \tag{1.5}$$

We now study so-called *spectral decompositions*, which are one of the most important features of spectral theory. First, we recall briefly their construction in the bounded case (see, e.g., [DS58, Sec. VII.3], [GGK90, I.2], or [TL80, Sec. V.9]).

Let $T \in \mathcal{L}(X)$ be a bounded operator and assume that the spectrum $\sigma(T)$ can be decomposed as

$$\sigma(T) = \sigma_c \cup \sigma_u, \tag{1.6}$$

where $\sigma_c$, $\sigma_u$ are closed and disjoint sets. From the functional calculus (already used in Section I.3) one obtains the associated *spectral projection*

$$P := P_c := \frac{1}{2\pi \mathrm{i}} \int_\gamma R(\lambda, T)\, d\lambda, \tag{1.7}$$

where $\gamma$ is a Jordan path in the complement of $\sigma_u$ and enclosing $\sigma_c$. This projection commutes with $T$ and yields the *spectral decomposition*

$$X = X_c \oplus X_u$$

with the $T$-invariant spaces $X_c := \operatorname{rg} P$, $X_u := \ker P$. The restrictions $T_c \in \mathcal{L}(X_c)$ and $T_u \in \mathcal{L}(X_u)$ of $T$ satisfy

$$\sigma(T_c) = \sigma_c \text{ and } \sigma(T_u) = \sigma_u, \tag{1.8}$$

a property that characterizes the above decomposition of $X$ and $T$ in a unique way.

For unbounded operators $A$ and an arbitrary decomposition of the spectrum $\sigma(A)$ into closed sets it is not always possible to find an associated spectral decomposition (for counterexamples see Exercise 2.30 or [Nag86, A-III, Expl. 3.2]). However, if one of these sets is compact, the spectral mapping theorem for the resolvent allows us to deduce the result from the bounded case. To prove this, we first need the following lemma. For later use in Section 2.b it is stated more generally than is needed here.

**1.15 Lemma.** *Let $Y$ be a Banach space continuously embedded in $X$. If $\lambda \in \rho(A)$ such that $R(\lambda, A)Y \subset Y$, then $\lambda \in \rho(A_|)$ and $R(\lambda, A_|) = R(\lambda, A)_|$.*

PROOF. By the definition of $D(A_|)$ and since $R(\lambda, A)Y \subseteq Y$, we already know that $R(\lambda, A)_|$ maps $Y$ onto $D(A_|)$ and therefore is the algebraic inverse of $\lambda - A_|$. To show that it is bounded in $Y$, it suffices to observe that it is a closed, everywhere defined operator. □

**1.16 Proposition.** *Let $A : D(A) \subset X \to X$ be a closed operator such that its spectrum $\sigma(A)$ can be decomposed into the disjoint union of two closed subsets $\sigma_c$ and $\sigma_u$, i.e.,*

$$\sigma(A) = \sigma_c \cup \sigma_u.$$

*If $\sigma_c$ is compact, then there exists a spectral decomposition $X = X_c \oplus X_u$ for $A$ in the following sense.*

(i) *The restriction $A_c := A_{|X_c}$ is bounded on the Banach space $X_c$.*

(ii) *$X_1^A = X_c \oplus (X_u)_1^{A_u}$, where $A_u := A_{|X_u}$ (and $X_1^A$ denotes the first Sobolev space with respect to $A$ as introduced in Exercise* II.5.9.(1)*).*

(iii) *$A = A_c \oplus A_u$.*

(iv) *$\sigma(A_c) = \sigma_c$ and $\sigma(A_u) = \sigma_u$.*

PROOF. When $A$ is bounded, we have already indicated a proof based on formula (1.7). Therefore, we may assume $A$ to be unbounded and fix some $\lambda \in \rho(A)$. Then $0 \in \sigma(R(\lambda, A))$. Hence, by Theorem 1.13, we obtain

$$\begin{aligned}\sigma\big(R(\lambda, A)\big) &= (\lambda - \sigma_c)^{-1} \bigcup \big((\lambda - \sigma_u)^{-1} \cup \{0\}\big) \\ &=: \tau_c \cup \tau_u,\end{aligned} \tag{1.9}$$

where $\tau_c$, $\tau_u$ are compact and disjoint subsets of $\mathbb{C}$. Now let $P$ be the spectral projection for $R(\lambda, A)$ associated to the decomposition (1.9) and put $X_c := \operatorname{rg} P$, $X_u := \ker P$. Since $R(\lambda, A)$ and $P$ commute, we have $R(\lambda, A)X_c \subseteq X_c$, and Lemma 1.15 implies

$$\lambda \in \rho(A_c) \qquad \text{and} \qquad R(\lambda, A_c) = R(\lambda, A)_{|X_c}. \tag{1.10}$$

Moreover, we know that $\sigma\big(R(\lambda, A_c)\big) = \tau_c \not\ni 0$. Therefore, the operator $A_c = \lambda - R(\lambda, A_c)^{-1}$ is bounded on $X_c$, and we obtain (i).

To verify (ii), observe that by similar arguments as above we obtain

$$(1.11) \qquad \lambda \in \rho(A_u) \qquad \text{and} \qquad R(\lambda, A_u) = R(\lambda, A)_{|X_u}.$$

Combining this with (1.10) yields

$$\begin{aligned} X_c + D(A_u) &= R(\lambda, A_c)X_c + R(\lambda, A_u)X_u \\ &\subseteq D(A) = R(\lambda, A)(X_c + X_u) \\ &\subseteq R(\lambda, A_c)X_c + R(\lambda, A_u)X_u \\ &= X_c + D(A_u), \end{aligned}$$

i.e., $X_1^A = X_c + D(A_u)$. Since $P \in \mathcal{L}(X)$, the restriction $P_{|X_1^A} : X_1^A \to X_1^A$ is closed and therefore bounded by the closed graph theorem. This proves (ii), while assertion (iii) then follows from (i) and (ii).

Finally, (iv) is a consequence of the Spectral Mapping Theorem 1.13 and (1.9), (1.10), (1.11). □

**1.17 Isolated Singularities.** We now sketch a particularly important case of the above decomposition that occurs when $\sigma_c = \{\mu\}$ consists of a single point only. This means that $\mu$ is isolated in $\sigma(A)$ and therefore the holomorphic function $\lambda \mapsto R(\lambda, A)$ can be expanded as a Laurent series

$$R(\lambda, A) = \sum_{n=-\infty}^{\infty} (\lambda - \mu)^n U_n$$

for $0 < |\lambda - \mu| < \delta$ and some sufficiently small $\delta > 0$. The coefficients $U_n$ of this series are bounded operators given by the formulas

$$(1.12) \qquad U_n = \frac{1}{2\pi i} \int_\gamma \frac{R(\lambda, A)}{(\lambda - \mu)^{n+1}} \, d\lambda, \qquad n \in \mathbb{Z},$$

where $\gamma$ is, for example, the positively oriented boundary of the disc with radius $\delta/2$ centered at $\mu$. The coefficient $U_{-1}$ is exactly the spectral projection $P$ corresponding to the decomposition $\sigma(A) = \{\mu\} \cup (\sigma(A) \setminus \{\mu\})$ of the spectrum of $A$ (cf. (1.7)). It is called the *residue* of $R(\cdot, A)$ at $\mu$. From (1.12) (or using the multiplicativity of the functional calculus in [TL80, Thm. V.8.1]), one deduces the identities

$$(1.13) \qquad \begin{aligned} U_{-(n+1)} &= (A - \mu)^n P \qquad \text{and} \\ U_{-(n+1)} \cdot U_{-(m+1)} &= U_{-(n+m+1)} \end{aligned}$$

for $n, m \geq 0$. If there exists $k > 0$ such that $U_{-k} \neq 0$ while $U_{-n} = 0$ for all $n > k$, then the spectral value $\mu$ is called a *pole of* $R(\cdot, A)$ *of order* $k$. In view of (1.13), this is true if and only if $U_{-k} \neq 0$ and $U_{-(k+1)} = 0$. Moreover, we can obtain $U_{-k}$ as

$$U_{-k} = \lim_{\lambda \to \mu} (\lambda - \mu)^k R(\lambda, A).$$

The dimension of the spectral subspace $\operatorname{rg} P$ is called the *algebraic multiplicity* $m_a$ of $\mu$, while $m_g := \dim \ker(\mu - A)$ is the *geometric multiplicity*. In the case $m_a = 1$, we call $\mu$ an *algebraically simple* (or *first-order*) *pole*.

If $k$ is the order of the pole, where we set $k = \infty$ if $R(\cdot, A)$ has an essential singularity at $\mu$, one can show the inequalities

$$m_g + k - 1 \le m_a \le m_g \cdot k \tag{1.14}$$

if we put $\infty \cdot 0 := \infty$. This implies that

(i) $m_a < \infty$ if and only if $\mu$ is a pole with $m_g < \infty$, and

(ii) if $\mu$ is a pole of order $k$, then $\mu \in P\sigma(A)$ and $\operatorname{rg} P = \ker(\mu - A)^k$.

For proofs of these facts we refer to [GGK90, Chap. II], [Kat80, III.5], [TL80, V.10], or [Yos65, VIII.8].

We now prove the following relationship between isolated spectral values of $A$ and those of its resolvent.

**1.18 Proposition.** *Let $A$ be a closed linear operator having nonempty resolvent set $\rho(A)$ and take some $\lambda_0 \in \rho(A)$. Then $\mu \in \mathbb{C}$ is an isolated point of $\sigma(A)$ if and only if $(\lambda_0 - \mu)^{-1}$ is isolated in $\sigma(R(\lambda_0, A))$. In this case, the residues and the orders of the poles of $R(\cdot, A)$ at $\mu$ and of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \mu)^{-1}$ coincide.*

Proof. The first claim follows easily from the Spectral Mapping Theorem 1.13 and the fact that the map $z \mapsto (\lambda_0 - z)^{-1}$ is homeomorphic between $\mathbb{C} \setminus \{\lambda_0\}$ and $\mathbb{C} \setminus \{0\}$.

In order to prove the assertion concerning the residues, we choose a positively oriented circle $\gamma \subset \rho(A)$ with center $\mu$ such that $\lambda_0$ lies in the exterior of $\gamma$. Then the residue $P$ of $R(\cdot, A)$ at $\mu$ is given by

$$\begin{aligned} P &= \frac{1}{2\pi i} \int_\gamma R(\lambda, A)\, d\lambda \\ &= \frac{1}{2\pi i} \int_\gamma \frac{R((\lambda_0 - \lambda)^{-1}, R(\lambda_0, A))}{(\lambda_0 - \lambda)^2}\, d\lambda - \frac{1}{2\pi i} \int_\gamma \frac{d\lambda}{(\lambda_0 - \lambda)} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{R((\lambda_0 - \lambda)^{-1}, R(\lambda_0, A))}{(\lambda_0 - \lambda)^2}\, d\lambda. \end{aligned}$$

Here we used the identities

$$R(\lambda, A) = \frac{R((\lambda_0 - \lambda)^{-1}, R(\lambda_0, A))}{(\lambda_0 - \lambda)^2} - \frac{1}{(\lambda_0 - \lambda)},$$

which follow from the resolvent equation (1.2) and Cauchy's integral theorem. The substitution $z := (\lambda_0 - \lambda)^{-1}$ then yields a path $\widetilde{\gamma}$ around $(\lambda_0 - \mu)^{-1}$, and we obtain

$$P = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} R\big(z, R(\lambda_0, A)\big)\, dz,$$

which is the residue of $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \mu)^{-1}$.

The final assertion concerning the orders of the poles follows from the identity

$$V_{-n} = \big((\lambda_0 - \mu)^{-1} R(\lambda_0, A)\big)^{n-1} U_{-n}, \quad n = 1, 2, 3, \dots ,$$

where $U_{-n}$ and $V_{-n}$ stand for the $-n$th coefficients in the Laurent expansions of $R(\cdot, A)$ at $\mu$ and $R(\cdot, R(\lambda_0, A))$ at $(\lambda_0 - \mu)^{-1}$, respectively. This has been shown above for $n = 1$ and follows for $n > 1$ by induction using the relations

$$U_{-(n+1)} = (A - \mu) U_{-n} \qquad \text{and} \qquad V_{-(n+1)} = \big(R(\lambda_0, A) - (\lambda_0 - \mu)^{-1}\big) V_{-n};$$

cf. formulas (1.13). ☐

If $A$ has compact resolvent, the above facts in combination with the Riesz–Schauder theory for compact operators, cf. [Yos65, X.5], [TL80, Sec. V.7], or [Lan93, Chap. XVII], yield the following result.

**1.19 Corollary.** *If the operator $A$ has compact resolvent, then every spectral value in $\sigma(A)$ is a pole of finite algebraic multiplicity. In particular, we have*

$$\sigma(A) = P\sigma(A).$$

**1.20 The Essential Spectrum.** As we already mentioned above, spectral decomposition is a powerful method to split an operator on a Banach space into two, hopefully simpler, parts acting on invariant subspaces. In this paragraph we present the tools for a decomposition in which one of these subspaces will be finite-dimensional. The results will be used frequently in the sequel, e.g., in Proposition 2.10, Theorem V.3.1, Theorem VI.8.24, and Theorem VI.2.6. We start with the following notion.

An operator $S \in \mathcal{L}(X)$ on a Banach space $X$ is called a *Fredholm operator* if

$$\dim \ker S < \infty \qquad \text{and} \qquad \dim {}^{X}\!/_{\operatorname{rg} S} < \infty.$$

For $T \in \mathcal{L}(X)$, we then define its *Fredholm domain* $\rho_{\mathrm{F}}(T)$ by

$$\rho_{\mathrm{F}}(T) := \big\{\lambda \in \mathbb{C} : \lambda - T \text{ is a Fredholm operator}\big\},$$

and call its complement

$$\sigma_{\mathrm{ess}}(T) := \mathbb{C} \setminus \rho_{\mathrm{F}}(T)$$

the *essential spectrum* of the operator $T$. One can show, see for instance [GGK90, Chap. XI, Thm. 5.1], that

$$\text{(1.15)} \quad S \text{ is a Fredholm operator} \iff \begin{cases} \text{there exists } T \in \mathcal{L}(X) \text{ such that} \\ I - TS \text{ and } I - ST \text{ are compact.} \end{cases}$$

Using this fact, an equivalent characterization of $\sigma_{\mathrm{ess}}(T)$ is obtained through the *Calkin algebra* $\mathcal{C}(X) := {}^{\mathcal{L}(X)}\!/_{\mathcal{K}(X)}$, where $\mathcal{K}(X)$ stands for the two-sided closed ideal in $\mathcal{L}(X)$ of all compact operators. In fact, $\mathcal{C}(X)$ equipped with the quotient norm

$$\big\|\widehat{T}\big\| := \operatorname{dist}\big(T, \mathcal{K}(X)\big) = \inf\big\{\|T - K\| : K \in \mathcal{K}(X)\big\}$$

for $\widehat{T} := T + \mathcal{K}(X) \in \mathcal{C}(X)$ is a Banach algebra with unit. Then, by the equivalence in (1.15), we have

$$\rho_{\mathrm{F}}(T) = \rho(\widehat{T})$$

and

$$\sigma_{\mathrm{ess}}(T) = \sigma(\widehat{T})$$

for all $T \in \mathcal{L}(X)$, where the spectrum of $\widehat{T}$ is defined in the Banach algebra $\mathcal{C}(X)$ (see [CPY74, Chap. 1]). In particular, this implies that $\sigma_{\mathrm{ess}}(T)$ is closed and, if $X$ is infinite-dimensional, nonempty.

In the sequel, we will also use the notation

$$\|T\|_{\mathrm{ess}} := \|\widehat{T}\|$$

and

$$\mathrm{r}_{\mathrm{ess}}(T) := \mathrm{r}(\widehat{T}) = \sup\big\{|\lambda| : \lambda \in \sigma_{\mathrm{ess}}(T)\big\}$$

for the *essential norm* and the *essential spectral radius*, respectively, of the operator $T$. Since $\|T\|_{\mathrm{ess}} = \|T+K\|_{\mathrm{ess}}$ for every compact operator $K$ on $X$, we have

$$\mathrm{r}_{\mathrm{ess}}(T+K) = \mathrm{r}_{\mathrm{ess}}(T)$$

for all $K \in \mathcal{K}(X)$. Moreover, using the Hadamard formula for the spectral radius of $\widehat{T}$, cf. [TL80, Chap. V, Thm. 3.5] or [Yos65, XIII.2, Thm. 3], we obtain the equality

$$\mathrm{r}_{\mathrm{ess}}(T) = \lim_{n\to\infty} \|T^n\|_{\mathrm{ess}}^{1/n}.$$

For a detailed analysis of the essential spectrum of an operator, we refer to [Kat80, Sec. IV.5.6], [GGK90, Chap. XVII], or [Gol66, Sec. IV.2]. Here, we only recall that the poles of $R(\cdot,T)$ with finite algebraic multiplicity belong to $\rho_{\mathrm{F}}(T)$. Conversely, an element of the unbounded connected component of $\rho_{\mathrm{F}}(T)$ either belongs to $\rho(T)$ or is a pole of finite algebraic multiplicity. Thus $\mathrm{r}_{\mathrm{ess}}(T)$ can be characterized by

$$\mathrm{r}_{\mathrm{ess}}(T) = \inf\left\{ r>0 : \begin{matrix} \lambda\in\sigma(T),\ |\lambda|>r \text{ is a pole of} \\ \text{finite algebraic multiplicity} \end{matrix} \right\}. \tag{1.16}$$

**1.21 Exercises.** (1) Let $A$ be a complex $n \times n$ matrix. Show that for $\lambda \in \sigma(A)$

(i) the geometric multiplicity $M_g$ is the number of Jordan blocks corresponding to $\lambda$,

(ii) the algebraic multiplicity $m_a$ is the sum of the sizes of all Jordan blocks corresponding to $\lambda$,

(iii) the pole order $k$ of $R(\cdot, A)$ in $\lambda$ is the size of the largest Jordan block corresponding to $\lambda$.

(2) Verify the inequalities in (1.14). (Hint: Assume first $k < \infty$. Then use the identity $R(\lambda,A)x = \sum_{j=0}^{k-1} \frac{(A-\mu_0)^j x}{(\lambda-\mu_0)^{j+1}}$ valid for all $x \in \ker(\mu_0 - A)^k$ in order to show that $\ker(\mu_0-A)^{k-1} \subsetneqq \ker(\mu_0-A)^k = \mathrm{rg}\, P$.)

(3) Compute the spectrum $\sigma(A)$ for the following operators on the Banach space $X := \mathrm{C}[0,1]$.

(i) $Af := \frac{1}{s(1-s)} \cdot f(s)$, $D(A) := \{f \in X : Af \in X\}$.

(ii) $Bf(s) := \mathrm{i}s^2 \cdot f(s)$, $D(B) := X$.

(iii) $Cf(s) := f'(s)$, $D(C) := \{\mathrm{C}^1[0,1] : f(0) = 0\}$.

(iv) $Df(s) := f'(s)$, $D(D) := \{f \in \mathrm{C}^1[0,1] : f'(1) = 0\}$.

(v) $Ef(s) := f'(s)$, $D(E) := \{f \in \mathrm{C}^1[0,1] : f(0) = f(1)\}$.

(vi) $Ff(s) := f'(s)$, $D(F) := \{f \in \mathrm{C}^1[0,1] : f'(0) = f'(1)\}$.

(vii) $Gf(s) := f''(s)$, $D(G) := \mathrm{C}^2[0,1]$.

(viii) $Hf(s) := f''(s)$, $D(H) := \{f \in \mathrm{C}^2[0,1] : f(0) = f(1) = 0\}$.

(ix) $If(s) := f''(s)$, $D(I) := \{f \in \mathrm{C}^2[0,1] : f'(0) = f''(1) = 0\}$.

(x) $Jf(s) := f''(s)$, $D(J) := \{f \in \mathrm{C}^2[0,1] : f''(0) = 0\}$.

Which of these operators are generators on $X$? (Hint: For (vi) and (ix) see Section VI.4.b.)

(4) Consider $X := \mathrm{C}_0(\mathbb{R}, \mathbb{C}^2)$ and

$$Af(s) := f'(s) + Mf(s), \quad s \in \mathbb{R},$$

where $M := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $D(A) := \mathrm{C}_0^1(\mathbb{R}, \mathbb{C}^2)$. Show that $\sigma(A)$ decomposes into $-1 + \mathrm{i}\mathbb{R}$ and $1 + \mathrm{i}\mathbb{R}$ and that there exists a corresponding spectral decomposition. (Hint: Transform $M$ into a diagonal matrix.)

(5) Let $A$ be an operator on a Banach space $X$ and let $B$ be a restriction of $A$. If $B$ is surjective and $A$ is injective, then $A = B$. This is the case if $B \subset A$ and $\rho(A) \cap \rho(B) \neq \emptyset$.

# 2. Spectrum of Semigroups and Generators

The Hille–Yosida theorem already ensures that the spectrum of the generator of a strongly continuous semigroup always lies in a proper left half-plane and thus satisfies a property not shared by arbitrary closed operators. In this section we are going to study the spectrum of generators and its relation to the spectrum of the semigroup operators more closely. In addition, we introduce some basic techniques used in the spectral theory of semigroups. In Section 2.c these techniques lead to a detailed description of periodic groups.

## *a. Basic Theory*

For (unbounded) semigroup generators, the role played by the spectral radius in the case of bounded operators is taken over by the following quantity.

**2.1 Definition.** *Let $A : D(A) \subset X \to X$ be a closed operator. Then*

$$\mathrm{s}(A) := \sup\bigl\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\bigr\}$$

*is called the spectral bound of $A$.*

Note that $\mathrm{s}(A)$ can be any real number including $-\infty$ (if $\sigma(A) = \emptyset$) and $+\infty$. For the generator $A$ of a strongly continuous semigroup $\mathcal{T} = \bigl(T(t)\bigr)_{t \geq 0}$, however, the spectral bound $\mathrm{s}(A)$ is always dominated by the *growth bound*

$$\omega_0 := \omega_0(\mathcal{T}) := \inf\left\{ w \in \mathbb{R} : \begin{array}{c} \text{there exists } M_w \geq 1 \text{ such that} \\ \|T(t)\| \leq M_w \mathrm{e}^{wt} \text{ for all } t \geq 0 \end{array} \right\}$$

of the semigroup[1] (see Definition I.5.6 and Corollary II.1.13).

[1] Occasionally, we will write "$\omega_0(A)$" instead of "$\omega_0(\mathcal{T})$," since by Theorem II.1.4 the semigroup $\mathcal{T}$ is uniquely determined by its generator $A$.

**2.2 Proposition.** *For the spectral bound* $\mathrm{s}(A)$ *of a generator* $A$ *and for the growth bound* $\omega_0$ *of the generated semigroup* $(T(t))_{t\geq 0}$, *one has*

$$(2.1)\qquad \begin{aligned} -\infty \leq \mathrm{s}(A) \leq \omega_0 &= \inf_{t>0} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| \\ &= \frac{1}{t_0} \log \mathrm{r}\big(T(t_0)\big) < \infty \end{aligned}$$

*for each* $t_0 > 0$. *In particular, the spectral radius of the semigroup operator* $T(t)$ *is given by*

$$(2.2)\qquad \mathrm{r}\big(T(t)\big) = \mathrm{e}^{\omega_0 t} \qquad \text{for all } t \geq 0.$$

For the proof we need the following elementary fact.

**2.3 Lemma.** *Let* $\xi : \mathbb{R}_+ \to \mathbb{R}$ *be bounded on compact intervals and subadditive, i.e.,* $\xi(s+t) \leq \xi(s) + \xi(t)$ *for all* $s, t \geq 0$. *Then*

$$\inf_{t>0} \frac{\xi(t)}{t} = \lim_{t\to\infty} \frac{\xi(t)}{t}$$

*exists.*

Proof. Fix $t_0 > 0$ and write $t = kt_0 + s$ with $k \in \mathbb{N}$, $s \in [0, t_0)$. The subadditivity implies

$$\frac{\xi(t)}{t} \leq \frac{1}{kt_0}\big(\xi(kt_0) + \xi(s)\big) \leq \frac{\xi(t_0)}{t_0} + \frac{\xi(s)}{kt_0}.$$

Since $k \to \infty$ if $t \to \infty$, we obtain

$$\overline{\lim_{t\to\infty}} \frac{\xi(t)}{t} \leq \frac{\xi(t_0)}{t_0}$$

for each $t_0 > 0$ and therefore

$$\overline{\lim_{t\to\infty}} \frac{\xi(t)}{t} \leq \inf_{t>0} \frac{\xi(t)}{t} \leq \underline{\lim_{t\to\infty}} \frac{\xi(t)}{t},$$

which proves the assertion. ☐

Proof of Proposition 2.2. Since the function

$$t \mapsto \xi(t) := \log \|T(t)\|$$

satisfies the assumptions of Lemma 2.3, we can define

$$v := \inf_{t>0} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|.$$

From this identity, it follows that

$$e^{vt} \le \|T(t)\|$$

for all $t \ge 0$; hence $v \le \omega_0$ by the definition of $\omega_0$. Now choose $w > v$. Then there exists $t_0 > 0$ such that

$$\frac{1}{t} \log \|T(t)\| \le w$$

for all $t \ge t_0$; hence $\|T(t)\| \le e^{wt}$ for $t \ge t_0$. On $[0, t_0]$, the norm of $T(t)$ remains bounded, so we find $M \ge 1$ such that

$$\|T(t)\| \le M e^{wt}$$

for all $t \ge 0$, i.e., $\omega_0 \le w$. Since we have already proved that $v \le \omega_0$, this implies $\omega_0 = v$.

To prove the identity $\omega_0 = {}^1\!/_{t_0} \log \mathrm{r}\big(T(t_0)\big)$, we use the Hadamard formula for the spectral radius, i.e.,

$$\begin{aligned} \mathrm{r}\big(T(t)\big) &= \lim_{n\to\infty} \|T(nt)\|^{1/n} = \lim_{n\to\infty} e^{t \cdot 1/nt \log \|T(nt)\|} \\ &= e^{t \cdot \lim_{n\to\infty} (1/nt \log \|T(nt)\|)} = e^{t\,\omega_0}. \end{aligned}$$

The remaining inequalities have already been proved in Corollary II.1.13. ☐

We now state two simple consequences of this proposition.

**2.4 Corollary.** *For a uniformly continuous semigroup $\big(T(t)\big)_{t\ge0}$ and its (bounded) generator $A$ one has*

$$\mathrm{s}(A) = \omega_0 \,. \tag{2.3}$$

Proof. From the spectral mapping theorem for uniformly continuous semigroups (see Lemma I.3.13), it follows that

$$\mathrm{r}\big(T(t)\big) = e^{\mathrm{s}(A)\cdot t};$$

hence $\mathrm{s}(A) = \omega_0$ by Proposition 2.2. ☐

**2.5 Corollary.** *For the generator $A$ of a strongly continuous semigroup $\big(T(t)\big)_{t\ge0}$ with growth bound $\omega_0 = -\infty$ (e.g., for a nilpotent semigroup) one has*

$$\mathrm{r}\big(T(t)\big) = 0 \quad \textit{for all } t > 0 \qquad \textit{and} \qquad \sigma(A) = \emptyset.$$

The inequalities in (2.1) establish an interesting relation between spectral properties of the generator $A$, expressed by the spectral bound $\mathrm{s}(A)$, and the qualitative behavior of the semigroup $\big(T(t)\big)_{t\geq 0}$, expressed by the growth bound $\omega_0$. In particular, if spectral and growth bound coincide, we obtain Liapunov stability theorems like Theorem I.2.10 and Theorem I.3.14. For general strongly continuous semigroups, however, the situation is more complex, as will be shown by the following examples and counterexamples.

**2.6 Examples.** We first discuss (left) translation semigroups on various function spaces (see Section I.4.c and Paragraph II.2.10) and show that the spectra heavily depend on the choice of the Banach space. Before starting the discussion, it is useful to observe that the exponential functions

$$\varepsilon_\lambda(s) := \mathrm{e}^{\lambda s}, \quad s \in \mathbb{R},$$

satisfy

$$\tfrac{d}{ds}\varepsilon_\lambda = \lambda\varepsilon_\lambda \qquad \text{for each } \lambda \in \mathbb{C}.$$

Since the generator $A$ of a translation semigroup is the first derivative with appropriate domain (see Paragraph II.2.10), it follows that $\lambda$ is an eigenvalue of $A$ if and only if $\varepsilon_\lambda$ belongs to the domain $D(A)$.
(i) Consider the (left) translation semigroup $\big(T(t)\big)_{t\geq 0}$ on the space $X := \mathrm{C}_0(\mathbb{R}_+)$. Its generator is

$$Af = f'$$

with domain

$$D(A) = \big\{f \in \mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{C}^1(\mathbb{R}_+) : f' \in \mathrm{C}_0(\mathbb{R}_+)\big\}.$$

Therefore, we have $\varepsilon_\lambda \in D(A)$ if and only if $\lambda \in \mathbb{C}$ satisfies $\operatorname{Re}\lambda < 0$. This shows that

$$P\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}.$$

We have that $\mathrm{s}(A) \leq \omega_0 \leq 0$, since $\big(T(t)\big)_{t\geq 0}$ is a contraction semigroup. This implies, since the spectrum is closed, that

$$\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}.$$

The same eigenfunctions $\varepsilon_\lambda$ yield eigenvalues $\mathrm{e}^{\lambda t}$ for the operators $T(t)$, and we obtain that

$$P\sigma\big(T(t)\big) = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\sigma\big(T(t)\big) = \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{for } t > 0.$$

(ii) Next, we consider the (left) translation group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $X := \mathrm{C}_0(\mathbb{R})$. Then $P\sigma(A) = \emptyset$, since no $\varepsilon_\lambda$ belongs to $D(A)$. However, for each $\alpha \in \mathbb{R}$, the functions

$$f_n(s) := \mathrm{e}^{\mathrm{i}\alpha s} \cdot \mathrm{e}^{-s^2/n}, \quad n \in \mathbb{N},$$

form an approximate eigenvector of $A$ for the approximate eigenvalue $\mathrm{i}\alpha$. This shows that

$$A\sigma(A) = \sigma(A) = \mathrm{i}\mathbb{R},$$

and analogously

$$\sigma\big(T(t)\big) = \{z \in \mathbb{C} : |z| = 1\}.$$

(iii) Since $\omega_0 = -\infty$ for the nilpotent right translation semigroup $\big(T(t)\big)_{t\geq 0}$ on $X := \mathrm{C}_0(0,1]$, see Example II.3.19, it follows from Corollary 2.5 that

$$\sigma\big(T(t)\big) = \{0\} \qquad \text{and} \qquad \sigma(A) = \emptyset.$$

In addition, for each $\lambda \in \mathbb{C}$, the resolvent is given by

$$\big(R(\lambda, A)f\big)(s) = \int_0^s \mathrm{e}^{-\lambda(s-\tau)} f(\tau)\, d\tau, \qquad s \in (0,1],\ f \in X. \tag{2.4}$$

(iv) For the periodic translation group on, e.g., $X = \mathrm{C}_{2\pi}(\mathbb{R})$ (see Paragraph I.4.15), the functions $\varepsilon_\lambda$ belong to $D(A)$ if and only if $\lambda \in \mathrm{i}\mathbb{Z}$. Since $A$ has compact resolvent (use Example II.4.26), we obtain from Corollary 1.19

$$\sigma(A) = P\sigma(A) = \mathrm{i}\mathbb{Z}.$$

The spectra of the operators $T(t)$ are always contained in $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues $\mathrm{e}^{\mathrm{i}kt}$ for $k \in \mathbb{Z}$. Since $\sigma\big(T(t)\big)$ is closed, it follows from Theorem 3.16 below, that

$$\sigma\big(T(t)\big) = \begin{cases} \Gamma & \text{if } t/2\pi \notin \mathbb{Q}, \\ \Gamma_q & \text{if } t/2\pi = p/q \in \mathbb{Q} \text{ with } p \text{ and } q \text{ coprime}, \end{cases}$$

where $\Gamma_q := \{z \in \mathbb{C} : z^q = 1\}$.

In each of these examples there is a close relationship between the spectrum $\sigma(A)$ and the spectra $\sigma\big(T(t)\big)$. As we show next this is not always the case.

**2.7 Counterexample.** Consider the Banach space

$$X := \mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^1(\mathbb{R}_+, \mathrm{e}^s ds)$$

of all continuous functions on $\mathbb{R}_+$ that vanish at infinity and are integrable for $\mathrm{e}^s\, ds$ endowed with the norm

$$\|f\| := \|f\|_\infty + \|f\|_1 = \sup_{s\geq 0} |f(s)| + \int_0^\infty |f(s)|\mathrm{e}^s\, ds.$$

The (left) translations define a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$ whose generator is

$$\begin{aligned} Af &= f', \\ D(A) &= \big\{ f \in X : f \in \mathrm{C}^1(\mathbb{R}_+), f' \in X \big\} \end{aligned}$$

(use Proposition II.2.3). As a first observation, we note that $\|T(t)\| = 1$ for all $t \geq 0$. Thus, we have $\omega_0 = 0$, and hence $\mathrm{s}(A) \leq 0$. On the other hand, $\varepsilon_\lambda \in D(A)$ only if $\mathrm{Re}\,\lambda < -1$. Hence, we obtain for the point spectrum

$$P\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < -1\}$$

and for the spectral bound $\mathrm{s}(A) \geq -1$.

We now show that $\lambda \in \rho(A)$ if $\mathrm{Re}\,\lambda > -1$. In fact, for every $f \in X$ we have that

$$\|\cdot\|_1\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s) f \, ds$$

exists, since $\|T(s)f\|_1 \leq \mathrm{e}^{-s} \|f\|_1$ for all $s \geq 0$. Moreover, the limit

$$\|\cdot\|_\infty\text{-}\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s) f \, ds$$

exists, since $\int_0^\infty \mathrm{e}^s |f(s)| \, ds < \infty$. Consequently, the improper integral

$$\int_0^\infty \mathrm{e}^{-\lambda s} T(s) f \, ds \tag{2.5}$$

exists in $X$ for every $f \in X$ and yields the inverse of $\lambda - A$ (see Theorem II.1.10.(i)). We conclude that

$$\sigma(A) = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq -1\}, \quad \text{whence} \quad \mathrm{s}(A) = -1,$$

while $\omega_0 = 0$ and $\mathrm{r}\big(T(t)\big) = 1$ by (2.2). In particular, for $t > 0$, $T(t)$ has spectral values that are not the exponential of a spectral value of $A$.

The above phenomenon makes the spectral theory of semigroups interesting and nontrivial. Before analyzing carefully what we will call the "spectral mapping theorem" for semigroups in Section 3, we first discuss another informative example.

**2.8 Delay Differential Operators.** We return to the delay differential operator from Paragraph II.3.29 defined as

$$Af := f' \qquad \text{on} \qquad D(A) := \big\{ f \in \mathrm{C}^1[-1,0] : f'(0) = Lf \big\}$$

on the Banach space $X := \mathrm{C}[-1,0]$ for some linear form $L \in X'$ and try to compute its point spectrum $P\sigma(A)$. As for the above translation

semigroups, we see that a function $f \in \mathrm{C}[-1,0]$ is an eigenfunction of $A$ only if it is (up to a scalar factor) of the form $f = \varepsilon_\lambda$, where

$$\varepsilon_\lambda(s) := \mathrm{e}^{\lambda s}, \quad s \in [-1,0],$$

for some $\lambda \in \mathbb{C}$. However, such a function $\varepsilon_\lambda$ belongs to $D(A)$ if and only if it satisfies the *boundary condition*

$$\varepsilon'_\lambda(0) = L\varepsilon_\lambda,$$

which becomes

$$\lambda = L\varepsilon_\lambda.$$

Therefore, if we define $\xi(\lambda) := \lambda - L\varepsilon_\lambda$, we obtain the point spectrum $P\sigma(A)$ as

$$P\sigma(A) = \big\{\lambda \in \mathbb{C} : \xi(\lambda) = 0\big\}.$$

Since $\xi(\cdot)$ is an analytic function on $\mathbb{C}$, its zeros are isolated, and therefore $P\sigma(A)$ is a discrete subset of $\mathbb{C}$.

In order to identify the entire spectrum $\sigma(A)$, we observe that $X_1 := \big(D(A), \|\cdot\|_A\big)$ is a closed subspace of $\mathrm{C}^1[-1,0]$ and that the canonical injection

$$i : \mathrm{C}^1[-1,0] \to \mathrm{C}[-1,0]$$

is compact by the Arzelà–Ascoli theorem. Therefore, it follows from Proposition II.4.25 that $R(\lambda, A)$ is a compact operator, and by Corollary 1.19 we obtain

$$\sigma(A) = P\sigma(A).$$

**Proposition.** *The spectrum of the above delay differential operator consists of isolated eigenvalues only. More precisely, we call*

$$\lambda \mapsto \xi(\lambda) := \lambda - L\varepsilon_\lambda$$

*the corresponding characteristic function and obtain*

$$\sigma(A) = \big\{\lambda \in \mathbb{C} : \xi(\lambda) = 0\big\}.$$

In other words, the spectrum of $A$ consists of the zeros of the *characteristic equation*

$$\xi(\lambda) = 0.$$

For concrete $L \in \mathrm{C}[-1,0]'$, it is still difficult to determine all complex zeros of the analytic function $\xi(\cdot)$. However, for applications to stability theory as in Section V.1, it suffices to know the spectral bound $\mathrm{s}(A)$. To determine it, we now assume that the linear form $L$ is decomposed as

$$L = L_0 + a\delta_0,$$

where $L_0$ is a *positive* linear form on $\mathrm{C}[-1,0]$ having no *atomic part* in 0. This means that $\lim_{n\to\infty} L_0(f_n) = 0$ whenever $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $X$ satisfying $\lim_{n\to\infty} f_n(s) = 0$ for all $-1 \le s < 0$. As usual, $\delta_0$ denotes the point evaluation at 0, and we take $a \in \mathbb{R}$. In this case, we can determine $\mathrm{s}(A)$ by discussing the characteristic equation as an equation on $\mathbb{R}$ only.

**Corollary.** *Consider the above delay differential operator $\big(A, D(A)\big)$ on $X := \mathrm{C}[-1,0]$ and assume that the linear form $L \in X'$ is of the form*

$$L = L_0 + a\delta_0$$

*for positive $L_0 \in X'$ having no atomic part in 0 and some $a \in \mathbb{R}$. Then the spectral bound $\mathrm{s}(A)$ is given by*

$$\mathrm{s}(A) = \sup\{\lambda \in \mathbb{R} : \lambda = L_0\varepsilon_\lambda + a\},$$

*and one has the equivalence*

$$\mathrm{s}(A) < 0 \iff \|L_0\| + a < 0.$$

Proof. The characteristic function $\lambda \mapsto \xi(\lambda) := \lambda - L_0\varepsilon_\lambda - a$, considered as a function on $\mathbb{R}$, is continuous and strictly increasing from $-\infty$ to $+\infty$. This holds, since we assumed $L_0$ to be positive having no atomic part in 0, hence satisfying

$$L_0\varepsilon_\lambda \downarrow 0 \qquad \text{as} \qquad \lambda \to \infty.$$

Therefore, $\xi$ has a unique real zero $\lambda_0$ satisfying

$$\lambda_0 < 0 \iff 0 < \xi(0).$$

To show that $\lambda_0 = \mathrm{s}(A)$, we take $\lambda = \mu + \mathrm{i}\nu \in \sigma(A)$. Using the above characteristic equation, this can be restated as

$$\mu + \mathrm{i}\nu = L_0(\varepsilon_\mu \varepsilon_{\mathrm{i}\nu}) + a.$$

By taking the real parts in this identity and using the positivity of $L_0$, we obtain

$$\mu = \mathrm{Re}(L_0(\varepsilon_\mu\varepsilon_{\mathrm{i}\nu}) + a) \le |L_0(\varepsilon_\mu\varepsilon_{\mathrm{i}\nu})| + a \le L_0(\varepsilon_\mu) + a,$$

which, by the above properties of $\xi$ on $\mathbb{R}$, implies $\mu \le \lambda_0$. Therefore, we conclude that

$$\mu = \mathrm{Re}\,\lambda \le \lambda_0 = \mathrm{s}(A)$$

for all $\lambda \in \sigma(A)$. □

It is recommended that the reader restate the above results for

$$L_1 f := af(0) + bf(-1)$$

or

$$L_2 f := af(0) + \int_{-1}^{0} k(s) f(s)\, ds$$

with $a \in \mathbb{R}$, $0 \le b$, and $0 \le k \in \mathrm{L}^\infty[-1,0]$.

After these examples, we return to the general theory and study the essential spectrum from Paragraph 1.20 for a strongly continuous semigroup $(T(t))_{t\ge 0}$. In particular, we apply Lemma 2.3 to the function

$$t \mapsto \xi(t) := \log \|T(t)\|_{\mathrm{ess}} = \log \|\widehat{T}(t)\|_{\mathcal{C}(X)}$$

to justify the following definition and the subsequent proposition.

**2.9 Definition.** *The essential growth bound of the semigroup $\mathcal{T} = (T(t))_{t\ge 0}$ with generator $A$ is given by*

$$\omega_{\mathrm{ess}} := \omega_{\mathrm{ess}}(\mathcal{T}) := \omega_{\mathrm{ess}}(A) := \inf_{t>0} \frac{1}{t} \log \|T(t)\|_{\mathrm{ess}}.$$

The analogue of Proposition 2.2 then reads as follows.

**2.10 Proposition.** *With the above notions, one has*

$$-\infty \le \omega_{\mathrm{ess}} = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|_{\mathrm{ess}} = \frac{1}{t_0} \log \mathrm{r}_{\mathrm{ess}}\bigl(T(t_0)\bigr) \le \omega_0 < \infty$$

*for each $t_0 > 0$.*

As stated above, one always has $\omega_{\mathrm{ess}} \le \omega_0$, and equality holds if and only if $\mathrm{r}_{\mathrm{ess}}(T(t)) = \mathrm{r}(T(t))$ for some/all $t > 0$. In the case that $\omega_{\mathrm{ess}} < \omega_0$, it follows from (1.16) and (ii) in Paragraph 1.17 that there is an eigenvalue $\lambda$ of $T(t)$ satisfying $|\lambda| = \mathrm{r}(T(t)) = \mathrm{e}^{t\,\omega_0}$, and hence by Theorem 3.7 below there exists $\widetilde{\lambda} \in P\sigma(A)$ such that $\mathrm{Re}\,\widetilde{\lambda} = \omega_0$. Thus $\omega_{\mathrm{ess}} < \omega_0$ implies $\mathrm{s}(A) = \omega_0$, i.e., we have proved the first part of the following result.

**2.11 Corollary.** *Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup on the Banach space $X$ with generator $A$. Then*

$$\omega_0 = \max\{\omega_{\mathrm{ess}}, \mathrm{s}(A)\}. \tag{2.6}$$

*Moreover, for every $w > \omega_{\mathrm{ess}}$ the set $\sigma_c := \sigma(A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \ge w\}$ is finite and the corresponding spectral projection has finite rank.*

PROOF. Assume that $\sigma_c$ is infinite. Then, by the Spectral Inclusion Theorem 3.6 below, there exists $t > 0$ such that the set $\mathrm{e}^{t\sigma_c} \subseteq \mathrm{e}^{t\sigma(A)} \subseteq \sigma(T(t))$ has an accumulation point $s_0$. Since $|s_0| \ge \mathrm{e}^{tw} > \mathrm{e}^{t\,\omega_{\mathrm{ess}}} = \mathrm{r}_{\mathrm{ess}}(T(t))$, this contradicts (1.16), and therefore $\sigma_c$ is finite. The fact that the corresponding spectral projection is of finite rank follows from the second part of Theorem 3.6 below. □

Finally, we consider compact perturbations of generators and show that they do not change the essential growth bound.

**2.12 Proposition.** *Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup on a Banach space $X$ and take a compact operator $K \in \mathcal{K}(X)$. If $(S(t))_{t\ge 0}$ denotes the semigroup generated by $A + K$, then $T(t) - S(t)$ is compact for all $t \ge 0$. In particular,*

$$\omega_{\mathrm{ess}}(A) = \omega_{\mathrm{ess}}(A + K).$$

PROOF. By Corollary III.1.7, the semigroups $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ are related by the variation of parameters formula

$$S(t) = T(t) + \int_0^t T(t-s)KS(s)\,ds, \quad t \ge 0.$$

Since $\int_0^t T(t-s)KS(s)\,ds$ is compact by Theorem C.7, the assertion follows. □

**2.13 Exercises.** (1) Use the rescaling procedure and Counterexample 2.7 to show that for arbitrary real numbers $\alpha < \beta$, there exists a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$ such that

$$\mathrm{s}(A) = \alpha \quad \text{and} \quad \omega_0 = \beta.$$

(2) Let $(T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on $X$ with generator $A$. Then there exist constants $m,\ M \geq 1,\ v,\ w \in \mathbb{R}$ such that

$$\frac{1}{m}\,\mathrm{e}^{-vt}\|x\| \leq \|T(t)x\| \leq M\mathrm{e}^{wt}\|x\| \qquad \text{for all } t\geq 0,\ x\in X.$$

Show that

$$-v \leq -\mathrm{s}(-A) \leq \mathrm{s}(A) \leq w.$$

(3*) Find a strongly continuous group $(T(t))_{t\in\mathbb{R}}$ with generator $A$ such that $\sigma(A) = \emptyset$. Then construct an example of an analytic semigroup whose generator also has empty spectrum. What is the growth bound in each case? (Hint: Use Corollary II.4.9. See also [Hua94].)

(4) Let $(T(t))_{t\geq 0}$ be the semigroup from Counterexample 2.7. Find an approximate eigenvector $(f_n)_{n\in\mathbb{N}}$ corresponding to the approximative eigenvalue $\lambda = 1$ of $T(t)$ for $t > 0$.

(5) Modify Counterexample 2.7 to obtain $\mathrm{s}(A) = -\infty$, $\omega_0 = 0$. (Hint: Consider $X := \mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^1(\mathbb{R}_+, \mathrm{e}^{x^2}\,dx)$.)

(6*) Consider the translations on

$$X := \left\{ f \in \mathrm{C}(\mathbb{R}) : \lim_{s\to\infty} f(s) = \lim_{s\to-\infty} \mathrm{e}^{3s} f(s) = 0 \text{ and } \int_{-\infty}^{\infty} \mathrm{e}^{2s}|f(s)|\,ds < \infty \right\}$$

endowed with the norm

$$\|f\| := \sup_{s\geq 0}|f(s)| + \sup_{s\leq 0}\mathrm{e}^{3s}|f(s)| + \int_{-\infty}^{\infty}\mathrm{e}^{2s}|f(s)|\,ds.$$

Show that this yields a strongly continuous group on $X$ with growth bound $\omega_0 = 0$, but spectral bound $\mathrm{s}(A) < -1$. (Hint: See [Wol81].)

## b. Spectrum of Induced Semigroups

In this subsection we return to the standard constructions from Sections I.5.b and II.2.a and discuss in the first part of this subsection how the spectrum of the generator of a subspace, quotient, and dual semigroup is related to the original generator. Again, this is rather technical but quite useful for later applications.

We start with a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space $X$ with generator $(A, D(A))$. If $Y$ is a closed, $(T(t))_{t\geq 0}$-invariant subspace of $X$, there are canonically induced semigroups $(T(t)_|)_{t\geq 0}$ and $(T(t)_/)_{t\geq 0}$ with generators $(A_|, D(A_|))$ and $(A_/, D(A_/))$ on the subspace $Y$ and the quotient space ${}^X\!/_Y$ (see Paragraphs I.5.12 and I.5.13, Paragraphs II.2.3 and II.2.4). The following example shows that the spectra of the operators $A$, $A_|$, and $A_/$ may differ drastically.

**2.14 Example.** Consider $X := \mathrm{L}^1(\mathbb{R})$, the closed subspace

$$Y := \big\{ f \in X : f(s) = 0 \text{ for } s \geq 1 \big\} \cong \mathrm{L}^1(-\infty, 1],$$

and the quotient space $X_/ := {}^X/_Y \cong \mathrm{L}^1[1,\infty)$. On these spaces the left translations induce strongly continuous semigroups with generators $A$, $A_|$, and $A_/$, respectively. For their spectra, one has

$$\sigma(A) = \mathrm{i}\mathbb{R}$$

(in analogy to Example 2.6.(ii)), while

$$\sigma(A_|) = \sigma(A_/) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$$

(compare Example 2.6.(i)). If we take

$$Z := \big\{ f \in Y : f(s) = 0 \text{ for } 0 \leq s \leq 1 \big\},$$

we obtain ${}^Y/_Z \cong \mathrm{L}^1[0,1]$, and the induced semigroup becomes the nilpotent translation semigroup. By Corollary 2.5, its generator $A_{|/}$ has spectrum

$$\sigma(A_{|/}) = \emptyset.$$

After this example, we show which relations do hold in general between the spectrum $\sigma(A)$, the subspace spectrum $\sigma(A_|)$, and the quotient spectrum $\sigma(A_/)$.

**2.15 Proposition.** *With the above notation, the inclusions*

$$\rho_+(A) \underset{(1)}{\subset} \rho(A_|) \cap \rho(A_/) \underset{(2)}{\subset} \rho(A) \underset{(3)}{\subset} [\rho(A_|) \cap \rho(A_/)] \cup [\sigma(A_|) \cap \sigma(A_/)]$$

*hold, where $\rho_+(A)$ denotes the connected component of $\rho(A)$ that is unbounded to the right.*

PROOF. We start with the observation that for $\lambda \in \rho(A)$ the operator $\lambda - A_|$ is always injective, and $\lambda - A_/$ is always surjective. Moreover, $\lambda - A_|$ is surjective if and only if $R(\lambda, A)Y \subseteq Y$ if and only if $\lambda - A_/$ is injective.

This observation immediately implies inclusion (3). To prove inclusion (1), we conclude from the integral representation of the resolvent that $R(\lambda, A)Y \subset Y$ for all $\lambda > \omega_0$. Due to the power series expansion of $R(\cdot, A)$, this inclusion also holds for all $\lambda \in \rho_+(A)$, and we obtain (1).

Finally, for the inclusion (2), we take $\lambda \in \rho(A_|) \cap \rho(A_/)$. Then $(\lambda - A)$ must be injective, since $(\lambda - A)x = 0$ implies $(\lambda - A_/)\widehat{x} = 0$; hence $\widehat{x} = 0$, i.e., $x \in Y$ and therefore $x = 0$. Moreover, $(\lambda - A)$ must be surjective. In fact, for $z \in X$, there exists $\widehat{x} \in X_/$ such that $(\lambda - A_/)\widehat{x} = \widehat{z} = z + Y$. This means that we find $u \in Y$ such that $(\lambda - A)x - z = u = (\lambda - A)v$ for some $v \in D(A_|)$. This shows that $(\lambda - A)(x - v) = z$. □

A particularly useful application of the above inclusions can be made to spectral points in the closure of $\rho_+(A)$.

**2.16 Corollary.** *Keep the above assumptions and take $\mu \in \overline{\rho_+(A)}$. Then the following equivalences hold.*

(i) *$\mu \in \sigma(A)$ if and only if $\mu \in \sigma(A_|) \cup \sigma(A_/)$.*

(ii) *$\mu$ is a pole of $R(\cdot, A)$ if and only if $\mu$ is a pole of each $R(\cdot, A_|)$ and $R(\cdot, A_/)$. In that case, the estimates*

$$\max(k_|, k_/) \le k \le k_| + k_/$$

*hold for the respective orders of the poles.*

PROOF. Since (i) is clear from the inclusions (1) and (2) in Proposition 2.15, it suffices to show (ii). To that purpose, we may, by the previous assertion, assume that for some $\delta > 0$ the punctured disc $\{\lambda \in \mathbb{C} : 0 < |\lambda - \mu| < \delta\}$ is contained in $\rho(A) \cap \rho(A_|) \cap \rho(A_/)$. Let $U_n$, $n \in \mathbb{N}$, denote the coefficients of the Laurent expansion of $R(\cdot, A)$ at $\mu$. Then the invariance of $Y$ for each $R(\lambda, A)$, $\lambda \in \rho_+(A)$, implies the same for each $U_n$. Therefore, we obtain (with obvious notation)

$$R(\lambda, A) = \sum_{n \ge -k} U_n (\lambda - \mu)^n$$

and

$$R(\lambda, A)_| = \sum_{n \ge -k_|} U_{n_|} (\lambda - \mu)^n, \qquad R(\lambda, A)_/ = \sum_{n \ge -k_/} U_{n_/} (\lambda - \mu)^n,$$

which shows that $\max(k_|, k_/) \le k$. If $R(\cdot, A)_|$ has a pole in $\mu$ of order $k_|$, then $U_{-(k_|+1)|} = 0$, i.e., $U_{-(k_|+1)} Y = \{0\}$. Similarly, one obtains $U_{-(k_/+1)} X \subset Y$ whenever $R(\cdot, A)_/$ has a pole in $\mu$ of order $k_/$. Therefore, $U_{-(k_|+1)} \cdot U_{-(k_/+1)} = 0$, and the identity

$$U_{-(k_|+1)} \cdot U_{-(k_/+1)} = U_{-(k_|+k_/+1)}$$

(see (1.12)) implies that the order of the pole is dominated by $k_| + k_/$. ☐

These results show which parts of $\sigma(A)$ can be recovered from $\sigma(A_|)$ and $\sigma(A_/)$.

In many situations, however, the invariant subspace $Y$ is not closed but only continuously embedded in $X$. If $Y$ contains the Sobolev space $X_1 := (D(A), \|\cdot\|_A)$ (as defined in Exercise II.5.9.(1)), we obtain coincidence of the spectra $\sigma(A)$ and $\sigma(A_|)$, where $(A_|, D(A_|))$ is the part of $(A, D(A))$ in $Y$ (see Proposition II.2.3). More precisely, the following holds.

**2.17 Proposition.** *Let $A$ be an operator with nonempty resolvent set $\rho(A)$ and domain $D(A) = X_1$. If $Y$ is a Banach space such that $X_1 \hookrightarrow Y \hookrightarrow X$, then one has*

$$\sigma(A_|) = \sigma(A),$$

*where $A_|$ is the part of $A$ in $Y$.*

PROOF. Since $R(\lambda, A)Y \subseteq R(\lambda, A)X = X_1 \subseteq Y$ for each $\lambda \in \rho(A)$, the inclusion $\rho(A) \subseteq \rho(A_|)$ follows from Lemma 1.15. For the proof of the converse inclusion, we first observe that for $Y_1 := (D(A_|), \|\cdot\|_{A_|})$ we have $Y_1 \hookrightarrow X_1 \hookrightarrow Y$. Moreover, we easily verify that the part $A_1$ of $A$ in $X_1$ coincides with the part of $A_|$ in $X_1$. Again by Lemma 1.15 this implies $\rho(A_|) \subseteq \rho(A_1)$. Since $\rho(A) \neq \emptyset$, the operators $A$ and $A_1$ are similar, and therefore $\rho(A_1) = \rho(A)$, i.e., $\rho(A_|) = \rho(A)$. ☐

Another standard construction is obtained in passing from the semigroup $(T(t))_{t\geq 0}$ and its generator $(A, D(A))$ to the adjoint semigroup $(T(t)')_{t\geq 0}$ and the adjoint operator $(A', D(A'))$ on the dual Banach space $X'$. This semigroup is not strongly continuous in general (see Paragraph II.2.5), but its restriction to $X^\odot := \overline{D(A')}$ is the strongly continuous semigroup $(T(t)^\odot)_{t\geq 0}$ whose generator is given by the part $(A^\odot, D(A^\odot))$ of $A'$ in $X^\odot$ (see Paragraph II.2.6). This yields continuous embeddings

$$\big(D(A'), \|\cdot\|_{A'}\big) \hookrightarrow X^\odot \hookrightarrow X',$$

and by the same arguments as above, one obtains the coincidence of the spectra of $A'$ and $A^\odot = A'_|$.

**2.18 Proposition.** *For the generator $A$ on $X$, its adjoint $A'$ on $X'$, and its part $A^\odot$ on $X^\odot$, the following hold.*

(i) $\sigma(A) = \sigma(A') = \sigma(A^\odot)$.

(ii) $R\sigma(A) = P\sigma(A') = P\sigma(A^\odot)$.

(iii) $\mathrm{s}(A) = \mathrm{s}(A') = \mathrm{s}(A^\odot)$.

(iv) $\omega_0(A) = \omega_0(A^\odot)$.

*Moreover, for the associated semigroups $(T(t))_{t\geq 0}$, $(T(t)')_{t\geq 0}$, and $(T(t)^\odot)_{t\geq 0}$, the following is true.*

(v) $\sigma(T(t)) = \sigma(T(t)') = \sigma(T(t)^\odot)$.

(vi) $R\sigma(T(t)) = P\sigma(T(t)') = P\sigma(T(t)^\odot)$.

PROOF. Assertion (i) follows by the above considerations, since $\sigma(A) = \sigma(A')$ (see Corollary B.12). Assertion (ii) holds by Proposition 1.12 and since an eigenvector of $A'$ always belongs to $D(A^\odot)$. Assertion (iii) is a consequence of (i), and (iv) follows from the estimate (2.2) in Chapter II.

Finally, the assertions (v) and (vi) follow by essentially the same arguments as those for (i) and (ii). The details are left as Exercise 2.22.(4). □

Partly to familiarize the reader with certain semigroup constructions, partly for later use (see the proof of Theorem V.2.21), we now show how to construct an *isometric limit semigroup* starting from a contraction semigroup. As a first step, we show that semigroups of isometries have special spectral properties.

**2.19 Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup of isometries on a Banach space $X$ and denote its generator by $\big(A, D(A)\big)$. Then one has*

$$\|(\lambda - A)x\| \geq |\operatorname{Re}\lambda| \cdot \|x\| \qquad \textit{for all } x \in D(A), \lambda \in \mathbb{C}, \tag{2.7}$$

*and one of the following two cases holds.*

(i) $\sigma(A) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$.

(ii) $\sigma(A) \subset \mathrm{i}\mathbb{R}$, *and the above semigroup extends to a strongly continuous group of isometries on $X$.*

Proof. For $\operatorname{Re}\lambda \neq 0$ and $x \in D(A)$, the identity (1.11) in Chapter II implies

$$\begin{aligned} e^{-\operatorname{Re}\lambda t}\,\|x\| &= e^{-\operatorname{Re}\lambda t}\,\|T(t)x\| \\ &\leq \|x\| + \int_0^t e^{-\operatorname{Re}\lambda s}\,\|T(s)(A-\lambda)x\|\,ds \\ &= \|x\| + \Bigl(\int_0^t e^{-\operatorname{Re}\lambda s}\,ds\Bigr)\,\|(A-\lambda)x\| \\ &= \|x\| + \frac{e^{-\operatorname{Re}\lambda t}-1}{-\operatorname{Re}\lambda}\,\|(A-\lambda)x\|. \end{aligned}$$

This proves (2.7). Using (2.7) and Lemma 1.9, we see that

$$A\sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} = \emptyset,$$

and hence by Proposition 1.10 the open half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ contains no boundary point of $\sigma(A)$. Since $\sigma(A)$ is contained in the closed half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$, one has either case (i) or $\sigma(A) \subset \mathrm{i}\mathbb{R}$. In this second case, it follows from (2.7) that also the resolvent of $-A$ satisfies the Hille–Yosida estimate

$$\|R(\lambda,-A)\| = \|R(-\lambda,A)\| \leq \frac{1}{|-\lambda|} = \frac{1}{\lambda} \qquad \text{for all } \lambda > 0.$$

By Corollary II.3.7 we conclude that $A$ generates a strongly continuous group of isometries. □

We now start from a strongly continuous contraction semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ with generator $\bigl(A, D(A)\bigr)$ on some Banach space $X$. Then for each $x \in X$ the map $t \mapsto \|T(t)x\|$ is decreasing, and we can define a seminorm on $X$ by

$$p(x) := \lim_{t\to\infty} \|T(t)x\| = \inf_{t\geq 0} \|T(t)x\|.$$

If we consider its null space $Y := p^{-1}\{0\}$, we obtain a norm

$$|||x + Y||| := p(x)$$

on the quotient space ${}^{X}/_{Y}$. Its completion will be denoted by

$$Z := \bigl({}^{X}/_{Y}, |||\cdot|||\bigr)^{\sim}.$$

Next, we take the operators $T(t) \in \mathcal{L}(X)$ and observe that they leave $Y$ invariant, hence induce quotient operators on ${}^{X}/_{Y}$. It follows from the above definitions that these quotient operators are isometries for $|||\cdot|||$, hence their continuous extensions are isometries on $Z$ and will be denoted by $S(t)$. Clearly, these operators form a semigroup $\bigl(S(t)\bigr)_{t\geq 0}$ on $Z$. Its strong continuity is then a consequence of

$$\begin{aligned} \lim_{s\downarrow 0} |||S(s)(x+Y) - (x+Y)||| &= \lim_{s\downarrow 0}\Bigl(\lim_{t\to\infty} \|T(t+s)x - T(t)x\|\Bigr) \\ &\leq \lim_{s\downarrow 0} \|T(s)x - x\| = 0 \end{aligned}$$

for all $x + Y$ in the dense subspace ${}^{X}/_{Y}$. This new semigroup $\bigl(S(t)\bigr)_{t\geq 0}$ will now be called the *isometric limit semigroup* corresponding to $\bigl(T(t)\bigr)_{t\geq 0}$.

**2.20 Proposition.** *For the generator $\big(B, D(B)\big)$ of the isometric limit semigroup $\big(S(t)\big)_{t\geq 0}$ on $Z$ corresponding to the strongly continuous contraction semigroup $\big(T(t)\big)_{t\geq 0}$ with generator $\big(A, D(A)\big)$, one has*

$$\sigma(B) \subset \sigma(A).$$

*In addition, if $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \mathrm{i}\mathbb{R}$, then $\big(S(t)\big)_{t\geq 0}$ can be extended to a strongly continuous group of isometries on $Z$.*

Proof. The first observation is that for every $x \in D(A)$ one has $x + Y \in D(B)$ and $B(x+Y) = Ax + Y$. Now take $\lambda \in \rho(A)$ and define

$$R(\lambda)(x+Y) := R(\lambda, A)x + Y, \quad x \in X.$$

This operator is well-defined on ${}^{X}/{}_{Y}$, its norm is dominated by $\|R(\lambda, A)\|$, and hence it extends continuously to a bounded operator $R(\lambda)$ on $Z$. The identities

$$(\lambda - B)R(\lambda)(x+Y) = (x+Y) \qquad \text{for all } x \in X$$

and

$$R(\lambda)(\lambda - B)(x+Y) = (x+Y) \qquad \text{for all } x \in D(A)$$

follow directly from the definition of $R(\lambda)$. Since

$$D := \big\{x + Y : x \in D(A)\big\} \subseteq D(B) \subset {}^{X}/{}_{Y}$$

is a core for $B$ (use Proposition II.1.7), it follows that $R(\lambda)$ is the inverse of $\lambda - B$; hence

$$\lambda \in \rho(B) \quad \text{and} \quad R(\lambda) = R(\lambda, B).$$

If $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is a proper subset of $\mathrm{i}\mathbb{R}$, the isometric limit semigroup $\big(S(t)\big)_{t\geq 0}$ and its generator $\big(B, D(B)\big)$ satisfy (ii) in Lemma 2.19. Therefore, $\big(S(t)\big)_{t\geq 0}$ extends to a group. □

The isometric limit semigroup will be used in Chapter V, when we discuss the asymptotic behavior of semigroups (e.g., Theorem V.2.21). Here, we show by an example that the extension to the completion $Z$ of ${}^{X}/{}_{Y}$ is necessary in general. In particular, the norm $|\!|\!|\cdot|\!|\!|$ does not coincide with the quotient norm on ${}^{X}/{}_{Y}$.

**2.21 Example.** Take the left translation semigroup $\big(T(t)\big)_{t\geq 0}$ on $X := \mathrm{L}^1(\mathbb{R}, m(s)\, ds)$, where

$$m(s) := \begin{cases} 1 & \text{for } s < 0, \\ \mathrm{e}^s & \text{for } s \geq 0. \end{cases}$$

This is a contraction semigroup for which

$$p(f) := \lim_{t\to\infty} \|T(t)f\| = \int_{\mathbb{R}} |f(s)|\, ds > 0$$

for every $0 \neq f \in X$. Therefore, $p^{-1}(0) = \{0\}$, the completion $(X, p)^{\sim}$ becomes $Z := \mathrm{L}^1(\mathbb{R}, ds)$, and the isometric limit semigroup is the translation (semi) group on $\mathrm{L}^1(\mathbb{R}, ds)$.

**2.22 Exercises.** (1) Reformulate the inclusions in Proposition 2.15 as

$$\mathbb{C} \setminus \rho_+(A) \supset \sigma(A_|) \cup \sigma(A_/) \supset \sigma(A) \supset \sigma(A_|) \cup \sigma(A_/) \setminus \big[\sigma(A_|) \cap \sigma(A_/)\big].$$

(2) Take the multiplication semigroup on $\mathrm{C}_0(\Omega)$ induced by a continuous function $q : \Omega \to \mathbb{C}$ satisfying $\operatorname{Re} q \le 0$. Show that the isometric limit semigroup is (isomorphic to) the multiplication group on $\mathrm{C}_0(K)$, $K := \{s \in \Omega : \operatorname{Re} q(s) = 0\}$, induced by the restriction of $q$ to $K$.

(3) We start from the left translation semigroup $(T(t))_{t\ge0}$ on $X := \mathrm{C_{ub}}(\mathbb{R}_+)$ and observe that $\mathrm{C}_0(\mathbb{R}_+)$ is a closed, $(T(t))_{t\ge0}$-invariant subspace.

(i) Show that the quotient semigroup $(T(t)_/)_{t\ge0}$ on $X_/ := {}^{\mathrm{C_{ub}}(\mathbb{R}_+)}/_{\mathrm{C}_0(\mathbb{R}_+)}$ extends to a strongly continuous group of isometries on $X_/$.

(ii) Determine the spectrum of the generator of this group.

(4) Prove parts (v) and (vi) of Proposition 2.18 (Hint: For the second equality in (v) observe the diagram after Corollary II.5.21, and use Proposition 2.17. To prove the second equality in (vi) use the fact that for every eigenvector $x'$ of $T(t)'$ the element $x^\odot := R(\mu, A')x'$ for some fixed $\mu \in \rho(A')$ is an eigenvector of $T(t)^\odot$.)

(5) Use the notation and definition of the semigroups $\mathcal{T} = (T(t))_{t\ge0}$ on $X$ and $\widehat{\mathcal{T}} = (\widehat{T}(t))_{t\ge0}$ on $\widehat{X}_\mathcal{T}$ from Exercises I.5.16.(3) and II.2.8.(3). Then the following identities hold for the corresponding spectra.

(i) $\sigma(A) = \sigma(\widehat{A})$.

(ii) $A\sigma(A) = A\sigma(\widehat{A}) = P\sigma(\widehat{A})$.

Use the counterexamples to the spectral mapping theorem (e.g., Counterexample 2.7) to show that in general, $\sigma(\widehat{T}(t)) \ne \sigma(T(t))$.

(6) Let

$$X := \Big\{ f : \mathbb{R}_+ \to \mathbb{C} : f \text{ is locally integrable, } \|f\|_X := \int_0^\infty |f(s)|\, \mathrm{e}^{-s}\, ds < \infty \Big\}$$

and let $Y := \mathrm{L}^1(\mathbb{R}_+)$ be endowed with its natural norm. On $X$ and $Y$ we define for $t \ge 0$ the left and right translations by

$$\big(T_l(t)f\big)(s) = f(s+t)$$

and

$$\big(T_r(t)f\big)(s) = \begin{cases} f(s-t), & s \ge t, \\ 0, & 0 \le s < t, \end{cases}$$

respectively (cf. Paragraph I.4.16).

(i) Show that $T_l(t)$ and $T_r(t)$ define strongly continuous semigroups on $X$ with $\|T_l(t)\|_X = \mathrm{e}^t$ and $\|T_r(t)\|_X = \mathrm{e}^{-t}$.

(ii) Denote by $A_\nu^Z$ the generator of $(T_\nu(t))_{t\ge0}$, $\nu = l, r$ on $Z = X, Y$. Show that $\sigma(A_l^Y) = \sigma(A_r^Y) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$, $\sigma(A_l^X) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 1\}$, and $\sigma(A_r^X) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le -1\}$. (Hint: In the case of the right translations try to find eigenfunctions of the adjoint of $A_r$.)

(iii) Why does (ii) not contradict Lemma 1.15 and Proposition 2.17 although $Y \hookrightarrow X$?

## c. Spectrum of Periodic Semigroups

In this subsection we present a first example for the power of spectral theory. In fact, we succeed in characterizing *periodic semigroups* by their spectral properties. This is not only interesting in itself, but will be useful for the investigation of arbitrary semigroups (see the proof of Theorem 3.7).

**2.23 Definition.** *A strongly continuous semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called* periodic *if $T(t_0) = I$ for some $t_0 > 0$. The* period *$\tau$ of $\mathcal{T}$ is*

$$\tau := \inf\big\{t_0 > 0 : T(t_0) = I\big\}.$$

Periodic semigroups are always groups with inverses $T(t)^{-1} = T(n\tau - t)$ for $0 \leq t \leq n\tau$, $\tau$ the period of $\mathcal{T}$. Moreover, they are bounded, and hence their growth bound is zero, and one has $\sigma(A) \subset \mathrm{i}\mathbb{R}$ as a consequence of the generation theorem for groups in Paragraph II.3.11.

In the matrix situation, this implies (see Exercise I.2.12.(4)) that $A$ and $T(t)$ are similar to the diagonal matrices given by $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $\operatorname{diag}(\mathrm{e}^{\lambda_1 t}, \dots, \mathrm{e}^{\lambda_n t})$ with $\lambda_i \in {}^{2\pi\mathrm{i}}/_{\tau}\,\mathbb{Z}$ for some $\tau > 0$.

As a first step towards an analogous characterization in the infinite-dimensional case, we observe the following.

**2.24 Lemma.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$. Assume that*

(i) *$P\sigma(A) \subset 2\pi\mathrm{i}\alpha\mathbb{Z}$ for some $\alpha > 0$ and that*

(ii) *the corresponding eigenvectors span a dense subspace of $X$.*

*Then the semigroup $\big(T(t)\big)_{t\geq 0}$ is periodic.*

PROOF. Take $0 \neq x \in D(A)$ and $n \in \mathbb{Z}$ such that $Ax = (2\pi\mathrm{i}\alpha n)x$. Define a function $\xi(s) := \mathrm{e}^{2\pi\mathrm{i}\alpha n(t-s)}T(s)x$ for $0 \leq s \leq t$. Then $\xi(0) = \mathrm{e}^{2\pi\mathrm{i}\alpha n t}x$, $\xi(t) = T(t)x$, while $\xi'(s) \equiv 0$. Therefore, the semigroup acts as $T(t)x = \mathrm{e}^{2\pi\mathrm{i}\alpha n t}x$ on the eigenvectors from (ii). Since these eigenvectors span a dense subspace, we obtain that $\big(T(t)\big)_{t\geq 0}$ is periodic with period $\tau \leq {}^1/_{\alpha}$. □

The relation between eigenvalues $2\pi\mathrm{i}\alpha n$ of $A$ and eigenvalues $\mathrm{e}^{2\pi\mathrm{i}\alpha n t}$ of $T(t)$ found in this proof will be our main concern in the following section (see Theorem 3.7). Here, we show that the above conditions are necessary even for periodic semigroups and start with the following lemma.

**2.25 Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ be a periodic strongly continuous semigroup with period $\tau > 0$ and generator $A$ on a Banach space $X$. Then*

$$\sigma(A) \subset \frac{2\pi\mathrm{i}}{\tau}\cdot\mathbb{Z} \qquad \textit{and}$$

$$R(\mu, A) = (1 - \mathrm{e}^{-\mu\tau})^{-1}\int_0^\tau \mathrm{e}^{-\mu s}T(s)\,ds \tag{2.8}$$

*for $\mu \notin {}^{2\pi\mathrm{i}}/_{\tau}\cdot\mathbb{Z}$.*

PROOF. It follows from the identities (1.10) and (1.11) in Lemma II.1.9 with $t = \tau$ that $(\mu - A)$ has a two-sided inverse if $\mu \neq {}^{2\pi i n}/_{\tau}$, $n \in \mathbb{Z}$, and that the inverse is given by the above expression. □

The above representation of $R(\mu, A)$ shows that the resolvent of the generator of a $\tau$-periodic semigroup is a meromorphic function having only poles of maximal order one with residues

$$P_n := \lim_{\mu \to \mu_n} (\mu - \mu_n) R(\mu, A) = \frac{1}{\tau} \int_0^\tau \mathrm{e}^{-\mu_n s} T(s)\, ds \in \mathcal{L}(X) \tag{2.9}$$

in $\mu_n := {}^{2\pi i n}/_{\tau}$. Moreover, it follows from Paragraph 1.17 that each $P_n$ is the spectral projection belonging to $\mu_n$ that by (1.13) satisfies $\operatorname{rg} P_n = \ker(\mu_n - A)$. In particular, this implies that the spectrum of $A$ consists of eigenvalues only.

Another way of looking at $P_n$ is to interpret it as the $n$th Fourier coefficient of the $\tau$-periodic function $s \mapsto T(s)$. A simple argument on Fourier series then completes the proof of the following characterization.

**2.26 Theorem.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$. Then the following assertions are equivalent.*

(a) *$(T(t))_{t \geq 0}$ is periodic.*

(b) *$\sigma(A) = P\sigma(A) \subset 2\pi i \alpha \mathbb{Z}$ for some $\alpha > 0$, and the corresponding eigenvectors span a dense subspace of $X$.*

PROOF. The implication (b) ⇒ (a) is Lemma 2.24, and we have seen above that the inclusion $\sigma(A) = P\sigma(A) \subset 2\pi i \alpha \mathbb{Z}$ in (b) follows from (a) by Lemma 2.25. It remains to show that for a periodic semigroup one has

$$\overline{\operatorname{lin}} \bigcup_{n \in \mathbb{Z}} P_n X = X.$$

If not, there exists $0 \neq x' \in X'$ vanishing on each $P_n X$, $n \in \mathbb{Z}$. This means that for each $x \in X$ all Fourier coefficients $\langle P_n x, x' \rangle$ of the function $\xi_{x,x'} : s \mapsto \langle T(s)x, x' \rangle$ vanish. If we take $x \in X$ such that $\langle x, x' \rangle \neq 0$, this cannot be true, since $\xi_{x,x'} \not\equiv 0$. □

Condition (b) above not only characterizes periodicity, but it even allows us to describe the action of $(T(t))_{t \geq 0}$. In fact, since

$$AP_n = \mu_n P_n, \tag{2.10}$$

it follows by (1.11) in Lemma II.1.9 (or as in the proof of Lemma 2.24) that

$$T(t) P_n = \mathrm{e}^{\mu_n t} P_n \qquad \text{for } t \geq 0. \tag{2.11}$$

Thus, the action of $(T(t))_{t\ge 0}$ is described on the dense subspace $\operatorname{lin}_{n\in\mathbb{Z}} P_nX$. Moreover,

$$\begin{aligned} P_mP_nx &= \frac{1}{\tau}\cdot\int_0^\tau \mathrm{e}^{-\mu_m s}T(s)P_nx\,ds \\ &= \frac{1}{\tau}\cdot\int_0^\tau \mathrm{e}^{(\mu_n-\mu_m)s}\,ds\,P_nx = 0 \qquad \text{for } n\neq m, \end{aligned}$$

i.e., the subspaces $P_nX$ are in a certain sense "orthogonal," and we could hope for a representation

$$T(t)x = \sum_{-\infty}^{+\infty} \mathrm{e}^{\mu_n t}P_nx \qquad \text{for each } x\in X.$$

As one can see from Exercise 2.30, this is not true in general, and only the following weaker statement holds.

**2.27 Theorem.** *Let $(T(t))_{t\ge 0}$ be a periodic semigroup with period $\tau>0$ on a Banach space $X$ with generator $A$ and take the associated spectral projections*

$$P_n := \frac{1}{\tau}\cdot\int_0^\tau \mathrm{e}^{-\mu_n s}T(s)\,ds, \quad \mu_n := \frac{2\pi\mathrm{i}n}{\tau}, \quad n\in\mathbb{Z}.$$

*For every $x\in D(A)$, one has $x=\sum_{-\infty}^{+\infty}P_nx$, and therefore*

$$T(t)x = \sum_{-\infty}^{+\infty} \mathrm{e}^{\mu_n t}P_nx \qquad \text{if } x\in D(A), \tag{2.12}$$

$$Ax = \sum_{-\infty}^{+\infty} \mu_n P_nx \qquad \text{if } x\in D(A^2). \tag{2.13}$$

Proof. We assume $\tau=2\pi$ and show first that $\sum_{-\infty}^{+\infty}P_nx$ is summable for $x\in D(A)$. For $y:=Ax$ we obtain $P_ny=P_nAx=AP_nx=\mathrm{i}nP_nx$. Take $H$ to be a finite subset of $\mathbb{Z}\setminus\{0\}$ and $x'\in X'$. Then

$$\begin{aligned} \Big|\sum_{n\in H}\langle P_nx,x'\rangle\Big| &= \Big|\sum_{n\in H}(\mathrm{i}n)^{-1}\langle P_ny,x'\rangle\Big| \\ &\le \Big(\sum_{n\in H}n^{-2}\Big)^{1/2}\cdot\Big(\sum_{n\in H}|\langle P_ny,x'\rangle|^2\Big)^{1/2}. \end{aligned}$$

From Bessel's inequality applied to the function $s\mapsto\langle T(s)y,x'\rangle$ belonging to $\mathrm{L}^2[0,2\pi]$ we obtain for the second factor

$$\begin{aligned} \sum_{n\in H}|\langle P_ny,x'\rangle|^2 &\le \frac{1}{2\pi}\cdot\int_0^{2\pi}|\langle T(s)y,x'\rangle|^2\,ds \\ &\le \|x'\|^2\cdot\frac{1}{2\pi}\cdot\int_0^{2\pi}\|T(s)y\|^2\,ds. \end{aligned}$$

With the constant $c := \left({}^1\!/_{2\pi} \cdot \int_0^{2\pi} \|T(s)y\|^2\, ds\right)^{1/2}$ we obtain

$$\Big\| \sum_{n\in H} P_n x \Big\| \le c\Big( \sum_{n\in H} n^{-2} \Big)^{1/2}$$

for every finite subset $H$ of $\mathbb{Z}$, i.e., $\sum_{-\infty}^{+\infty} P_n x$ is summable. Next, we set $z := \sum_{-\infty}^{+\infty} P_n x$ and observe that for every $x' \in X'$, the Fourier coefficients of the continuous, $2\pi$-periodic functions

$$s \mapsto \langle T(s)z, x'\rangle \qquad \text{and} \qquad s \mapsto \langle T(s)x, x'\rangle$$

coincide. Therefore, these functions are identical for $s \ge 0$ and in particular for $s = 0$. This implies $\langle z, x'\rangle = \langle x, x'\rangle$, and by the Hahn–Banach theorem we obtain $z = \sum_{-\infty}^{+\infty} P_n x = x$. By replacing $x$ by $T(t)x$ and $Ax$, respectively, the identities (2.11) and (2.10) then yield (2.12) and (2.13). □

For semigroups with a bounded generator $A$, we know that $\sigma(A)$ is bounded. Therefore, if the semigroup is also periodic, only a finite number of spectral projections $P_n$ are distinct from 0, and we arrive at the following characterization.

**2.28 Corollary.** *Let $(T(t))_{t\ge 0}$ be a semigroup with bounded generator on some Banach space $X$. This semigroup has period ${}^\tau\!/_k$ for some $k \in \mathbb{N}$ if and only if there exist finitely many pairwise orthogonal projections $P_n$, $-m \le n \le m$, $P_{-m} \ne 0$ or $P_m \ne 0$, such that*

(i) $\sum_{n=-m}^{+m} P_n = I$,

(ii) $T(t) = \sum_{n=-m}^{+m} \mathrm{e}^{2\pi i n t/\tau} P_n$,

(iii) $A = \sum_{n=-m}^{+m} \left({}^{2\pi i n}\!/_\tau\right) P_n$.

We close these considerations with a concrete, but typical, example.

**2.29 Example.** Let $(T(t))_{t\ge 0}$ be the rotation group on $X := \mathrm{L}^p(\Gamma)$ for $1 < p < \infty$ (see Paragraph I.4.18 or Example 2.6.(iv)). It is periodic with period $2\pi$, and the spectrum of its generator is $\sigma(A) = \mathrm{i}\mathbb{Z}$. The eigenfunctions $\varepsilon_n(z) := z^n$ yield the projections

$$P_n = \frac{1}{2\pi \mathrm{i}} \cdot \varepsilon_{-(n+1)} \otimes \varepsilon_n, \qquad \text{i.e.,}$$

$$P_n f(z) = \frac{1}{2\pi \mathrm{i}} \cdot \left( \int_\Gamma f(w) w^{-(n+1)}\, dw \right) \cdot z^n.$$

It is left as an exercise to compute the norms of $Q_m := \sum_{-m}^{+m} P_n$ in $\mathrm{L}^p(\Gamma)$ for various $p$ and then check the assertions of Theorem 2.27. By doing so, one proves some classical convergence theorems for Fourier series (compare also [LT79, Thm. 2.c.15]).

**2.30 Exercise.** Consider the translation group on the space $C_{2\pi}(\mathbb{R})$ of all $2\pi$-periodic continuous functions on $\mathbb{R}$ and denote its generator by $A$. Prove the following statements.

(i) $\sigma(A) = P\sigma(A) = \mathrm{i}\mathbb{Z}$, and each spectral projection $P_n$, $n \in \mathbb{Z}$, has norm one and rank one.

(ii) For $Q_n := \sum_{k=-n}^{n} P_k$, one has $\lim_{n\to\infty} \|Q_n\| = \infty$. (Hint: Use the representation

$$(Q_n f)(s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(n + {}^1\!/\!_2)(s-t)}{\sin({}^1\!/\!_2(s-t))} f(t)\, dt$$

and estimate $\|Q_n g_n\|$ for $g_n(s) := {}^s\!/\!_{|s|} \sin(n + {}^1\!/\!_2)s$, $-\pi \le s \le \pi$.)

(iii) There is no spectral projection $Q$ corresponding to $\sigma_1 : \{0, 1, 2, \ldots\}$, i.e., satisfying $QP_n = P_n$ for $n \ge 0$ and $QP_n = 0$ for $n < 0$. (Hint: Define isometries $V_n$ by $(V_n f)(s) := \mathrm{e}^{-\mathrm{i}ns} f(s)$ and use the identity

$$Q_n = V_{-n} Q V_n - V_{n+1} Q V_{-n-1}$$

to estimate $\|Q_n\|$. See [Dav80, Sec. 8.1].)

# 3. Spectral Mapping Theorems

It is our ultimate goal to describe the semigroup $(T(t))_{t\ge0}$ by the spectrum $\sigma(A)$ of its generator $A$. We achieved this in a very satisfactory way for periodic semigroups in Theorem 2.27. However, as we have already seen in Counterexample 2.7, the general case is much more complex. As a first, but essential, step, we now study in great detail the relation between the spectrum $\sigma(A)$ of the generator $A$ and the spectrum $\sigma(T(t))$ of the semigroup operators $T(t)$. The intuitive interpretation of $T(t)$ as the exponential "$\mathrm{e}^{tA}$" of $A$ and the spectral mapping theorem for bounded operators in the form of Lemma I.3.13 lead us to the following principle.

**3.1 Leitmotif.** *The spectra $\sigma(T(t))$ of the semigroup operators $T(t)$ should be obtained from the spectrum $\sigma(A)$ of the generator $A$ by a relation of the form*

$$\text{“}\sigma(T(t)) = \mathrm{e}^{t\sigma(A)} := \{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\}.\text{”} \tag{3.1}$$

## *a. Examples and Counterexamples*

If (3.1), or a similar relation, holds, we say that the semigroup $(T(t))_{t\ge0}$ and its generator $A$ satisfy a *spectral mapping theorem*. However, before proving results in this direction, we explain in a series of examples and counterexamples what might go wrong.

**3.2 Examples.** (i) Take a strongly continuous semigroup $(T(t))_{t\ge0}$ that cannot be extended to a group (e.g., the left translation semigroup on $C_0(\mathbb{R}_+)$; see Paragraph I.4.16). Then $0 \in \sigma(T(t))$ for all $t > 0$, while evidently 0 is never contained in $\mathrm{e}^{t\sigma(A)}$.

Therefore, we are led to modify (3.1) and will call a *spectral mapping theorem* the relation

$$\text{(SMT)} \qquad \sigma\big(T(t)\big) \setminus \{0\} = \mathrm{e}^{t\sigma(A)} \quad \text{for } t \geq 0.$$

(ii) The periodic rotation group from Example 2.29 satisfies $\sigma(A) = \mathrm{i}\mathbb{Z}$, and each $\mathrm{e}^{\mathrm{i}kt}$, $k \in \mathbb{Z}$, is an eigenvalue of $T(t)$ by (2.11). If $t/2\pi$ is irrational, these eigenvalues form a dense subset of $\Gamma$. Since the spectrum is always closed, we obtain $\sigma\big(T(t)\big) = \Gamma$ for these $t > 0$. (See also Example 2.6.(iv).)

The phenomenon appearing in this example will be referred to as a *weak spectral mapping theorem*, meaning that only

$$\text{(WSMT)} \qquad \sigma\big(T(t)\big) \setminus \{0\} = \overline{\mathrm{e}^{t\sigma(A)}} \setminus \{0\} \quad \text{for } t \geq 0$$

holds.

The above modifications of the spectral mapping theorem are simply caused by properties of the complex exponential map $z \mapsto \mathrm{e}^z$ and will have no serious consequences for our applications in Chapter V. Much more problematic is the failure of (SMT) or (WSMT) due to the particular form of the operator $A$ and the semigroup $\big(T(t)\big)_{t\geq 0}$.

Such a breakdown occurs for generators $A$ for which the spectral bound $\mathrm{s}(A)$ does not coincide with the growth bound $\omega_0$. In fact, if

$$\mathrm{s}(A) < \omega_0,$$

then

$$\mathrm{e}^{t\sigma(A)} \subset \Big\{\lambda \in \mathbb{C} : |\lambda| \leq \mathrm{e}^{t\,\mathrm{s}(A)}\Big\},$$

while $\mathrm{r}\big(T(t)\big) = \mathrm{e}^{t\,\omega_0} > \mathrm{e}^{t\,\mathrm{s}(A)}$ (use Proposition 2.2). Therefore, the generator and the semigroup in Counterexample 2.7 do not satisfy (WSMT).

While the semigroup in this example was the well-known translation semigroup, the chosen Banach space seems to be artificial. Therefore, we present more examples for a drastic failure of (WSMT) on more natural spaces.

**3.3 Counterexample (on Reflexive Banach Spaces).** Take $1 < p < q < \infty$ and the Banach space

$$X := \mathrm{L}^p[1,\infty) \cap \mathrm{L}^q[1,\infty)$$

with norm $\|f\| := \|f\|_p + \|f\|_q$. On this space we define a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ by

$$T(t)f(s) := f(s\mathrm{e}^t)$$

for $s \geq 1$, $t \geq 0$, and $f \in X$. Its generator is given by

$$Af(s) = sf'(s), \qquad s \geq 1,$$

on the domain

$$D(A) = \left\{ f \in X : \begin{array}{l} f \text{ is absolutely continuous} \\ \text{and } s \mapsto sf'(s) \text{ belongs to } X \end{array} \right\}.$$

(See Exercise I.5.9.(3).)

Then the following holds.

**Proposition.** *For the generator $\bigl(A, D(A)\bigr)$ of the semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ on the Banach space $X$, we have*

$$\mathrm{s}(A) = -\frac{1}{p} < -\frac{1}{q} = \omega_0\,.$$

Proof. For each $1 < r < \infty$, consider the strongly continuous semigroup $\bigl(T_r(t)\bigr)_{t\geq 0}$ on $\mathrm{L}^r[1,\infty)$ defined by

$$T_r(t)f(s) := f(s\mathrm{e}^t),$$

and denote its generator by $\bigl(A_r, D(A_r)\bigr)$. Then we can estimate

$$\begin{aligned}\|T_r(t)f\|_r &= \left(\int_1^\infty |f(s\mathrm{e}^t)|^r\,ds\right)^{1/r}\\ &= \mathrm{e}^{-t/r}\left(\int_{\mathrm{e}^t}^\infty |f(s)|^r\,ds\right)^{1/r} \leq \mathrm{e}^{-t/r}\|f\|_r \qquad \text{for } f \in \mathrm{L}^r[1,\infty).\end{aligned}$$

Since for each $\beta > 1/r$ the function $s \mapsto f_\beta(s) := s^{-\beta}$ belongs to $D(A_r)$ and satisfies

$$(A_r f_\beta)(s) = s f_\beta'(s) = -\beta s^{-\beta},$$

we obtain that $-\beta \in P\sigma(A_r)$; hence

$$\mathrm{s}(A_r) = \omega_0(A_r) = -\frac{1}{r}. \tag{3.2}$$

We now determine the norm of $T(t)$ on $X$. Observe first that

$$\begin{aligned}\|T(t)f\| = \|T_p(t)f\|_p + \|T_q(t)f\|_q &\leq \mathrm{e}^{-t/p}\|f\|_p + \mathrm{e}^{-t/q}\|f\|_q\\ &\leq \mathrm{e}^{-t/q}\|f\|,\end{aligned} \tag{3.3}$$

and hence the growth bound satisfies

$$\omega_0 \leq -\frac{1}{q}.$$

On the other hand, for $f_t := \mathbb{1}_{[\mathrm{e}^t,\mathrm{e}^t+1]}$ and arbitrary $1 < r < \infty$, we have

$$\|T_r(t)f_t\|_r = \|\mathbb{1}_{[1,1+\mathrm{e}^{-t}]}\|_r = \mathrm{e}^{-t/r} = \mathrm{e}^{-t/r}\|f_t\|_r.$$

Therefore, it follows that

$$\begin{aligned}\|T(t)f_t\| = \|T_p(t)f_t\|_p + \|T_q(t)f_t\|_q &= \mathrm{e}^{-t/p} + \mathrm{e}^{-t/q}\\ &\geq \mathrm{e}^{-t/q} = \tfrac{1}{2}\,\mathrm{e}^{-t/q}\|f_t\|.\end{aligned} \tag{3.4}$$

The combination of (3.3) and (3.4) yields

$$\tfrac{1}{2}\,\mathrm{e}^{-t/q} \le \|T(t)\| \le \mathrm{e}^{-t/q};$$

hence

$$\omega_0 = -\frac{1}{q}. \tag{3.5}$$

Next, we observe that $\big(T(t)\big)_{t\ge 0}$ is the restriction of $\big(T_p(t)\big)_{t\ge 0}$ to $X \hookrightarrow \mathrm{L}^p[1,\infty)$. Therefore, it follows from the proposition in II.2.3 that its generator $A$ is the part of $A_p$ in $X$. Moreover, since $\big(T_p(t)\big)_{t\ge 0}$ has negative growth bound, we obtain

$$\big(R(0,A_p)f\big)(s) = \int_0^\infty f(s\mathrm{e}^t)\,dt = \int_s^\infty f(t)\,\frac{dt}{t}$$

for $f \in \mathrm{L}^p[1,\infty)$ and (almost) all $s > 1$. This yields the estimate

$$\begin{aligned}\big|\big(R(0,A_p)f\big)(s)\big| &\le \left(\int_s^\infty \frac{1}{t^{p'}}\,dt\right)^{1/p'} \|f\|_p\\ &= \left(\frac{s^{1-p'}}{p'-1}\right)^{1/p'} \|f\|_p = s^{-1/p}\,\frac{\|f\|_p}{(p'-1)^{1/p'}}\end{aligned}$$

with $1/p + 1/p' = 1$. Since $s \mapsto s^{-1/p} \in \mathrm{L}^q[1,\infty)$, this implies $D(A_p) \subset \mathrm{L}^q[1,\infty)$ and $\|g\|_q \le c\|A_p g\|_p$ for all $g \in D(A_p)$ and a suitable constant $c > 0$. Hence, $D(A_p) \subset X$ and $\|g\| = \|g\|_p + \|g\|_q \le \|g\|_p + c\|A_p g\|_p$ for all $g \in D(A_p)$. In conclusion we obtain

$$D(A_p) \hookrightarrow X \hookrightarrow \mathrm{L}^p[1,\infty).$$

By Proposition 2.17, this implies that $\sigma(A) = \sigma(A_p)$ and, by (3.2), that

$$\mathrm{s}(A) = -\frac{1}{p}.$$

□

Even for semigroups on Hilbert spaces the spectral mapping theorem may fail.

**3.4 Counterexample (on Hilbert Spaces).** We start by considering the $n$-dimensional Hilbert space $X_n := \mathbb{C}^n$ (with the $\|\cdot\|_2$-norm) and the $n \times n$ matrix

$$A_n := \begin{pmatrix} 0 & 1 & 0 & 0\\ \vdots & \ddots & \ddots & 0\\ \vdots & & \ddots & 1\\ 0 & \cdots & \cdots & 0\end{pmatrix}.$$

Since $A_n$ is nilpotent, we obtain $\sigma(A_n) = \{0\}$. Moreover, the semigroups $\big(\mathrm{e}^{tA_n}\big)_{t\ge 0}$ generated by $A_n$ satisfy

$$\big\|\mathrm{e}^{tA_n}\big\| \le \mathrm{e}^t$$

for $t \ge 0$. We now collect some elementary facts about these matrices.

**Lemma.** *For the elements* $x_n := n^{-1/2}(1,\dots,1) \in X_n$ *we have* $\|x_n\| = 1$ *and*

(i) $\|A_n x_n - x_n\| \le n^{-1/2}$,
(ii) $\|e^{tA_n}x_n - e^t x_n\| \le te^t n^{-1/2}$ *for* $t \ge 0$ *and* $n \in \mathbb{N}$.

Proof. Assertion (i) follows directly from the definition, while (ii) is obtained from

$$e^{tA_n}x_n - e^t x_n = \int_0^t e^{t-s}e^{sA_n}(A_n x_n - x_n)\,ds$$

(see (1.10) in Lemma II.1.9) and the estimate $\|e^{tA_n}\| \le e^t$. □

Consider now the Hilbert space

$$X := \bigoplus_{n\in\mathbb{N}}^{2} X_n$$

with inner product

$$\big((x_n)\,|\,(y_n)\big) := \sum_{n\in\mathbb{N}} (x_n\,|\,y_n)$$

(cf. (A.1) in Appendix A), on which we define $A := \oplus_{n\in\mathbb{N}}(A_n + in)$ with maximal domain $D(A)$ in $X$. This operator generates the strongly continuous semigroup $\big(T(t)\big)_{t\ge0}$ given by

$$T(t) := \bigoplus_{n\in\mathbb{N}} \big(e^{int}e^{tA_n}\big)$$

and satisfying

$$\|T(t)\| \le \sup_{n\in\mathbb{N}} \big\|e^{int}e^{tA_n}\big\| \le e^t$$

for $t \ge 0$. This implies that its growth bound satisfies

$$\omega_0 \le 1.$$

We now show that $s(A) = 0$. For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, we have

$$R(\lambda, A_n + in) = R(\lambda - in, A_n) = \sum_{k=0}^{n-1} \frac{A_n^k}{(\lambda - in)^{k+1}}.$$

Since $\|A_n\| = 1$, we conclude that

$$\|R(\lambda, A_n + in)\| \le \sum_{k=0}^{n-1} \frac{1}{|\lambda - in|^{k+1}} \le \frac{1}{|\lambda - in| - 1}$$

for $n \in \mathbb{N}$ sufficiently large. This implies $\sup_{n\in\mathbb{N}} \|R(\lambda, A_n + in)\| < \infty$, and therefore

$$\bigoplus_{n\in\mathbb{N}} \big(R(\lambda, A_n + in)\big)$$

is a bounded operator on $X$, which evidently gives the inverse of $(\lambda - A)$. Hence, $s(A) \le 0$, while $s(A) \ge 0$ follows from the fact that each $in$ is an eigenvalue of $A$.

To prove $\omega_0 \geq 1$, we show that $\mathrm{r}\big(T(t_0)\big) \geq \mathrm{e}^{t_0}$ for $t_0 = 2\pi$. Take $x_n$ as in the lemma, identify it with the element $(0, \ldots, x_n, 0, \ldots) \in X$, and consider the sequence $(x_n)_{n\in\mathbb{N}}$ in $X$. Then $(x_n)_{n\in\mathbb{N}}$ is an approximate eigenvector of $T(2\pi)$ with eigenvalue $\mathrm{e}^{2\pi}$. So we have proved the following.

**Proposition.** *For the strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with*

$$T(t) := \bigoplus_{n\in\mathbb{N}} (\mathrm{e}^{int}\mathrm{e}^{tA_n})$$

*and its generator*

$$A := \bigoplus_{n\in\mathbb{N}} (A_n + \mathrm{i}n)$$

*on the Hilbert space $X := \oplus^2_{n\in\mathbb{N}} X_n$, one has*

$$\mathrm{s}(A) = 0 < \omega_0 = 1.$$

**3.5 Exercises.** (1) Show that the semigroup in Counterexample 3.4 is in fact a group whose generator has compact resolvent.

(2) Use the bijection between $\mathbb{R}_+$ and $[1, \infty)$ given by $\varphi(s) := \mathrm{e}^s$ to define a translation semigroup on a function space on $\mathbb{R}_+$ that is similar to the semigroup in Counterexample 3.3.

(3) On the space $\mathrm{L}^2_{2\pi}$ of all $2\pi$-periodic functions on $\mathbb{R}^2$ that are square integrable on $[0, 2\pi]^2$ consider the second-order partial differential equation

$$\text{(3.6)} \qquad \begin{cases} \dfrac{\partial^2 u(t,x,y)}{\partial t^2} = \dfrac{\partial^2 u(t,x,y)}{\partial x^2} + \dfrac{\partial^2 u(t,x,y)}{\partial y^2} + \mathrm{e}^{\mathrm{i}y}\dfrac{\partial u(t,x,y)}{\partial x}, \\ u(0,x,y) = u_0(x,y), \qquad \dfrac{\partial u(0,x,y)}{\partial t} = u_1(x,y) \end{cases}$$

for $(x, y) \in [0, 2\pi]^2$ and $t \geq 0$.

(i) Show that (3.6) is equivalent to the abstract Cauchy problem (ACP) for the operator $(A, D(A))$ defined by

$$A(u,v) := \Big(v, \tfrac{d^2}{dx^2}u + \tfrac{d^2}{dy^2}u + \mathrm{e}^{\mathrm{i}\cdot}\tfrac{d}{dx}u\Big), \qquad D(A) := \mathrm{H}^2_{2\pi} \times \mathrm{H}^1_{2\pi}$$

on $X := \mathrm{H}^1_{2\pi} \times \mathrm{L}^2_{2\pi}$ and for the initial value $(u_0, u_1)$. (Hint: See Section VI.3.c.)

(ii) Show that $A$ generates a strongly continuous semigroup on $X$.

(iii*) Show that $\mathrm{s}(A) = 0$, while $\omega_0 \geq 1/2$. (Hint: See [Ren94].)

## b. Spectral Mapping Theorems for Semigroups

After having seen so many failures of our Leitmotif 3.1, it is now time to present some positive results. Surprisingly, "most" of (SMT) still holds.

**3.6 Spectral Inclusion Theorem.** *For the generator $(A, D(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$, we have the inclusions*

$$\sigma\big(T(t)\big) \supset \mathrm{e}^{t\sigma(A)} \qquad \text{for} \qquad t \geq 0. \tag{3.7}$$

*More precisely, for the point, approximate point, and residual spectra the following inclusions hold for all $t \geq 0$:*

$$P\sigma\big(T(t)\big) \supset \mathrm{e}^{tP\sigma(A)}, \tag{3.8}$$

$$A\sigma\big(T(t)\big) \supset \mathrm{e}^{tA\sigma(A)}, \tag{3.9}$$

$$R\sigma\big(T(t)\big) \supset \mathrm{e}^{tR\sigma(A)}. \tag{3.10}$$

*Moreover, for $\lambda_0 \in \mathbb{C}$ such that $\mathrm{e}^{t\lambda_0}$ is an isolated singularity of $R\big(\cdot, T(t)\big)$, it follows that $\lambda$ is an isolated singularity of $R(\cdot, A)$ and, with obvious notation,*

$$m_g\big(\mathrm{e}^{\lambda_0 t}, T(t)\big) \geq m_g(\lambda_0, A), \tag{3.11}$$

$$m_a\big(\mathrm{e}^{\lambda_0 t}, T(t)\big) \geq m_a(\lambda_0, A), \tag{3.12}$$

$$k\big(\mathrm{e}^{\lambda_0 t}, T(t)\big) \geq k(\lambda_0, A). \tag{3.13}$$

Proof. Recalling the identities

$$\begin{aligned} \mathrm{e}^{\lambda t}x - T(t)x &= (\lambda - A)\int_0^t \mathrm{e}^{\lambda(t-s)}T(s)x\,ds \quad \text{for } x \in X,\\ &= \int_0^t \mathrm{e}^{\lambda(t-s)}T(s)(\lambda - A)x\,ds \quad \text{for } x \in D(A) \end{aligned} \tag{3.14}$$

from Lemma II.1.9, we see that $\big(\mathrm{e}^{\lambda t} - T(t)\big)$ is not bijective if $(\lambda - A)$ fails to be bijective. This proves (3.7).

We now prove (3.9) and, by the same arguments, (3.8). Take $\lambda \in A\sigma(A)$ and a corresponding approximate eigenvector $(x_n)_{n\in\mathbb{N}} \subset D(A)$. Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n := \mathrm{e}^{\lambda t}x_n - T(t)x_n = \int_0^t \mathrm{e}^{\lambda(t-s)}T(s)(\lambda - A)x_n\,ds.$$

These vectors satisfy for some constant $c > 0$ the estimate

$$\|y_n\| = \int_0^t \Big\|\mathrm{e}^{\lambda(t-s)}T(s)(\lambda - A)x_n\Big\|\,ds \leq c\,\|(\lambda - A)x_n\| \to 0 \text{ as } n \to \infty.$$

Hence, $\mathrm{e}^{\lambda t}$ is an approximate eigenvalue of $T(t)$, and $(x_n)_{n\in\mathbb{N}}$ serves as the same approximate eigenvector for all $t \geq 0$.

Next, take $\lambda \in R\sigma(A)$ and use (3.14) to obtain that

$$\operatorname{rg}\big(e^{\lambda t} - T(t)\big) \subset \operatorname{rg}(\lambda - A)$$

is not dense in $X$. Hence (3.10) holds.

The inequality (3.11) for the geometric multiplicities $m_g$ follows immediately from (3.14).

We now prove (3.12) for the algebraic multiplicities $m_a$. To this end, we introduce the operator $I(\lambda, t) := \int_0^t e^{\lambda(t-s)} T(s)\, ds$ and observe that by (3.14) we have

$$\begin{aligned} R(\lambda, A) &= I(\lambda, t)\, R\big(e^{\lambda t}, T(t)\big) \\ &= R\big(e^{\lambda t}, T(t)\big)\, I(\lambda, t) \qquad \text{for } e^{\lambda t} \in \rho\big(T(t)\big). \end{aligned} \tag{3.15}$$

Next, we substitute $\mu = e^{\lambda t}$ and obtain for $\mu_0 := e^{\lambda_0 t}$ the identity

$$(\mu - \mu_0)^n = \big(e^{\lambda t} - e^{\lambda_0 t}\big)^n, \qquad n \in \mathbb{Z}.$$

The term on the right-hand side can be written as $h_n(\lambda)(\lambda - \lambda_0)^n$ for an analytic function $h_n$. By writing $h_n$ at $\lambda = \lambda_0$ as its power series, we obtain in this way from the Laurent expansion of the function $\mu \mapsto R\big(\mu, T(t)\big)$ in $\mu_0$ the Laurent series of $\lambda \mapsto R\big(e^{\lambda t}, T(t)\big)$ in $\lambda_0$. Now we denote the spectral projection of $A$ in $\lambda_0$ by $P$ and the one of $T(t)$ in $\mu_0 = e^{\lambda_0 t}$ by $Q$. Then, by Paragraph 1.17, all coefficients in the principal part of the Laurent series expansion of $\mu \mapsto R\big(\mu, T(t)\big)$ in $\mu_0$ are contained in $Q\mathcal{L}(X)$. Since the map $\lambda \mapsto I(\lambda, t)$ is analytic, the previous considerations and (3.15) show that also all coefficients in the main part of the Laurent series expansion of $\lambda \mapsto R(\lambda, A)$ in $\lambda_0$ are elements of $Q\mathcal{L}(X)$. In particular, $P \in Q\mathcal{L}(X)$, and therefore $\operatorname{rg} P \subseteq \operatorname{rg} Q$, which proves (3.12).

Finally, the inequality in (3.13) for the orders of the poles $k$ follows from (3.15). □

It follows from the above examples and counterexamples that not all converse inclusions hold in general. In fact, we will see that it is only the approximate point spectrum that is responsible for the failure of (SMT). For the point spectrum and the residual spectrum, however, we are able to prove a spectral mapping formula.

**3.7 Spectral Mapping Theorem for Point and Residual Spectrum.** *For the generator $\big(A, D(A)\big)$ of a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ on a Banach space $X$, we have the identities*

$$P\sigma\big(T(t)\big) \setminus \{0\} = e^{tP\sigma(A)}, \tag{3.16}$$

$$R\sigma\big(T(t)\big) \setminus \{0\} = e^{tR\sigma(A)} \tag{3.17}$$

*for all $t \ge 0$.*

Proof. Take $t_0 > 0$ and $0 \neq \lambda \in P\sigma\big(T(t_0)\big)$. According to Paragraph I.5.11 and Paragraph II.2.2, we can pass from the semigroup $\big(T(t)\big)_{t\geq 0}$ to the rescaled semigroup $\big(S(t)\big)_{t\geq 0} := \big(\mathrm{e}^{-t\log\lambda}T(t_0t)\big)_{t\geq 0}$ having the generator $B = t_0A - \log\lambda$. Since for this rescaled semigroup 1 is an eigenvalue of $S(1)$, we can assume that $t_0 = 1$ and $\lambda = 1$ from the beginning. Consider now the $\big(T(t)\big)_{t\geq 0}$-invariant, closed subspace

$$Y := \big\{y \in X : T(1)y = y\big\},$$

which is nontrivial by assumption. The semigroup $\big(T(t)_|\big)_{t\geq 0}$ restricted to $Y$ is periodic with period $\tau \in \mathbb{N}^{-1}$. The characterization of periodic semigroups in Theorem 2.26 implies the existence of at least one $n \in \mathbb{Z}$ such that

$$\mu := 2\pi \mathrm{i} n \in P\sigma(A_|).$$

Since $P\sigma(A_|) \subset P\sigma(A)$, we obtain that

$$1 \in \mathrm{e}^{P\sigma(A)}.$$

This and (3.8) proves (3.16).

The identity for the residual spectrum follows from (3.16) if we consider the sun dual semigroup $\big(T(t)^{\odot}\big)_{t\geq 0}$ and use that $R\sigma(A) = P\sigma(A^{\odot})$ and $R\sigma\big(T(t)\big) = P\sigma\big(T(t)^{\odot}\big)$ (see Proposition 2.18). □

Based on our understanding of periodic semigroups expressed in Theorem 2.27, we can even relate the eigenvectors of $A$ and $T(t)$. In fact, if $\lambda = 1$ is an eigenvalue of $T(t_0)$ with eigenvector $y \in X$, we put

$$y_n := P_n y = \frac{1}{t_0}\int_0^{t_0} \mathrm{e}^{-2\pi \mathrm{i} ns/t_0}T(s)y\,ds \ \in Y := \big\{x \in X : T(t_0)x = x\big\}$$

(cf. the projections $P_n$ defined in (2.9)). Then $y_n$ is an eigenvector of $A_{|Y}$ and hence of $A$ with eigenvalue ${}^{2\pi \mathrm{i} n}/_{t_0}$ as soon as $y_n \neq 0$. The series expansion (2.12) implies that this must hold for at least one $n \in \mathbb{Z}$. It then follows from (1.11) in Lemma II.1.9 that this same $y_n$ is an eigenvector for each $T(t)$, $t \geq 0$. We state this and more information on the eigenspaces of $A$ and $T(t)$ in the following corollary.

**3.8 Corollary.** *For the eigenspaces of the generator $A$ and of the semigroup operators $T(t)$, respectively, the following identities hold for every $\mu \in \mathbb{C}$.*

(i) $\ker(\mu - A) = \bigcap_{s\geq 0} \ker\big(\mathrm{e}^{\mu s} - T(s)\big)$,

(ii) $\ker\big(\mathrm{e}^{\mu t_0} - T(t_0)\big) = \overline{\mathrm{lin}}_{n\in\mathbb{Z}} \ker(\mu + {}^{2\pi \mathrm{i} n}/_{t_0} - A)$ *for each* $t_0 > 0$.

Proof. It remains to show assertion (ii). After assuming $\mu = 0$, we observe that $\big(T(t)\big)_{t\geq 0}$ restricted to $\ker\big(1 - T(t_0)\big)$ becomes periodic. Hence the assertion has been proved in Theorem 2.26. □

Since we have proved spectral mapping theorems for the point as well as for the residual spectrum, it follows that in the Counterexamples 3.3 and 3.4 there must be approximate eigenvalues $\mu$ of $T(t)$ that do not stem from some $\lambda \in \sigma(A)$ via the exponential map. In order to overcome this failure and to obtain a spectral mapping theorem for the entire spectrum, we could exclude the existence of such approximate eigenvalues and assume

$$\sigma\big(T(t)\big) = P\sigma\big(T(t)\big) \cup R\sigma\big(T(t)\big)$$

(e.g., if $\big(T(t)\big)_{t\ge 0}$ is eventually compact). A more interesting and useful way to save the validity of (SMT), however, is to look for additional properties of the semigroup that guarantee even

$$A\sigma\big(T(t)\big) \setminus \{0\} = \mathrm{e}^{tA\sigma(A)}. \tag{3.18}$$

Eventual norm continuity seems to be the most general hypothesis doing this job.

However, we first characterize those approximate eigenvalues that satisfy the spectral mapping property.

**3.9 Lemma.** *For an approximate eigenvalue $\lambda \neq 0$ of the operator $T(t_0)$ the following statements are equivalent.*

(a) *There exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ satisfying $\|x_n\| = 1$ and $\|T(t_0)x_n - \lambda x_n\| \to 0$ such that $\lim_{t\downarrow 0}\sup_{n\in\mathbb{N}} \|T(t)x_n - x_n\| = 0$.*

(b) *There exists $\mu \in A\sigma(A)$ such that $\lambda = \mathrm{e}^{\mu t_0}$.*

Proof. The implication (b) $\Rightarrow$ (a) follows from identity (3.14).

To show the converse implication it suffices, as in the proof of Theorem 3.7, to consider the case $\lambda = 1$ and $t_0 = 1$ only. To this end we take an approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ as in (a). The uniform continuity of $\big(T(t)\big)_{t\ge 0}$ on the vectors $x_n$ implies that the maps $[0,1] \ni t \mapsto T(t)x_n$, $n \in \mathbb{N}$, are equicontinuous. Choose now $x_n' \in X'$, $\|x_n'\| \le 1$, satisfying $\langle x_n, x_n'\rangle \ge 1/2$ for all $n \in \mathbb{N}$. Then the functions

$$[0,1] \ni s \mapsto \xi_n(s) := \langle T(s)x_n, x_n'\rangle$$

are uniformly bounded and equicontinuous. Hence, there exists, by the Arzelà–Ascoli theorem, a convergent subsequence, still denoted by $(\xi_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty}\xi_n =: \xi \in \mathrm{C}[0,1]$. From $\xi(0) = \lim_{n\to\infty}\xi_n(0) \ge 1/2$ we obtain that $\xi \neq 0$. Therefore, this function has a non-zero Fourier coefficient, i.e., there exists $\mu_m := 2\pi i m$, $m \in \mathbb{Z}$, such that

$$\int_0^1 \mathrm{e}^{-\mu_m s}\xi(s)\,ds \neq 0.$$

If we set

$$z_n := \int_0^1 \mathrm{e}^{-\mu_m s} T(s) x_n \, ds,$$

we have $z_n \in D(A)$ by Lemma II.1.3. In addition, the elements $z_n$ satisfy

$$(\mu_m - A) z_n = \big(1 - \mathrm{e}^{-\mu_m} T(1)\big) x_n = \big(1 - T(1)\big) x_n \to 0$$

and

$$\begin{aligned} \varliminf_{n\to\infty} \|z_n\| &\geq \varliminf_{n\to\infty} |\langle z_n, x_n' \rangle| \\ &\geq \varliminf_{n\to\infty} \left| \int_0^1 \mathrm{e}^{-\mu_m s} \langle T(s) x_n, x_n' \rangle \, ds \right| \\ &\geq \left| \int_0^1 \mathrm{e}^{-\mu_m s} \xi(s) \, ds \right| > 0. \end{aligned}$$

This shows that $\big({}^{z_n}/_{\|z_n\|}\big)_{n\in\mathbb{N}}$ is an approximate eigenvector of $A$ corresponding to the approximate eigenvalue $\mu_m = 2\pi i m$. □

For eventually norm-continuous semigroups we can always construct approximate eigenvectors satisfying condition (a) of the previous lemma. Therefore, we obtain (SMT).

**3.10 Spectral Mapping Theorem for Eventually Norm-Continuous Semigroups.** *Let $\big(T(t)\big)_{t\geq 0}$ be an eventually norm-continuous semigroup with generator $\big(A, D(A)\big)$ on the Banach space $X$. Then the spectral mapping theorem*

$$\text{(SMT)} \qquad \sigma\big(T(t)\big) \setminus \{0\} = \mathrm{e}^{t\sigma(A)}, \quad t \geq 0,$$

*holds.*

Proof. Taking into account all our previous theorems such as 3.6 and 3.7 and using the rescaling technique, we have to show the following.

If $1 \in A\sigma\big(T(1)\big)$, then there exists $m \in \mathbb{Z}$
such that $\mu_m := 2\pi i m \in A\sigma(A)$.

To prove this claim, we take an approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ of $T(1)$, i.e., we assume $\|x_n\| = 1$ and $\|T(1)x_n - x_n\| \to 0$. Moreover, we assume that $t \mapsto T(t)$ is norm continuous for $t \geq t_0$. Now choose $t_0 < k \in \mathbb{N}$ and observe that

$$\|T(k)x_n - x_n\| = \big\|T(k)x_n - T(k-1)x_n + T(k-1)x_n - \cdots - x_n\big\| \to 0$$

as $n \to \infty$. The semigroup $\big(T(t)\big)_{t\geq 0}$ is then uniformly continuous on $\big(T(k)x_n\big)_{n\in\mathbb{N}}$ by assumption and on $\big(T(k)x_n - x_n\big)_{n\in\mathbb{N}}$, since this is a null sequence (use Proposition A.3). Therefore, $\big(T(t)\big)_{t\geq 0}$ is uniformly continuous on $(x_n)_{n\in\mathbb{N}} = \big(T(k)x_n\big)_{n\in\mathbb{N}} - \big(T(k)x_n - x_n\big)_{n\in\mathbb{N}}$, and the assertion follows from Lemma 3.9. □

For later reference, it is useful to state the following simple consequence of (SMT), sometimes called the *spectral bound equal growth bound condition.*

**3.11 Corollary.** *For an eventually norm-continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with generator $\big(A, D(A)\big)$ on a Banach space $X$, we have*

$$\text{(SBeGB)} \qquad \mathrm{s}(A) = \omega_0 \,.$$

Finally, we know from Diagram (4.26) in Chapter II that many important regularity properties of semigroups imply eventual norm continuity. We state the spectral mapping theorem for these semigroups.

**3.12 Corollary.** *The spectral mapping theorem*

$$\text{(SMT)} \qquad \mathrm{e}^{t\sigma(A)} = \sigma\big(T(t)\big) \setminus \{0\}, \quad t \geq 0,$$

*and the spectral bound equal growth bound condition*

$$\text{(SBeGB)} \qquad \mathrm{s}(A) = \omega_0$$

*hold for the following classes of strongly continuous semigroups:*

(i) *eventually compact semigroups,*
(ii) *eventually differentiable semigroups,*
(iii) *analytic semigroups,*
(iv) *uniformly continuous semigroups.*

It is the above condition (SBeGB) that will be used in Chapter V (e.g., in Theorem V.1.10) to characterize stability of semigroups. However, not all of (SMT) is needed to derive (SBeGB). The weaker property (WSMT), already encountered in Example 3.2.(ii), is sufficient. Therefore, the following simple result on multiplication operators (see Section I.4.a and Paragraph II.2.9) is a useful addition to the above corollaries.

**3.13 Proposition.** *Let $M_q$ be the generator of a multiplication semigroup $\big(T_q(t)\big)_{t\geq 0}$ on $X := \mathrm{C}_0(\Omega)$ (or $X := \mathrm{L}^p(\Omega, \mu)$) defined by an appropriate function $q : \Omega \to \mathbb{C}$. Then*

$$\text{(WSMT)} \qquad \sigma\big(T_q(t)\big) = \overline{\mathrm{e}^{t\sigma(M_q)}} \qquad \text{for } t \geq 0$$

*holds.*

Proof. In Proposition I.4.2.(iv), we stated that the spectrum of a multiplication operator is the closed (essential) range of the corresponding function. Therefore, we obtain

$$\sigma\big(T_q(t)\big) = \overline{\mathrm{e}^{tq_{(\mathrm{ess})}(\Omega)}} = \overline{\mathrm{e}^{t\overline{q_{(\mathrm{ess})}(\Omega)}}} = \overline{\mathrm{e}^{t\sigma(M_q)}}$$

for all $t \geq 0$. □

A simple, but typical, example is given by the multiplication operator

$$M_q(x_n)_{n\in\mathbb{Z}} := (\mathrm{i}nx_n)_{n\in\mathbb{Z}}$$

for $(x_n)_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$. Then $\sigma(M_q) = \mathrm{i}\mathbb{Z}$ and $\sigma\big(T_q(t)\big) = \Gamma$ whenever $t/2\pi \notin \mathbb{Q}$. Therefore, only (WSMT) but not (SMT) holds. See also Example 3.2.(ii).

Most importantly, the above proposition can be applied to semigroups of normal operators on Hilbert spaces. In fact, due to the Spectral Theorem I.4.9, these semigroups are always isomorphic to multiplication semigroups on $\mathrm{L}^2$-spaces; hence (WSMT) holds.

**3.14 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup of normal operators on a Hilbert space and denote its generator by $\big(A, D(A)\big)$. Then*

$$\text{(WSMT)} \qquad \sigma\big(T(t)\big) = \overline{\mathrm{e}^{t\sigma(A)}} \qquad \textit{for } t \geq 0$$

*holds.*

**3.15 Exercises.** (1) Give an alternative proof of Corollary 3.8. (Hint: For (i) use only identity (3.14); for (ii) consider $\mu = 0$ and apply Theorem 2.26.)

(2) Assume that for some $t_0 > 0$ the spectral radius $\mathrm{r}(T(t_0))$ is an eigenvalue of $T(t_0)$ (or of its adjoint $T(t_0)'$). Show that in this case one has (SBeGB), i.e., $\mathrm{s}(A) = \omega_0$.

(3) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on some $\mathrm{L}^1(\Omega, \mu)$ and assume that $0 \leq T(t)f$ for all $0 \leq f \in \mathrm{L}^1(\Omega, \mu)$ and all $t \geq 0$. Show that (SBeGB) holds, that is, $\mathrm{s}(A) = \omega_0$. (Hint: Use Lemma VI.1.9.)

(4*) A strongly continuous semigroup $(T(t))_{t\geq 0}$ with growth bound $\omega_0$ is called *asymptotically norm continuous* if

$$\lim_{t\to\infty} \Big( \overline{\lim_{h\downarrow 0}}\, \mathrm{e}^{-\omega_0 t} \|T(t+h) - T(t)\| \Big) = 0.$$

(i) Show that the semigroup $(T(t))_{t\geq 0}$ is asymptotically norm continuous if $T(t) = U_0(t) + U_1(t)$ for operator families $(U_0(t))_{t\geq 0}$ and $(U_1(t))_{t\geq 0}$ where $(U_0(t))_{t\geq 0}$ is eventually norm continuous and $\lim_{t\to\infty} \mathrm{e}^{-\omega_0 t}\|U_1(t)\| = 0$.

(ii) Construct an example of such a decomposition using Theorem III.1.10.

(iii) For a semigroup $(T(t))_{t\geq 0}$ that is norm continuous at infinity, the spectral mapping theorems holds for the boundary spectrum, i.e.,

$$\sigma\big(T(t)\big) \cap \big\{\lambda \in \mathbb{C} : |\lambda| = \mathrm{r}\big(T(t)\big)\big\} = \mathrm{e}^{t(\sigma(A)\cap(\mathrm{s}(A)+\mathrm{i}\mathbb{R}))}$$

for $t \geq 0$ and $\mathrm{r}(T(t)) > 0$. See [MM96], [Bla99], and [NP99].

(5*) Let $A$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. If $\mathrm{e}^{\mu t}$ is a pole of order $k$ of the resolvent $R(\cdot, T(t))$ with residue $P$ and if $Q_k$ is the $k$th coefficient of the Laurent series, then the following properties hold.

(i) $\mu + {}^{2\pi i n}/_t$ is a pole of $R(\cdot, A)$ of order $\leq k$ for every $n \in \mathbb{Z}$.

(ii) The residues $P_n$ in $\mu + {}^{2\pi i n}/_t$ yield $\operatorname{rg} P = \overline{\operatorname{lin}}_{n\in\mathbb{Z}} P_n X$.

(iii) The $k$th coefficient of the Laurent series of $R(\cdot, A)$ at $\mu + {}^{2\pi i n}/_t$ is

$$Q_n = (t\mathrm{e}^{\mu t})^{1-k} \frac{1}{t} Q \int_0^t \mathrm{e}^{-(\mu + {}^{2\pi i n}/_t)s} T(s)\, ds.$$

(Hint: See [Gre81, Prop. 1.10].)

(6) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ and assume that $\lambda = \mathrm{e}^{\mu t_0}$ is an approximate eigenvalue of $T(t_0)$ for some $t_0 > 0$. Show that the following assertions are equivalent.

(a) There exist $x_n \in X$, $\|x_n\| = 1$, such that $\lim_{n\to\infty} \|T(t_0)x_n - \lambda x_n\| = 0$ and $(T(t))_{t\geq 0}$ is uniformly continuous on $\{x_n : n \in \mathbb{N}\}$.

(b) There exists $m \in \mathbb{Z}$ such that $\mu + {}^{2\pi i m}/_{t_0} \in A\sigma(A)$.

(c) There exist $m \in \mathbb{N}$ and $x_n \in X$, $\|x_n\| = 1$, such that

$$\lim_{n\to\infty} \|T(t_0)x_n - \mathrm{e}^{\mu t}\mathrm{e}^{2\pi i m t/t_0} x_n\| = 0$$

uniformly on compact $t$-intervals.

Restate assertion (c) by looking at the distance of the orbits $\xi_n(\cdot) := T(\cdot)x_n$ to the function space

$$\Big\{ f \in \mathrm{C}\big([0, t_0], X\big) : f(t) = \mathrm{e}^{\mu t}\mathrm{e}^{2\pi i m t/t_0} z \text{ for some } z \in X \Big\}.$$

(Hint: See Lemma 3.9 or [NP99].)

## c. Weak Spectral Mapping Theorem for Bounded Groups

We conclude this section with an important theorem on the spectrum of strongly continuous groups. Note that groups with unbounded generator have none of the regularity properties needed in Corollary 3.12 in order to obtain (SMT). However, if the group is bounded, the following is true.

**3.16 Theorem.** *Let $(T(t))_{t\in\mathbb{R}}$ be a bounded strongly continuous group on a Banach space $X$ with generator $A$. Then the weak spectral mapping theorem*

$$\text{(WSMT)} \qquad \sigma\big(T(t)\big) = \overline{\mathrm{e}^{t\sigma(A)}} \qquad \textit{for } t \in \mathbb{R}$$

*holds.*

In the proof we follow ideas of S. Huang (see [Hua96] or [NH94]) and divide it into several steps. First, we define a functional calculus for the group $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ on the convolution algebra $(\mathrm{L}^1(\mathbb{R}), *)$ introduced in Appendix C.b.

**3.17 Lemma.** (*Functional calculus*). *Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a bounded strongly continuous group with generator $A$ on a Banach space $X$. For $f \in \mathrm{L}^1(\mathbb{R})$, define*

$$\widehat{f}(\mathcal{T})x := \int_{-\infty}^{\infty} f(t)T(t)x\,dt$$

*for $x \in X$, where the integral is understood in the sense of Bochner. Then the following assertions are true.*

(i) *$\widehat{f}(\mathcal{T}) \in \mathcal{L}(X)$ and $\|\widehat{f}(\mathcal{T})\| \le \|f\|_1 \cdot \sup_{t\in\mathbb{R}} \|T(t)\|$ for all $f \in \mathrm{L}^1(\mathbb{R})$.*

(ii) *$\widehat{f * g}(\mathcal{T}) = \widehat{f}(\mathcal{T})\,\widehat{g}(\mathcal{T})$ for all $f, g \in \mathrm{L}^1(\mathbb{R})$.*

(iii) *If $f \in \mathrm{L}^1(\mathbb{R})$ and $\widehat{f} \in \mathrm{L}^1(\mathbb{R})$, then*

$$\widehat{f}(\mathcal{T})x = \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \widehat{f}(s)\big(R(\delta - \mathrm{i}s, A) - R(-\delta - \mathrm{i}s, A)\big)x\,ds$$

*for all $x \in X$.*

(iv) *If $f \in \mathcal{K} := \big\{f \in \mathrm{L}^1(\mathbb{R}) : \widehat{f}$ has compact support$\big\}$ and $\widehat{f} \equiv 0$ in a neighborhood of $\mathrm{i}\sigma(A)$, then $\widehat{f}(\mathcal{T}) = 0$.*

Proof. Assertion (i) is easily verified, while (ii) follows by calculations similar to those proving Lemma C.12.(i).

To prove (iii) we note that $\omega_0(A) = \omega_0(-A) = 0$. Recall next that by the inversion formula for the Fourier transform from Theorem C.9,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s)\,\mathrm{e}^{\mathrm{i}st}\,ds$$

for almost all $t \in \mathbb{R}$. Then from Lebesgue's dominated convergence theorem and Fubini's theorem we obtain

$$\begin{aligned}
\widehat{f}(\mathcal{T})x &= \lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\delta|t|} f(t)T(t)x\,dt \\
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \mathrm{e}^{-\delta|t|}\left(\int_{-\infty}^{\infty} \widehat{f}(s)\,\mathrm{e}^{\mathrm{i}st}T(t)x\,ds\right)dt \\
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \widehat{f}(s)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\delta|t|+\mathrm{i}st}T(t)x\,dt\right)ds \\
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \widehat{f}(s)\big(R(\delta - \mathrm{i}s, A) - R(-\delta - \mathrm{i}s, A)\big)x\,ds
\end{aligned}$$

for all $x \in X$. This proves (iii).

To prove (iv), let $V := \mathbb{R} \setminus \operatorname{supp}\widehat{f}$. Then $V$ is a neighborhood of $\mathrm{i}\sigma(A)$ such that $\widehat{f} \equiv 0$ on $V$ and $\mathbb{R} \setminus V = \operatorname{supp}\widehat{f}$ is compact. Moreover, by (iii) we have

$$\widehat{f}(\mathcal{T})x = \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{\mathbb{R}\setminus V} \widehat{f}(s)\big(R(\delta - \mathrm{i}s, A) - R(-\delta - \mathrm{i}s, A)\big)x\,ds$$

for all $x \in X$. Since $\mathbb{R} \setminus V \subset \mathrm{i}\rho(A)$, we find that for all $\delta > 0$ the functions $s \mapsto \widehat{f}(s)\big(R(\delta - \mathrm{i}s, A) - R(-\delta - \mathrm{i}s, A)\big)x$ are continuous on $\mathbb{R} \setminus V$ and satisfy

$$\lim_{\delta\downarrow 0} \widehat{f}(s)\big(R(\delta - \mathrm{i}s, A) - R(-\delta - \mathrm{i}s, A)\big)x = 0$$

for all $s \in \mathbb{R}\setminus V$. By Lebesgue's dominated convergence theorem this implies that $\widehat{f}(\mathcal{T})x = 0$ for each $x \in X$ and thus $\widehat{f}(\mathcal{T}) = 0$. □

With the aid of this functional calculus we now introduce the "Arveson spectrum" of a bounded group.

**3.18 Proposition.** *Let $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ be a bounded strongly continuous group with generator $A$ on a Banach space $X$. Then, for $I_{\mathcal{T}} := \{f \in \mathrm{L}^1(\mathbb{R}) : \widehat{f}(\mathcal{T}) = 0\}$ and the Arveson spectrum*

$$\mathrm{Sp}(\mathcal{T}) := \Big\{s \in \mathbb{R} : \widehat{f}(s) = 0 \textit{ for all } f \in I_{\mathcal{T}}\Big\},$$

*we have*

$$\mathrm{Sp}(\mathcal{T}) = \mathrm{i}\sigma(A).$$

*Moreover, $A$ is bounded if and only if $\sigma(A)$ is bounded.*

Proof. We first note that $\sigma(A) \subseteq \mathrm{i}\mathbb{R}$, hence $\sigma(A) = A\sigma(A)$ by Proposition 1.10. Using this fact and Lemma 1.9 we find for every $\lambda \in \sigma(A)$ an approximate eigenvector $(x_n)_{n\in\mathbb{N}} \subset D(A)$ of $A$. However, from the proof of Theorem 3.6 it follows that for all $t \in \mathbb{R}$, the sequence $(x_n)_{n\in\mathbb{N}}$ is also an approximate eigenvector of $T(t)$ for the approximate eigenvalue $\mathrm{e}^{\lambda t}$. Hence, for all $f \in \mathrm{L}^1(\mathbb{R})$ and $s = \mathrm{i}\lambda$ we have

$$\big(\widehat{f}(s) - \widehat{f}(\mathcal{T})\big)x_n = \int_{-\infty}^{\infty} f(t)\big(\mathrm{e}^{\lambda t} - T(t)\big)x_n\, dt,$$

where by Lebesgue's dominated convergence theorem the right-hand side of this equality converges to zero as $n \to \infty$. This implies $\widehat{f}(s) \in \sigma(\widehat{f}(\mathcal{T}))$ and therefore $|\widehat{f}(s)| \le \|\widehat{f}(\mathcal{T})\| = 0$ for all $s \in \mathrm{i}\sigma(A)$ and $f \in I_{\mathcal{T}}$. This proves the inclusion $\mathrm{i}\sigma(A) \subseteq \mathrm{Sp}(\mathcal{T})$.

On the other hand, if $s_0 \notin \mathrm{i}\sigma(A)$, then by Lemma C.12.(ii) there exists a function $f_0 \in \mathcal{K}$ such that $\widehat{f_0}(s_0) \neq 0$ and $\widehat{f_0} \equiv 0$ in a neighborhood of $\mathrm{i}\sigma(A)$. Applying Lemma 3.17.(iv) to $f_0$ we obtain $\widehat{f_0}(\mathcal{T}) = 0$. It follows that $f_0 \in I_{\mathcal{T}}$, while $\widehat{f_0}(s_0) \neq 0$. Therefore, $s_0 \notin \mathrm{Sp}(\mathcal{T})$ and thus $\mathrm{Sp}(\mathcal{T}) = \mathrm{i}\sigma(A)$.

To prove the "moreover" part, we assume $\sigma(A)$ to be bounded. Let $P$ be the corresponding spectral projection; cf. Proposition 1.16. Then $A_{|PX}$ is a bounded operator and $\sigma(B) = \emptyset$ for $B := A_{|(I-P)X}$. Let $\mathcal{S}$ be the restriction of $\mathcal{T}$ to $(I-P)X$. Then, since $(I-P)X$ is $(T(t))_{t\in\mathbb{R}}$-invariant, $\mathcal{S}$ is also a bounded strongly continuous group with generator $B$. Lemma 3.17.(iv) applies to $\mathcal{S}$ and yields, since $\sigma(B)$ is empty, that $\widehat{f}(\mathcal{S}) = 0$ for all $f \in \mathcal{K}$. However, by Lemma C.12.(iii), $\mathcal{K}$ is norm dense in $\mathrm{L}^1(\mathbb{R})$, and by Lemma 3.17.(i) the mapping $f \mapsto \widehat{f}(\mathcal{S})$ is continuous; hence $\widehat{f}(\mathcal{S}) = 0$ for all $f \in \mathrm{L}^1(\mathbb{R})$. In particular, $R(\lambda, B) = 0$ for all $\mathrm{Re}\,\lambda > 0$. This implies that $(I-P)X = \{0\}$, and thus $A = A_{|PX}$ is bounded. ☐

Next, we repeat the previous constructions by replacing $(\mathrm{L}^1(\mathbb{R}), *)$ by the algebra $(\ell^1(\mathbb{Z}), *)$ and the bounded group $\mathcal{T}$ by a doubly power bounded linear operator $U$.

**3.19 Lemma.** (*Functional calculus*). *Let $U \in \mathcal{L}(X)$ be a doubly power bounded operator, i.e., $U$ is invertible and $\sup_{n\in\mathbb{Z}} \|U^n\| < \infty$. For $f := (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, define*

$$\widehat{f}(U) := \sum_{n\in\mathbb{Z}} a_n U^n \in \mathcal{L}(X).$$

*Then the following assertions are true.*

(i) $\|\widehat{f}(U)\| \leq \|f\|_1 \cdot \sup_{n\in\mathbb{Z}} \|U^n\|$ *for all* $f \in \ell^1(\mathbb{Z})$.

(ii) $\widehat{f * g}(U) = \widehat{f}(U)\,\widehat{g}(U)$ *for all* $f, g \in \ell^1(\mathbb{Z})$.

(iii) *For all* $f \in \ell^1(\mathbb{Z})$ *we have*

$$\widehat{f}(U) = \frac{1}{2\pi \mathrm{i}} \lim_{r\uparrow 1} \int_\Gamma \widehat{f}(z^{-1}) U \left( r^{-1}z^{-1} R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U) \right) dz.$$

(iv) *If* $f \in \ell^1(\mathbb{Z})$ *and* $\widehat{f} \equiv 0$ *in a neighborhood of* $\sigma(U)$*, then* $\widehat{f}(U) = 0$.

PROOF. Assertion (i) follows from simple calculations, while (ii) can be proved as Lemma C.13.(i).

To prove (iii), we first verify, using the Neumann series and the resolvent equation, that

$$\sum_{n\in\mathbb{Z}} z^n r^{|n|} U^n = \left( r^{-1}z^{-1} R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U) \right)$$

for all $0 < r < 1$. Moreover, for $f = (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} a_n r^{|n-1|} U^{n-1} &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \Big( \sum_{n\in\mathbb{Z}} a_n z^{-n} \Big) \cdot \Big( \sum_{n\in\mathbb{Z}} z^n r^{|n|} U^n \Big)\, dz \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widehat{f}(z^{-1}) \left( r^{-1}z^{-1} R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U) \right) dz.\end{aligned}$$

Since by Abel's limit theorem $\lim_{r\uparrow 1} U \sum_{n\in\mathbb{Z}} a_n r^{|n-1|} U^{n-1} = \widehat{f}(U)$, the desired result follows.

Finally, to verify (iv) we choose some neighborhood $V \subseteq \Gamma$ of $\sigma(U)$ such that $\widehat{f} \equiv 0$ on $V$. Then, by (iii) we have for $V^* := \{z^{-1} : z \in V\}$

$$\widehat{f}(U) = \frac{1}{2\pi \mathrm{i}} \lim_{r\uparrow 1} \int_{\Gamma\setminus V^*} \widehat{f}(z^{-1}) U \left( r^{-1}z^{-1} R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U) \right) dz.$$

Since $\Gamma \setminus V \subset \rho(U)$, we find that the functions $z \mapsto r^{-1}z^{-1}R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U)$ for all $1 > r > 0$ are continuous on $\Gamma \setminus V^*$ and satisfy

$$\lim_{r\uparrow 1} \left( r^{-1}z^{-1} R(r^{-1}z^{-1}, U) - rz^{-1}R(rz^{-1}, U) \right) = 0$$

for all $z \in \Gamma \setminus V^*$. Hence, Lebesgue's dominated convergence theorem implies that $\widehat{f}(U) = 0$. ☐

We now introduce the "Arveson spectrum" for a doubly power bounded operator $U$ in a way analogous to the group case.

**3.20 Proposition.** *Let $U \in \mathcal{L}(X)$ be a doubly power bounded operator.*

(i) *For $I_U := \{f \in \ell^1(\mathbb{Z}) : \widehat{f}(U) = 0\}$ and the Arveson spectrum*

$$\mathrm{Sp}(U) := \{z \in \Gamma : \widehat{f}(z) = 0 \text{ for all } f \in I_U\}$$

*we have*

$$\mathrm{Sp}(U) = \sigma(U).$$

(ii) *Assume $(X_n)_{n\geq 1}$ to be a sequence of closed subspaces of $X$ that are invariant under $U$ and $U^{-1}$ such that $\overline{\bigcup_{n=1}^\infty X_n} = X$. Then*

$$\overline{\bigcup_{n=1}^\infty \sigma(U_{|X_n})} = \sigma(U).$$

Proof. (i) We first note that $\sigma(U) \subseteq \Gamma$, and therefore $\sigma(U) = A\sigma(U)$ by Proposition 1.10. Hence, for all $z \in \sigma(U)$ we can choose an approximate eigenvector $(x_k)_{k\in\mathbb{N}} \subset X$ and obtain for arbitrary $f = (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$

$$\big(\widehat{f}(z) - \widehat{f}(U)\big)x_k = \sum_{n\in\mathbb{Z}} a_n\,(z^n - U^n)\,x_k \to 0$$

as $k \to \infty$. This shows that $\widehat{f}(z) \in \sigma(\widehat{f}(U))$ and therefore $|\widehat{f}(z)| \leq \|\widehat{f}(U)\| = 0$ for all $f \in I_U$. This proves the inclusion $\sigma(U) \subseteq \mathrm{Sp}(U)$.

On the other hand, for $z_0 \in \Gamma \setminus \sigma(U)$ by Lemma C.13.(ii) we can find an $f_0 \in \ell^1(\mathbb{Z})$ such that $\widehat{f_0}(z_0) \neq 0$ and $\widehat{f_0} \equiv 0$ in a neighborhood of $\sigma(U)$. Applying Lemma 3.19.(iv) to $f_0$ we obtain $\widehat{f_0}(U) = 0$. Since $\widehat{f_0}(z_0) \neq 0$, this implies that $z_0 \notin \mathrm{Sp}(U)$ and hence $\mathrm{Sp}(U) = \sigma(U)$.

To prove (ii), we recall that

$$\sigma(U_{|X_n}) = A\sigma(U_{|X_n}) \subseteq \sigma(U),$$

where the equality follows from Proposition 1.10 and Lemma 1.15. Therefore, the inclusion

$$\overline{\bigcup_{n=1}^\infty \sigma(U_{|X_n})} \subseteq \sigma(U)$$

holds.

On the other hand, if $z_1 \in \Gamma \setminus \overline{\bigcup_{n=1}^\infty \sigma(U_{|X_n})}$, then again by Lemma C.13.(ii) we can find an $f_1 \in \ell^1(\mathbb{Z})$ such that $\widehat{f_1}(z_1) \neq 0$ and $\widehat{f_1} \equiv 0$ in a neighborhood of $\overline{\bigcup_{n=1}^\infty \sigma(U_{|X_n})}$. By Lemma 3.19.(iv) this implies that $\widehat{f_1}(U_{|X_n}) = 0$ for all $n \geq 1$. Since $\bigcup_{n=1}^\infty X_n$ is dense in $X$, it follows that $\widehat{f_1}(U) = 0$; hence $z_1 \notin \mathrm{Sp}(U) = \sigma(U)$. This completes the proof. □

After these preparations we are well prepared to prove the weak spectral mapping theorem.

PROOF OF THEOREM 3.16. We write $\mathcal{T} = (T(t))_{t\in\mathbb{R}}$ and define for each $n \in \mathbb{N}$ the subspaces

$$X_n := \Big\{x \in X : \widehat{f}(\mathcal{T})x = 0 \text{ for all } f \in \mathrm{L}^1(\mathbb{R}) \text{ satisfying } \widehat{f} \equiv 0 \text{ on } [-n,n]\Big\}$$

of $X$. We then claim that

(i) each $X_n$ is $\mathcal{T}$-invariant and closed,

(ii) each $A_n := A_{|X_n}$ is bounded,

(iii) $\overline{\bigcup_{n=1}^\infty X_n} = X$.

Assertion (i) follows directly from the definition of $X_n$.

To show (ii), we fix $n \in \mathbb{N}$ and denote by $\mathcal{T}_n$ the restriction of $\mathcal{T}$ to $X_n$. Moreover, we choose $r \in \mathbb{R} \setminus [-n,n]$. Then, by Lemma C.12.(ii), there exists $f \in \mathrm{L}^1(\mathbb{R})$ such that $\widehat{f} \equiv 0$ on $[-n,n]$ and $\widehat{f}(r) \neq 0$. By definition of $X_n$, we find that $\widehat{f}(\mathcal{T}_n)x = \widehat{f}(\mathcal{T})x = 0$ for all $x \in X_n$, i.e., $\widehat{f}(\mathcal{T}_n) = 0$. Hence, $r$ is not contained in the Arveson spectrum $\mathrm{Sp}(\mathcal{T}_n)$, and therefore $\mathrm{Sp}(\mathcal{T}_n)$ is contained in $[-n,n]$. Since by Corollary II.2.3 the generator of $\mathcal{T}_n$ is given by $A_n$, Proposition 3.18 implies

$$\mathrm{Sp}(\mathcal{T}_n) = i\sigma(A_n).$$

Hence, again from Proposition 3.18, it follows that $A_n$ is bounded, proving (ii).

To prove (iii), we take some $g \in \mathcal{K}$ and choose $n \in \mathbb{N}$ such that $\operatorname{supp}\widehat{g} \subseteq [-n,n]$. Then, by Lemma C.12.(i), we have

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} = 0$$

for all $f \in \mathrm{L}^1(\mathbb{R})$ such that $\widehat{f} \equiv 0$ on $[-n,n]$. The Inversion Theorem C.9 implies $f * g = 0$ almost everywhere. Combining this fact with Lemma 3.17.(ii), we see that the subspace

$$X_0 := \operatorname{lin}\big\{\widehat{g}(\mathcal{T})x : g \in \mathcal{K},\ x \in X\big\}$$

is contained in $\bigcup_{n=1}^\infty X_n$. Therefore, (iii) follows if we can show that $\overline{X_0} = X$. To this end, let $x' \in X'$ vanish on $X_0$. Then $\big\langle \widehat{f}(\mathcal{T})x, x'\big\rangle = 0$ for all $f \in \mathcal{K}$ and $x \in X$. Since $\mathcal{K}$ is norm dense in $\mathrm{L}^1(\mathbb{R})$ by Lemma C.12.(iii) and since the map $f \mapsto \widehat{f}(\mathcal{T})$ is continuous by Lemma 3.17.(i), we find that $\big\langle \widehat{f}(\mathcal{T})x, x'\big\rangle = 0$ for all $f \in \mathrm{L}^1(\mathbb{R})$ and $x \in X$. In particular, $\big\langle R(\lambda, A)x, x'\big\rangle = 0$ for $\operatorname{Re}\lambda > 0$ and all $x \in X$. Since $R(\lambda, A)X = D(A)$ is dense in $X$, this implies $x' = 0$, and thus (iii) follows from the Hahn–Banach theorem.

We now fix $t \in \mathbb{R}$ and define $U := T(t)$. Then $U$ is a doubly power bounded operator, and we have $U_{|X_n} = \mathrm{e}^{tA_n}$ for $n \geq 1$. Moreover, since $\sigma(A_n) \subset \mathrm{i}\mathbb{R}$ we conclude from Proposition 1.10 that $\sigma(A_n) = A\sigma(A_n) \subseteq A\sigma(A) = \sigma(A)$; hence

$$\sigma\big(U_{|X_n}\big) = \big\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A_n)\big\} \subseteq \big\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\big\}.$$

This implies, by Proposition 3.20, that

$$\sigma\big(T(t)\big) = \sigma(U) = \overline{\bigcup_{n=1}^{\infty} \sigma(U_{|X_n})} \subseteq \overline{\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\}}.$$

Since by Theorem 3.6 we already know that

$$\sigma\big(T(t)\big) \supseteq \overline{\{\mathrm{e}^{t\lambda} : \lambda \in \sigma(A)\}},$$

the proof is complete. ☐

Since the spectrum of a bounded operator is never empty, we obtain the following simple but interesting consequence of Theorem 3.16.

**3.21 Corollary.** *The generator of a bounded strongly continuous group has nonempty spectrum.*

**3.22 Exercises.** (1*) Let $\mathcal{T} := (T(t))_{t\in\mathbb{R}}$ be a strongly continuous group on a Banach space $X$.

(i) Show that the weak spectral mapping theorem (WSMT) even holds if $\mathcal{T}$ is polynomially bounded. (Hint: See [Nag86, A-III, Thm. 7.4].)

(ii) The assertion in (i) remains true if

$$\|T(t)\| \le w(t) \quad \text{for all } t \in \mathbb{R}$$

and a *non-quasi-analytic weight* $w$, i.e., $w$ satisfies $1 \le w(s+t) \le w(s)+w(t)$ for $s, t \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2}\, dt < \infty.$$

(Hint: See [Hua94] and [NH94].)

(2*) Show that for a matrix-valued multiplication semigroup $(\mathcal{M}(t))_{t\ge 0}$ on the space $\mathrm{C}_0(\mathbb{R}, X)$ (i.e., in the situation of Paragraph III.4.13 take $X$ with $\dim X < \infty$) the weak spectral mapping theorem (WSMT) holds. (Hint: See [Hol91].)

## 4. Spectral Theory and Perturbation

In this section, let $\mathcal{T} = (T(t))_{t\ge 0}$ be a strongly continuous semigroup with generator $(A, D(A))$ on a Banach space $X$ and assume that the spectra $\sigma(A)$ and $\sigma(T(t))$ and the corresponding resolvents are known. Then take an operator $B \in \mathcal{L}(X)$ and the semigroup $\mathcal{S} = (S(t))_{t\ge 0}$ generated by $A + B$.

**4.1 Problem.** *Determine the spectra $\sigma(A + B)$ and $\sigma(S(t))$.*

The first part of this problem is relatively easy, and at least a part of $\sigma(A+B)$ can be described by a condition involving only $B$ and the resolvent of $A$.

**4.2 Proposition.** *For $\lambda \in \rho(A)$ we have*

$$\lambda \in \sigma(A + B) \quad \Longleftrightarrow \quad 1 \in \sigma\big(BR(\lambda, A)\big) \quad \Longleftrightarrow \quad 1 \in \sigma\big(R(\lambda, A)B\big).$$

PROOF. The assertion follows from the identity

$$\lambda - A - B = \big(I - BR(\lambda, A)\big)(\lambda - A)$$

and the equality $\sigma(BR(\lambda, A)) \setminus \{0\} = \sigma(R(\lambda, A)B) \setminus \{0\}$; see [GGK90, III.2, (3)]. □

This result also solves the second part of Problem 4.1 if the spectral mapping theorem

(SMT) $$\sigma\big(S(t)\big) \setminus \{0\} = \mathrm{e}^{t\sigma(A+B)}, \quad t \ge 0,$$

holds. By Theorem 3.10 this is true, e.g., in the situation of Theorem III.1.16. Since these assumptions are quite strong, we discuss a new and simpler problem.

**4.3 Problem.** *Estimate the spectral radius* $\mathrm{r}(S(t))$ *by the spectral radius* $\mathrm{r}(T(t))$ *and by* $\mathrm{e}^{t\,\mathrm{s}(A+B)}$, *where* $\mathrm{s}(A+B)$ *denotes the spectral bound of* $A+B$.

Since $\mathrm{r}(S(t)) = \mathrm{e}^{t\,\omega_0(\mathcal{S})}$ (see Proposition 2.2) the answer to this problem gives stability conditions for the perturbed semigroup $(S(t))_{t\geq 0}$ (cf. Proposition V.1.7).

We now recall from Corollary 2.11 that

$$\omega_0(\mathcal{S}) = \max\bigl\{\omega_{\mathrm{ess}}(\mathcal{S}), \mathrm{s}(A+B)\bigr\}$$

and from Proposition 2.10 that

$$\mathrm{e}^{t\,\omega_{\mathrm{ess}}(\mathcal{S})} = \mathrm{r}_{\mathrm{ess}}\bigl(S(t)\bigr),$$

where $\omega_{\mathrm{ess}}(\mathcal{S})$ is the essential growth bound of $\mathcal{S}$ and $\mathrm{r}_{\mathrm{ess}}(S(t))$ denotes the essential spectral radius of $S(t)$. Hence, if we can estimate $\mathrm{r}_{\mathrm{ess}}(S(t))$ by the essential spectral radius $\mathrm{r}_{\mathrm{ess}}(T(t))$ of the unperturbed semigroup, we obtain an answer to Problem 4.3.

To that purpose, we use Exercise III.1.17.(3.ii) and obtain that the semigroup $(S(t))_{t\geq 0}$ is given by the Dyson–Phillips series

$$S(t) = \sum_{j=0}^{\infty} S_j(t), \qquad t \geq 0, \tag{4.1}$$

where

$$\begin{aligned} S_0(t) &:= T(t), \\ S_{j+1}(t) &:= \int_0^t S_j(t-s)BT(s)\,ds \qquad \text{for all } t\geq 0,\ j\in\mathbb{N}_0. \end{aligned} \tag{4.2}$$

We now introduce the set

$$\mathcal{K} := \bigl\{C \in \mathcal{L}(X) : \mathrm{r}_{\mathrm{ess}}(B-C) = \mathrm{r}_{\mathrm{ess}}(B) \text{ for all } B \in \mathcal{L}(X)\bigr\} \tag{4.3}$$

and recall that $\mathcal{K}$ contains all compact operators, or, more generally, all strictly power compact or strictly singular operators (cf. [Voi80] and [Kat58, Thm. 2, p. 285]). In particular, if $X = \mathrm{L}^1(\Omega,\mu)$ or $X = \mathrm{C}_0(\Omega)$, then $\mathcal{K}$ contains all weakly compact operators, see [DS58, Cor. VI.8.13].

With this notation we can state the main result.

**4.4 Theorem.** *Assume that there exist* $n \in \mathbb{N}_0$ *and a sequence* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ *with* $\lim_{k\to\infty} t_k = \infty$ *such that*

$$R_{n+1}(t_k) := \sum_{j=n+1}^{\infty} S_j(t_k) = S(t_k) - \sum_{j=0}^{n} S_j(t_k) \in \mathcal{K} \qquad \textit{for all } k \in \mathbb{N}. \tag{4.4}$$

*Then we have*

$$\mathrm{r}_{\mathrm{ess}}\bigl(S(t)\bigr) \leq \mathrm{r}_{\mathrm{ess}}\bigl(T(t)\bigr) \qquad \textit{for all } t \geq 0 \tag{4.5}$$

*and therefore* $\omega_{\mathrm{ess}}(\mathcal{S}) \leq \omega_{\mathrm{ess}}(\mathcal{T})$.

The proof is split into the following two lemmas. First, we show that the terms in the Dyson–Phillips series (4.1) satisfy the same exponential estimate as $\mathcal{T}$.

**4.5 Lemma.** *For every $w > \omega_0(\mathcal{T})$ and $j \in \mathbb{N}_0$ there exists a constant $M_j(w) > 0$ such that*

$$\|S_j(t)\| \le M_j(w) \cdot \mathrm{e}^{wt} \qquad \text{for all } t \ge 0. \tag{4.6}$$

PROOF. By the definition of the growth bound in I.5.6, for every $w > \omega_0(\mathcal{T})$ there exists $M(w) \ge 1$ such that $\|T(t)\| \le M(w)\,\mathrm{e}^{wt}$. In particular, the assertion is true for $j = 0$ with $M_0(w) := M(w)$. We now proceed by induction and assume (4.6) to be true for some $j \in \mathbb{N}_0$. Then, from the definition of $S_{j+1}(t)$ in (4.2) we obtain for $w > \widetilde{w} > \omega_0(\mathcal{T})$ that

$$\begin{aligned}\|S_{j+1}(t)\| &\le \int_0^t M_j(\widetilde{w})\mathrm{e}^{\tilde{w}(t-s)}\ \|B\|\ M(\widetilde{w})\mathrm{e}^{\tilde{w}s}\,ds \\ &= M_j(\widetilde{w})\|B\|M(\widetilde{w}) \cdot t\,\mathrm{e}^{\tilde{w}t} \le M_{j+1}(w)\,\mathrm{e}^{wt}\end{aligned}$$

for $M_{j+1}(w) := M_j(\widetilde{w})\,\|B\|\,M(\widetilde{w}) \cdot \sup\left\{t\,\mathrm{e}^{(\tilde{w}-w)t} : t \ge 0\right\}$. ☐

In the next and essential step we derive an estimate for the essential spectral radius of the partial sums of the Dyson–Phillips series.

**4.6 Lemma.** *For every $w > \omega_{\mathrm{ess}}(\mathcal{T})$ and $n \in \mathbb{N}_0$ there exists a constant $L_n > 0$ such that*

$$\mathrm{r}_{\mathrm{ess}}\Big(\sum_{j=0}^{n} S_j(t)\Big) \le L_n \mathrm{e}^{wt} \qquad \text{for all } t \ge 0. \tag{4.7}$$

PROOF. Let $\sigma_c := \sigma(A) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge w\}$ and denote by $P_c$ the corresponding spectral projection from (1.7). This yields the decomposition

$$X = X_c \oplus X_u, \tag{4.8}$$

where $X_c := P_c X$ is finite-dimensional (use Corollary 2.11) and $X_u = P_u X$ for $P_u = I - P_c$. Since $T(t)$ commutes with $P_c$ for all $t \ge 0$, we can, according to (4.8), represent it as a diagonal matrix

$$T(t) = \begin{pmatrix} T_c(t) & 0 \\ 0 & T_u(t) \end{pmatrix}.$$

Denoting the generator of the semigroup $\mathcal{T}_u = (T_u(t))_{t\ge 0}$ by $A_u$, we see from Corollary 2.11 and Proposition 1.16 that

$$\omega_0(\mathcal{T}_u) = \max\{\omega_{\mathrm{ess}}(\mathcal{T}_u), \mathrm{s}(A_u)\} = \max\{\omega_{\mathrm{ess}}(\mathcal{T}), \mathrm{s}(A_u)\} < w. \tag{4.9}$$

We now define $B_u := P_u B_{|X_u}$ and

$$\widetilde{B} := \begin{pmatrix} 0 & 0 \\ 0 & B_u \end{pmatrix} = P_u B P_u.$$

Moreover, we denote by $\widetilde{S}_j(t)$ the terms in (4.2) with $B$ replaced by $\widetilde{B}$. Then

$$\widetilde{S}_j(t) := \begin{pmatrix} 0 & 0 \\ 0 & S_{u,j}(t) \end{pmatrix} \qquad \text{for all } j \in \mathbb{N},\ t \ge 0, \tag{4.10}$$

where $S_{u,j}(t)$ denotes the terms in (4.2) with $T(t)$ and $B$ replaced by $T_u(t)$ and $B_u$, respectively. We proceed by verifying that

(i) $\widetilde{S}_j(t) - S_j(t)$ is compact for all $j \in \mathbb{N}_0$ and $t \ge 0$, and

(ii) $\mathrm{r}_{\mathrm{ess}}\left(\sum_{j=0}^{n} \widetilde{S}_j(t)\right) \le L_n \mathrm{e}^{wt}$ for some suitable constant $L_n > 0$.

Since by definition the essential spectral radius remains unchanged under compact perturbations, this will imply the desired assertion.

In order to prove (i) we proceed by induction. Since $\widetilde{S}_0(t) = S_0(t)$, the assertion is trivially satisfied for $j = 0$. Assume now (i) to be true for some $j \in \mathbb{N}_0$. Then, we first observe that by (4.10) the operators $\widetilde{S}_j(t)$ and $P_u = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I_{X_u} \end{smallmatrix}\right)$ commute for all $j \in \mathbb{N}_0$ and $t \geq 0$. Using this fact, we obtain

$$\begin{aligned}
\widetilde{S}_{j+1}(t) - S_{j+1}(t) &= \int_0^t \widetilde{S}_j(t-s)\widetilde{B}S_0(s)\,ds - \int_0^t S_j(t-s)BS_0(s)\,ds \\
&= \int_0^t \widetilde{S}_j(t-s)(P_u - I)BP_uS_0(s)\,ds + \int_0^t \widetilde{S}_j(t-s)B(P_u - I)S_0(s)\,ds \\
&\quad + \int_0^t \big(\widetilde{S}_j(t-s) - S_j(t-s)\big)BS_0(s)\,ds \\
&= -P_c \int_0^t \widetilde{S}_j(t-s)BP_uS_0(s)\,ds - \int_0^t \widetilde{S}_j(t-s)BS_0(s)\,ds\,P_c \\
&\quad + \int_0^t \big(\widetilde{S}_j(t-s) - S_j(t-s)\big)BS_0(s)\,ds. \qquad (4.11)
\end{aligned}$$

Since $P_c$ is of finite rank and the integral in (4.11) is compact (use Theorem C.7), this proves (i).

To verify (ii), we apply Lemma 4.5 to $\mathcal{T}_u = (T_u(t))_{t\geq 0}$ and $B_u$ and obtain from (4.9) constants $M_{u,j}(w) = M_{u,j} > 0$ such that

$$\|S_{u,j}(t)\| \leq M_{u,j}\,\mathrm{e}^{wt} \qquad \text{for all } j \in \mathbb{N}_0,\ t \geq 0.$$

Since

$$\sum_{j=0}^{n} \widetilde{S}_j(t) = \begin{pmatrix} T_c(t) & 0 \\ 0 & \sum_{j=0}^{n} S_{u,j}(t) \end{pmatrix},$$

where $T_c(t)$ is of finite rank, this implies

$$\mathrm{r}_{\mathrm{ess}}\Big(\sum_{j=0}^{n} \widetilde{S}_j(t)\Big) = \mathrm{r}_{\mathrm{ess}}\Big(\sum_{j=0}^{n} S_{u,j}(t)\Big) \leq \Big\|\sum_{j=0}^{n} S_{u,j}(t)\Big\| \leq L_n \mathrm{e}^{wt}$$

for $L_n := \sum_{j=0}^{n} M_{u,j}$. This gives (ii), and the proof of the lemma is complete. □

We are now in the condition to prove the above theorem.

Proof of Theorem 4.4. Let $w > \omega_{\mathrm{ess}}(\mathcal{T})$. Then by Lemma 4.6 there exists a constant $L_n > 0$ such that

$$\begin{aligned}
L_n \mathrm{e}^{wt_k} &\geq \mathrm{r}_{\mathrm{ess}}\Big(\sum_{j=0}^{n} S_j(t_k)\Big) = \mathrm{r}_{\mathrm{ess}}\Big(S(t_k) - \Big(S(t_k) - \sum_{j=0}^{n} S_j(t_k)\Big)\Big) \\
&= \mathrm{r}_{\mathrm{ess}}\big(S(t_k)\big) \geq 0 \qquad \text{for all } k \in \mathbb{N},
\end{aligned}$$

where we used in the second equality the assumption $S(t_k) - \sum_{j=0}^{n} S_j(t_k) \in \mathcal{K}$ for all $k \in \mathbb{N}$. Since by Proposition 2.10

$$\mathrm{r}_{\mathrm{ess}}\big(S(t)\big) = \mathrm{e}^{t\,\omega_{\mathrm{ess}}(\mathcal{S})},$$

this implies that

$$L_n \geq \mathrm{e}^{(\omega_{\mathrm{ess}}(\mathcal{S}) - w)t_k} \geq 0 \qquad \text{for all } k \in \mathbb{N},$$

and hence $\lim_{k\to\infty} t_k = \infty$ implies that $\omega_{\mathrm{ess}}(\mathcal{S}) \leq w$. Since this holds for all $\omega_{\mathrm{ess}}(\mathcal{T}) < w$, we obtain $\omega_{\mathrm{ess}}(\mathcal{S}) \leq \omega_{\mathrm{ess}}(\mathcal{T})$ as claimed. □

As explained above, Theorem 4.4 gives the following answer to Problem 4.3.

**4.7 Corollary.** *Assume that there exist* $n \in \mathbb{N}_0$ *and a sequence* $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ *with* $\lim_{k\to\infty} t_k = \infty$ *such that*

$$R_{n+1}(t_k) := \sum_{j=n+1}^{\infty} S_j(t_k) \in \mathcal{K} \qquad \text{for all } k \in \mathbb{N}.$$

*Then one has*

$$\mathrm{r}\big(S(t)\big) \le \max\left\{\mathrm{r}\big(T(t)\big),\, \mathrm{e}^{t\,\mathrm{r}(A+B)}\right\}$$

*for all* $t \ge 0$.

For an application of these results we refer to Section VI.2.

**4.8 Exercises.** (1) Let $\mathcal{T} = (T(t))_{t\ge 0}$ be a strongly continuous semigroup with generator $A$ on a Banach space $X$, and take a perturbing operator $B \in \mathcal{L}(X)$. Moreover, denote by $\mathcal{S} = (S(t))_{t\ge 0}$ the perturbed semigroup with generator $A+B$ and let $S_j(t)$ be defined by (4.2).

(i) Show that the original semigroup $(T(t))_{t\ge 0}$ is given by

$$T(t) = \sum_{i=0}^{\infty} T_i(t), \qquad t \ge 0,$$

where $T_0(t) := S(t)$ and

$$T_{i+1}(t) := -\int_0^t T_i(t-s)BS(s)\,ds, \qquad t \ge 0.$$

(ii) Show that

$$T_i(t) = (-1)^i \sum_{j=i}^{\infty} \binom{j}{i} S_j(t), \qquad t \ge 0.$$

(2) Take again a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\ge 0}$ with generator $A$ on a Banach space $X$ and a perturbation $B \in \mathcal{L}(X)$. If for some $n \in \mathbb{N}$ the Dyson–Phillips term $S_{n+1}(t)$ in (4.2) is compact for all $t \ge 0$, then we obtain

$$\omega_{\mathrm{ess}}(\mathcal{T}) = \omega_{\mathrm{ess}}(\mathcal{S}),$$

where $\mathcal{S} = (S(t))_{t\ge 0}$ denotes the semigroup generated by $A+B$. (Hint: Use Exercise 1.)

## Notes to Chapter IV

**Section 1.** There are many good references for the spectral theory of closed operators. We mention Dunford–Schwartz [DS58], Taylor–Lay [TL80], and Kato [Kat80]. For the essential spectrum, introduced in one way or another (cf. the footnote on p. 243 of [Kat80]), we refer to [Kat80, Sec. IV.5.6], [GGK90, Chap. XVII], or [Gol66, IV.2].

**Section** 2.a. Right from the beginning, spectral theory was an essential tool for the investigation of semigroups (see [Phi51]). Our presentation is inspired by Section A-III in [Nag86]. The failure of "$\mathrm{s}(A) = \omega_0$" was already known to Hille ([Hil48, Sec. 21.12]), while the simple Counterexample 2.7 is due to [GVW81]. The phenomenon, discussed in Paragraph 2.8, that the spectrum (or a part of it) is determined by some characteristic equation is typical for operators arising from functional differential equations (see Section VI.1 and Section VI.6) and can be studied in an abstract framework (see [Nag97]).

**Section** 2.b. The standard constructions are taken from [Nag86, A-III], while the isometric limit semigroups first appeared explicitly in [Vũ92].

**Section** 2.c. The systematic study of periodic (semi) groups goes back (at least) to [Bar77], and has been extended in numerous papers to almost periodic semigroups (see [Ves97] and the references therein).

**Section** 3.a. Counterexample 3.3 is due to Arendt (see [Are94, Sec. 3]), while Zabczyk [Zab75] found Counterexample 3.4 and Renardy gave a counterexample for partial differential operators; see Exercise 3.5.(3) and [Ren94]. A more detailed analysis of the failure of (SMT) is done in [Wro89], while Trefethen looks in [Tre97] at this phenomenon from the perspective of numerical analysis.

**Section** 3.b. Theorem 3.10 was already known to Hille–Phillips [HP57] with still another proof in [Dav80]. In [NP99], Lemma 3.9 is used to introduce the *critical spectrum* of a strongly continuous semigroup $(T(t))_{t\geq 0}$ as the part of $\sigma(T(t))$ not obtained from $\sigma(A)$. We also refer to the systematic treatment of spectral mapping theorems in Chapter 2 of [Nee96].

**Section** 3.c. The notions $\mathrm{Sp}(\mathcal{T})$ and $\mathrm{Sp}(U)$ used in Proposition 3.18 and Proposition 3.20, respectively, are due to Evans; see [Arv74]. The equality $\mathrm{Sp}(\mathcal{T}) = \mathrm{i}\sigma(A)$ in Proposition 3.18 was proved in [Eva76], while $\mathrm{Sp}(U) = \sigma(U)$ in Proposition 3.20 can be found in [Hua96]. Theorem 3.16 and the method of its proof can be extended to polynomially bounded groups (see [DLZ82], [Nag86, A-III, Thm. 7.4], [Mar86], [Hua96]) and even to groups satisfying non-quasi-analytic growth conditions [NH94], but not to general strongly continuous groups (see [Hua94]).

**Section** 4. The best reference for the spectral theory of perturbed operators is still Kato's classic [Kat80]. Spectral analysis of perturbed semigroups started with the paper [Vid70] by Vidav. Theorem 4.4 is due to Voigt [Voi80], [Voi94], who also gave conditions for the equality of the essential growth bounds of the original and the perturbed semigroup. For more recent results on the spectral mapping theorem for perturbed semigroups we refer to [Thi98b] and [BNP99].

# Chapter V

# Asymptotics of Semigroups

We now come to one of the most interesting aspects of semigroup theory. After having established generation, perturbation, and approximation theorems in the previous chapters, we will investigate the qualitative behavior of a given semigroup. We already dealt with this problem when we classified strongly continuous semigroups according to their regularity properties in Section II.4, but we will now concentrate on their "asymptotic" behavior. By this we mean the behavior of the semigroup $\big(T(t)\big)_{t\geq 0}$ for large $t > 0$ or, more precisely, the existence (or nonexistence) of

$$\lim_{t\to\infty} T(t),$$

where the limit will be understood in various ways and for different topologies. If we recall that the function $t \mapsto T(t)x$ yields the (mild) solutions of the corresponding abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot{x}(t) = Ax(t), & t \geq 0, \\ x(0) = x \end{cases}$$

(see Section II.6), it is evident that such results will be of utmost importance.

# 1. Stability and Hyperbolicity for Semigroups

Among the many interesting types of asymptotic behavior, we first study *stability* of strongly continuous semigroups $\big(T(t)\big)_{t\geq 0}$. By this we mean that the operators $T(t)$ should converge to *zero* as $t \to \infty$. However, as is to be expected in infinite-dimensional spaces, we have to distinguish different concepts of convergence.

## a. Stability Concepts

For a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with generator $A : D(A) \subseteq X \to X$ we make precise what we mean by

$$\text{“} \lim_{t\to\infty} T(t) = 0.\text{”}$$

A first stability concept, called *uniform exponential stability*, has already been introduced in Definition I.3.11. However, we now vary the topology and the "speed" of the convergence by proposing the following concepts.

**1.1 Definition.** *A strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$ *is called*

(a) *uniformly exponentially stable if there exists* $\varepsilon > 0$ *such that*

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t} \, \|T(t)\| = 0, \tag{1.1}$$

(b) *uniformly stable if*

$$\lim_{t\to\infty} \|T(t)\| = 0, \tag{1.2}$$

(c) *strongly stable if*

$$\lim_{t\to\infty} \|T(t)x\| = 0 \qquad \textit{for all } x \in X, \tag{1.3}$$

(d) *weakly stable if*

$$\lim_{t\to\infty} \langle T(t)x, x'\rangle = 0 \qquad \textit{for all } x \in X \textit{ and } x' \in X'. \tag{1.4}$$

We start our discussion of these concepts by noting that the two "uniform" properties coincide and are even equivalent to a "pointwise" condition.

**1.2 Proposition.** *For a strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$*, the following assertions are equivalent.*

(a) $\big(T(t)\big)_{t\geq 0}$ *is uniformly exponentially stable.*

(b) $\big(T(t)\big)_{t\geq 0}$ *is uniformly stable.*

(c) *There exists* $\varepsilon > 0$ *such that* $\lim_{t\to\infty} \mathrm{e}^{\varepsilon t} \, \|T(t)x\| = 0$ *for all* $x \in X$.

PROOF. Clearly, (a) implies (b) and (c). Since $e^{\omega_0 t} = r\big(T(t)\big) \le \|T(t)\|$ for all $t \ge 0$ (see Proposition IV.2.2), (b) implies $\omega_0 < 0$, hence (a). If (c) holds, then $\big(e^{\varepsilon t}T(t)\big)_{t\ge0}$ is strongly, hence uniformly, bounded, which implies $\lim_{t\to\infty} e^{\varepsilon/2 t}\|T(t)\| = 0$. □

It is obvious from the definition that uniform (exponential) stability implies strong stability, which again implies weak stability. The following examples show that none of the converse implications holds.

**1.3 Examples.** (i) The (left) translation semigroup $\big(T(t)\big)_{t\ge0}$ on $X := L^p(\mathbb{R}_+)$, $1 \le p < \infty$, is strongly stable, but one has

$$\|T(t)\| = 1$$

for all $t \ge 0$; hence it is not uniformly stable.

(ii) The (left) translation group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $X := L^p(\mathbb{R})$, $1 < p < \infty$, is a group of isometries, hence is not strongly stable. However, for functions $f \in X$, $g \in X' = L^q(\mathbb{R})$, $1/p + 1/q = 1$, with compact support and large $t$, one has that $T(t)f$ and $g$ have disjoint supports, whence

$$\langle T(t)f, g\rangle = \int_{-\infty}^{\infty} f(s+t)g(s)\,ds = 0.$$

For arbitrary $f \in X$, $g \in X'$ and for each $n \in \mathbb{N}$, we choose $f_n \in X$ and $g_n \in X'$ with compact support such that $\|f - f_n\|_p \le 1/n$ and $\|g - g_n\|_q \le 1/n$. Then

$$\begin{aligned} \big|\big\langle T(t)f, g\big\rangle\big| &\le \big|\big\langle T(t)(f - f_n), g_n\big\rangle\big| + \big|\big\langle T(t)f, g - g_n\big\rangle\big| + \big|\big\langle T(t)f_n, g_n\big\rangle\big| \\ &\le \tfrac{1}{n}\big(\|g\|_q + 1 + \|f\|_p\big) + \big|\big\langle T(t)f_n, g_n\big\rangle\big|. \end{aligned}$$

Since the last term is 0 for large $t$, we conclude that

$$\lim_{t\to\infty} \langle T(t)f, g\rangle = 0$$

for all $f \in X, g \in X'$, i.e., $\big(T(t)\big)_{t\ge0}$ is weakly stable.

As a brief intermezzo, we show that the above definitions do not exhaust the range of reasonable stability concepts. For example, it may happen that

$$\text{“}\, \|T(t)x\| \to 0 \quad \textit{for} \quad x \in D(A), \,\text{”}$$

i.e., for the (classical) solutions of (ACP) only, while the semigroup is not stable in the sense of Definition 1.1.

**1.4 Example.** Take the (left) translation semigroup $(T(t))_{t\geq 0}$ with generator $(A, D(A))$ on $X := \mathrm{C}_0(\mathbb{R}_+) \cap \mathrm{L}^1(\mathbb{R}_+, \mathrm{e}^x dx)$ as in Counterexample IV.2.7. There, we have shown that $\|T(t)\| = 1$ for all $t \geq 0$ and that

$$R(\lambda, A)g = \int_0^\infty \mathrm{e}^{-\lambda s} T(s)g\, ds = \lim_{t\to\infty} \int_0^t \mathrm{e}^{-\lambda s} T(s)g\, ds$$

exists for all $g \in X$ and all $\lambda > -1$. Now take $f \in D(A)$ and use identity (1.11) in Lemma II.1.9 to obtain

$$T(t)f = \mathrm{e}^{\lambda t} \left( f - \int_0^t \mathrm{e}^{-\lambda s} T(s)(\lambda - A)f\, ds \right).$$

These two identities imply that for each $\varepsilon < 1$ we have

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t}\, \|T(t)f\| = 0$$

for all $f \in D(A)$.

If we now take the rescaled semigroup $(S(t))_{t\geq 0}$ with $S(t) := \mathrm{e}^{t/2} T(t)$, we obtain $\|S(t)\| = \mathrm{e}^{t/2}$. This semigroup is unbounded, hence not weakly stable. On the other hand, it satisfies the following stability property.

**1.5 Definition.** *A strongly continuous semigroup $(T(t))_{t\geq 0}$ with the generator $(A, D(A))$ is called exponentially stable if there exists $\varepsilon > 0$ such that*

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t}\, \|T(t)x\| = 0 \tag{1.5}$$

*for all $x \in D(A)$.*

We refer to [Nee96] for more information on this and related concepts.

It is now our goal to characterize the above stability concepts, hopefully by properties of the generator. In the following subsection we try this for uniform exponential stability.

**1.6 Exercises.** (1) Discuss the above stability properties for multiplication semigroups on $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{C}_0(\mathbb{R})$. See also Examples 2.19.(ii) and (iii) below.

(2) Let $\mu$ be a probability measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure. Use the Riemann–Lebesgue lemma (see Theorem C.8) to show that the multiplication semigroup $(T(t))_{t\geq 0}$ with

$$\big(T(t)f\big)(s) := \mathrm{e}^{its} f(s), \qquad s \in \mathbb{R},$$

is weakly stable on $\mathrm{L}^p(\mathbb{R}, \mu)$ for $1 \leq p < \infty$.

(3) Show that the adjoint semigroup of a strongly stable semigroup is *weak*-stable*, that is, $\lim_{t\to\infty} \langle x, T(t)'x' \rangle = 0$ for all $x \in X$, $x' \in X'$, but not strongly stable in general.

(4) Show that an eventually compact semigroup that is weakly stable is necessarily uniformly exponentially stable.

## *b. Characterization of Uniform Exponential Stability*

We start by recalling the definition of the *growth bound*

$$
\begin{aligned}
\omega_0 : &= \omega_0(\mathcal{T}) := \omega_0(A) \\
(1.6) \qquad : &= \inf\left\{w \in \mathbb{R} : \exists\, M_w \ge 1 \text{ such that } \|T(t)\| \le M_w \mathrm{e}^{wt}\ \forall\, t \ge 0\right\} \\
&= \inf\left\{w \in \mathbb{R} : \lim_{t\to\infty} \mathrm{e}^{-wt}\, \|T(t)\| = 0\right\}
\end{aligned}
$$

of a semigroup $\mathcal{T} = \big(T(t)\big)_{t\ge 0}$ with generator $A$ (compare Definition I.5.6). From this definition it is immediately clear that $\big(T(t)\big)_{t\ge 0}$ is uniformly exponentially stable if and only if

$$
(1.7) \qquad \omega_0 < 0.
$$

Moreover, the identity

$$
(1.8) \qquad \omega_0 = \inf_{t>0} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| = \frac{1}{t_0} \log \mathrm{r}\big(T(t_0)\big)
$$

for each $t_0 > 0$, proved in Proposition IV.2.2, yields the following characterizations of uniform exponential stability (cf. Proposition I.3.12 in the case of uniformly continuous semigroups).

**1.7 Proposition.** *For a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$, the following assertions are equivalent.*

(a) $\omega_0 < 0$, *i.e., $\big(T(t)\big)_{t\ge 0}$ is uniformly exponentially stable.*
(b) $\lim_{t\to\infty} \|T(t)\| = 0$.
(c) $\|T(t_0)\| < 1$ *for some* $t_0 > 0$.
(d) $\mathrm{r}\big(T(t_1)\big) < 1$ *for some* $t_1 > 0$.

A much less obvious characterization is obtained by looking at the orbit maps $\xi_x : t \mapsto T(t)x$. Then the exponential estimate

$$
(1.9) \qquad \|T(t)x\| \le M\mathrm{e}^{-\varepsilon t}\, \|x\|
$$

for some constants $M \ge 1$, $\varepsilon > 0$ and all $x \in X$ (i.e., uniform exponential stability) implies that each $\xi_x(\cdot)$ belongs to $\mathrm{L}^p(\mathbb{R}_+, X)$ for all $1 \le p < \infty$, that is,

$$
\int_0^\infty \|T(t)x\|^p \, dt < \infty
$$

for each $x \in X$. The following theorem states that also the converse implication holds.

**1.8 Theorem.** (DATKO 1970, PAZY 1972). *A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ is uniformly exponentially stable if and only if for one/all $p \in [1,\infty)$ one has*

$$\int_0^\infty \|T(t)x\|^p \, dt < \infty \tag{1.10}$$

*for all $x \in X$.*

PROOF. If the semigroup is exponentially stable, then, as mentioned above, (1.10) is satisfied. In order to show the converse implication, it suffices by Proposition 1.7 to verify that

$$\lim_{t\to\infty} \|T(t)\| = 0. \tag{1.11}$$

To this end, we define for $n \in \mathbb{N}$ the operators $\mathcal{T}_n \in \mathcal{L}\big(X, \mathrm{L}^p(\mathbb{R}_+, X)\big)$ by

$$\mathcal{T}_n x := \mathbb{1}_{[0,n]}(\cdot)\, T(\cdot)x.$$

Then by assumption, the set $\{\mathcal{T}_n x : n \in \mathbb{N}\} \subset \mathrm{L}^p(\mathbb{R}_+, X)$ is bounded for each $x \in X$. Hence, by the uniform boundedness principle, there exists $C > 0$ such that

$$\int_0^t \|T(r)x\|^p \, dr \leq C^p \|x\|^p \qquad \text{for all } x \in X,\ t \geq 0.$$

On the other hand, by Proposition I.5.5 there exist $M \geq 1$ and $w > 0$ such that

$$\|T(t)\| \leq M\mathrm{e}^{wt} \qquad \text{for all } t \geq 0.$$

From the previous two inequalities, we obtain

$$\begin{aligned}
\frac{1-\mathrm{e}^{-pwt}}{pw} \cdot \|T(t)x\|^p &= \int_0^t \mathrm{e}^{-pwr} \|T(r)T(t-r)x\|^p \, dr \\
&\leq \int_0^t M^p \|T(t-r)x\|^p \, dr \\
&\leq M^p C^p \|x\|^p \qquad \text{for all } x \in X,\ t \geq 0.
\end{aligned}$$

Hence, there exists a constant $L > 0$ such that

$$\|T(t)\| \leq L \qquad \text{for all } t \geq 0.$$

Using this, we conclude that

$$\begin{aligned}
t\,\|T(t)x\|^p &= \int_0^t \|T(t-r)T(r)x\|^p \, dr \\
&\leq \int_0^t L^p \|T(r)x\|^p \, dr \\
&\leq L^p C^p \|x\|^p \quad \text{for all } x \in X,\ t \geq 0,
\end{aligned}$$

and therefore

$$\|T(t)\| \leq LCt^{-1/p} \qquad \text{for all } t > 0.$$

This implies (1.11) and completes the proof. □

All these stability criteria, as nice as they are, have the major disadvantage that they rely on the explicit knowledge of the semigroup $\big(T(t)\big)_{t\geq 0}$ and its orbits $t \mapsto T(t)x$. In most cases, however, only the generator (and its resolvent) is given. Therefore, direct characterizations of uniform exponential stability of the semigroup in terms of its generator are more desirable. Spectral theory provides the appropriate tool for this purpose, and the Liapunov theorem for matrix semigroups (Theorem I.2.10) and for uniformly continuous semigroups (Theorem I.3.14) are the prototypes for the results we are looking for. In particular, one hopes that the inequality

$$\mathrm{s}(A) < 0 \tag{1.12}$$

for the spectral bound $\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ of the generator $A$ (see Definition II.1.12) characterizes uniform exponential stability. Counterexample IV.2.7 (see also Counterexamples IV.3.3 and 3.4) shows that this fails drastically. The reason is the failure of the spectral mapping theorem (SMT) as discussed in Section IV.3. On the other hand, if some (weak) spectral mapping theorem holds for the semigroup $\big(T(t)\big)_{t\geq 0}$ and its generator $A$, then the growth bound $\omega_0$ and the spectral bound $\mathrm{s}(A)$ coincide, and hence the inequality (1.12) implies (1.7).

**1.9 Lemma.** *If for the strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and its generator $A$ the weak spectral mapping theorem*

$$\sigma\big(T(t)\big) \cup \{0\} = \overline{\mathrm{e}^{t\sigma(A)}} \cup \{0\} \quad \textit{for } t \geq 0 \tag{WSMT}$$

*holds, then growth bound $\omega_0$ and spectral bound $\mathrm{s}(A)$ coincide, i.e.,*

$$\mathrm{s}(A) = \omega_0\,. \tag{1.13}$$

Proof. It suffices to recall the identity (1.8) stating that

$$\omega_0 = \frac{1}{t}\log \mathrm{r}\big(T(t)\big) \quad \text{for each } t > 0.$$

Since $-\infty \leq \mathrm{s}(A) \leq \omega_0$ by Corollary II.1.13, we assume $\omega_0 > -\infty$ and obtain

$$\begin{aligned}
\omega_0 &= \frac{1}{t}\log\sup\big\{|\mu| : \mu \in \sigma\big(T(t)\big)\big\} = \frac{1}{t}\log\sup\big\{|\mathrm{e}^{t\lambda}| : \lambda \in \sigma(A)\big\} \\
&= \frac{1}{t}\log\sup\big\{\mathrm{e}^{t\operatorname{Re}\lambda} : \lambda \in \sigma(A)\big\} = \sup\Big\{\frac{1}{t}\log \mathrm{e}^{t\operatorname{Re}\lambda} : \lambda \in \sigma(A)\Big\} \\
&= \mathrm{s}(A).
\end{aligned}$$

□

The coincidence of growth and spectral bounds clearly implies that uniform exponential stability is equivalent to the negativity of the spectral bound. So in this case the inequality $\mathrm{s}(A) < 0$ yields a characterization of the long-term behavior of the semigroup $\big(T(t)\big)_{t\geq 0}$ in terms of its generator $A$ and its spectrum $\sigma(A)$. This is one reason for our thorough study of spectral mapping theorems in Section IV.3. The results obtained there, in particular Theorem IV.3.10 and its corollaries, pay off and yield the spectral bound equal growth bound condition (SBeGB) already stated in Corollary IV.3.11. We restate this as an infinite-dimensional version of Liapunov's stability theorem (cf. Theorem I.2.10 and Theorem I.3.14).

**1.10 Theorem.** *An eventually norm-continuous semigroup $\big(T(t)\big)_{t\geq 0}$ is uniformly exponentially stable if and only if the spectral bound* $\mathrm{s}(A)$ *of its generator $A$ satisfies*

$$\mathrm{s}(A) < 0.$$

Looking back at the stability results obtained so far, i.e., Proposition 1.7, Theorem 1.8, and Theorem 1.10, we observe that in each case we needed information on the semigroup itself in order to conclude its stability. This can be avoided by restricting our attention to semigroups on Hilbert spaces only.

**1.11 Theorem.** (Gearhart 1978, Prüss 1984, Greiner 1985). *A strongly continuous semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ on a Hilbert space $H$ is uniformly exponentially stable if and only if the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ is contained in the resolvent set $\rho(A)$ of the generator $A$ with the resolvent satisfying*

$$M := \sup_{\operatorname{Re}\lambda > 0} \|R(\lambda, A)\| < \infty. \tag{1.14}$$

Proof. If $\omega_0 < 0$, then the estimate (1.14) follows from Theorem II.1.10. For the converse implication, we first observe that by Corollary IV.1.14 we have $\mathrm{i}\mathbb{R} \subset \rho(A)$; hence the estimate (1.14) extends by continuity to $\operatorname{Re}\lambda \geq 0$. Next, we take $w > |\,\omega_0\,| + 1$ and consider the rescaled semigroup $\big(T_{-w}(t)\big)_{t\geq 0}$ with $T_{-w}(t) := \mathrm{e}^{-wt}T(t)$. Then, by Theorem II.1.10.(i) and for $x \in H$, $s \in \mathbb{R}$, we have

$$R(w + \mathrm{i}s, A)x = R(\mathrm{i}s, A - w)x = \int_0^\infty \mathrm{e}^{-\mathrm{i}st} T_{-w}(t)x \, dt.$$

Using the Fourier transform $\mathcal{F} : \mathrm{L}^2(\mathbb{R}, H) \to \mathrm{L}^2(\mathbb{R}, H)$ from Appendix C.b, we obtain

$$R(w + \mathrm{i}s, A)x = \mathcal{F}\big(T_{-w}(\cdot)x\big)(s), \tag{1.15}$$

where we extend $T_{-w}(\cdot)$ to $\mathbb{R}$ by setting $T_{-w}(t) := 0$ for $t < 0$. Since $(T_{-w}(t))_{t\geq 0}$ is exponentially stable, we have $T_{-w}(\cdot)x \in \mathrm{L}^2(\mathbb{R}, H)$.

It is at this point that we use the assumption that $H$ is a Hilbert space in order to conclude, from Plancherel's Theorem C.14, that

$$\int_{-\infty}^{+\infty} \|R(w+\mathrm{i}s,A)x\|^2 \, ds = 2\pi \int_0^{\infty} \|T_{-w}(t)x\|^2 \, dt \le L \cdot \|x\|^2$$

for some constant $L > 0$ and all $x \in H$. By the resolvent equation we have

$$R(\mathrm{i}s,A) = R(w+\mathrm{i}s,A) + wR(\mathrm{i}s,A)R(w+\mathrm{i}s,A)$$

for all $s \in \mathbb{R}$ and hence

$$\|R(\mathrm{i}s,A)x\| \le (1+Mw) \cdot \|R(w+\mathrm{i}s,A)x\| \tag{1.16}$$

for all $s \in \mathbb{R}$ and $x \in H$. Combining these facts, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \|R(\mathrm{i}s,A)x\|^2 \, ds &\le (1+Mw)^2 \cdot \int_{-\infty}^{+\infty} \|R(w+\mathrm{i}s,A)x\|^2 \, ds \\ &\le (1+Mw)^2 \cdot L^2 \cdot \|x\|^2\end{aligned} \tag{1.17}$$

for all $x \in H$. Since $\|T\| = \|T^*\|$ for every $T \in \mathcal{L}(H)$, by symmetry the same estimate is true for the resolvent of the generator $A^*$ of the adjoint semigroup $\big(T(t)^*\big)_{t\ge 0}$, i.e.,

$$\int_{-\infty}^{+\infty} \|R(\mathrm{i}s,A^*)y\|^2 \, ds \le (1+Mw)^2 \cdot L^2 \cdot \|y\|^2 \tag{1.18}$$

for all $y \in H$.

Next, we use the inversion formula in Corollary III.5.16 for $k = 2$ and conclude that

$$\begin{aligned}\big(tT(t)x \,|\, y\big) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{(w+\mathrm{i}s)t} \big(R(w+\mathrm{i}s,A)^2 x \,|\, y\big) \, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}st} \big(R(\mathrm{i}s,A)x \,|\, R(-\mathrm{i}s,A^*)y\big) \, ds\end{aligned}$$

for all $x \in D(A^2)$ and $y \in H$. For the second equality we used Cauchy's integral theorem, which is applicable since $R(\lambda,A)$ is uniformly bounded for $\operatorname{Re}\lambda \ge 0$ and hence

$$\|R(\lambda,A)x\| = \frac{1}{|\lambda|} \|R(\lambda,A)Ax + x\| \le \frac{1}{|\lambda|} (M\|Ax\| + \|x\|).$$

Together with (1.17), (1.18), and the Cauchy–Schwarz inequality this gives

$$\begin{aligned}\big|\big(tT(t)x \,|\, y\big)\big| &\le \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \|R(\mathrm{i}s,A)x\|^2 \, ds\right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \|R(\mathrm{i}s,A^*)y\|^2 \, ds\right)^{1/2} \\ &\le \frac{(1+Mw)^2 \cdot L^2}{2\pi} \|x\| \cdot \|y\|\end{aligned}$$

for all $x, y \in D(A^2)$. Since $D(A^2)$ is dense in $H$, this implies

$$\begin{aligned}\|tT(t)\| &= \sup\left\{\big|\big(tT(t)x \,|\, y\big)\big| : x,y \in D(A^2),\ \|x\| = \|y\| = 1\right\} \\ &\le \frac{(1+Mw)^2 \cdot L^2}{2\pi}.\end{aligned}$$

Hence $\lim_{t\to\infty} \|T(t)\| = 0$ and therefore $\omega_0(\mathcal{T}) < 0$ by Proposition 1.7. □

This stability criterion is extremely useful for the stability analysis of concrete equations; see, e.g., Theorem VI.3.18 and Theorem VI.8.35 or [Hua85]. Its theoretical significance is emphasized by the following comments.

**1.12 Comments.** (i) The theorem does not hold without the boundedness assumption on the resolvent in the right half-plane. Take the semigroup $\big(T(t)\big)_{t\geq 0}$ from Counterexample IV.3.4. Then $\big(\mathrm{e}^{-t/2}T(t)\big)_{t\geq 0}$ is a semigroup on a Hilbert space having spectral bound $\mathrm{s}(A) = -1/2$, and hence we have $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subset \rho(A)$, but its growth bound is $\omega_0 = 1/2$.

(ii) The theorem does not hold on arbitrary Banach spaces. In fact, for the semigroup in Counterexample IV.2.7 one has

$$\|R(\lambda + \mathrm{i}s, A)\| \leq \|R(\lambda, A)\|$$

for all $\lambda > \mathrm{s}(A) = -1$ and $s \in \mathbb{R}$ (use the integral representation (2.5) of the resolvent in Section IV.2). Since $\|T(t)\| = 1$ for all $t \geq 0$, this semigroup is not uniformly exponentially stable, but the resolvent of its generator exists and is uniformly bounded in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$.

**1.13 Exercises.** (1) Show that for a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $X$ with generator $A$ its growth bound is given by

$$\omega_0 = \inf\Big\{\lambda > \mathrm{s}(A) : \sup_{s\in\mathbb{R}} \|R(\lambda + \mathrm{i}s, A)\| < \infty\Big\}.$$

(2*) Extend the construction from Paragraph I.3.16 to an arbitrary strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $H$.

(i) Define $\mathfrak{U}(t)T := T(t) \cdot T \cdot T(t)^*$ for $t \geq 0$ and $T \in \mathcal{L}(H)$ and show that $(\mathfrak{U}(t))_{t\geq 0}$ is a semigroup on $\mathcal{L}(H)$ that is continuous for the weak operator topology on $\mathcal{L}(H)$.

(ii) Define $R(\lambda)T := \int_0^\infty \mathrm{e}^{-\lambda t}\mathfrak{U}(t)T\,dt$, $T \in \mathcal{L}(H)$ and $\lambda$ large, in the weak operator topology and show that $R(\lambda)$ is the resolvent of a Hille–Yosida operator $(G, D(G))$ on $\mathcal{L}(H)$.

(iii) Express $G$ by a formula analogous to (3.4) in Section I.3.

(iv) Show that the following assertions are equivalent.

(a) $(T(t))_{t\geq 0}$ is uniformly exponentially stable.

(b) $(\mathfrak{U}(t))_{t\geq 0}$ is uniformly exponentially stable.

(c) $\mathrm{s}(G) < 0$.

(d) $\int_0^\infty \mathfrak{U}(t)T\,dt$ exists for every $T \in \mathcal{L}(H)$.

(e) There exists a positive definite $R \in \mathcal{L}(H)$ such that $GR = -I$.

(Hint: See [Nag86, D-IV, Sec. 2].)

## c. Hyperbolic Decompositions

We now use the previous stability theorems in order to decompose a semigroup into a *stable* and an *unstable* part. More precisely, we try to decompose the Banach space into the direct sum of two closed subspaces such that the semigroup becomes "forward" exponentially stable on one subspace and "backward" exponentially stable on the other subspace. This has already been done for matrix semigroups (Exercise I.2.12.(5)) and for uniformly continuous semigroups (Exercise I.3.17.(4)). In the general case, however, an extra property appears in the definition.

**1.14 Definition.** *A semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called hyperbolic if $X$ can be written as a direct sum $X = X_s \oplus X_u$ of two $\big(T(t)\big)_{t\geq 0}$-invariant, closed subspaces $X_s$, $X_u$ such that the restricted semigroups $(T_s(t))_{t\geq 0}$ on $X_s$ and $(T_u(t))_{t\geq 0}$ on $X_u$ satisfy the following conditions.*

(i) *The semigroup $(T_s(t))_{t\geq 0}$ is uniformly exponentially stable on $X_s$.*

(ii) *The operators $T_u(t)$ are invertible on $X_u$, and $\big(T_u(t)^{-1}\big)_{t\geq 0}$ is uniformly exponentially stable on $X_u$.*

It is easy to see that a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ is hyperbolic if and only if there exists a projection $P$ and constants $M, \varepsilon > 0$ such that each $T(t)$ commutes with $P$, satisfies $T(t) \ker P = \ker P$, and

$$\|T(t)x\| \leq M\mathrm{e}^{-\varepsilon t}\|x\| \qquad \text{for } t \geq 0 \text{ and } x \in \operatorname{rg} P, \tag{1.19}$$

$$\|T(t)x\| \geq \frac{1}{M}\,\mathrm{e}^{+\varepsilon t}\|x\| \qquad \text{for } t \geq 0 \text{ and } x \in \ker P. \tag{1.20}$$

As in the case of uniform exponential stability, we look for a spectral characterization of hyperbolicity. Using the spectra $\sigma\big(T(t)\big)$ of the semigroup operators $T(t)$, this is easy.

**1.15 Proposition.** *For a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$, the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\geq 0}$ is hyperbolic.*

(b) *$\sigma\big(T(t)\big) \cap \Gamma = \emptyset$ for one/all $t > 0$.*

Proof. The proof of the implication (a) ⇒ (b) starts from the observation that $\sigma\big(T(t)\big) = \sigma\big(T_s(t)\big) \cup \sigma\big(T_u(t)\big)$ because of the direct sum decomposition. By assumption, $\big(T_s(t)\big)_{t\geq 0}$ is uniformly exponentially stable; hence $\mathrm{r}\big(T_s(t)\big) < 1$ for $t > 0$, and therefore $\sigma\big(T_s(t)\big) \cap \Gamma = \emptyset$.

By the same argument, we obtain that $\mathrm{r}\left(T_u(t)^{-1}\right) < 1$. Since

$$\sigma\big(T_u(t)\big) = \big\{\lambda^{-1} : \lambda \in \sigma\left(T_u(t)^{-1}\right)\big\},$$

we conclude that $|\lambda| > 1$ for each $\lambda \in \sigma\big(T_u(t)\big)$; hence $\sigma\big(T_u(t)\big) \cap \Gamma = \emptyset$.

To prove (b) $\Rightarrow$ (a), we fix $s > 0$ such that $\sigma\big(T(s)\big) \cap \Gamma = \emptyset$ and use the existence of a spectral projection $P$ corresponding to the spectral set $\big\{\lambda \in \sigma\big(T(s)\big) : |\lambda| < 1\big\}$. Then the space $X$ is the direct sum $X = X_s \oplus X_u$ of the $\big(T(t)\big)_{t\geq 0}$-invariant subspaces $X_s := \operatorname{rg} P$ and $X_u := \ker P$. The restriction $T_s(s) \in \mathcal{L}(X_s)$ of $T(s)$ in $X_s$ has spectrum

$$\sigma\big(T_s(s)\big) = \big\{\lambda \in \sigma\big(T(s)\big) : |\lambda| < 1\big\},$$

hence spectral radius $\mathrm{r}\big(T_s(s)\big) < 1$. From Proposition 1.7.(d), it follows that the semigroup $\big(T_s(t)\big)_{t\geq 0} := \big(PT(t)\big)_{t\geq 0}$ is uniformly exponentially stable on $X_s$. Similarly, the restriction $T_u(s) \in \mathcal{L}(X_u)$ of $T(s)$ in $X_u$ has spectrum

$$\sigma\big(T_u(s)\big) = \big\{\lambda \in \sigma\big(T(s)\big) : |\lambda| > 1\big\},$$

hence is invertible on $X_u$. Clearly, this implies that $T_u(t)$ is invertible for $0 \leq t \leq s$, while for $t > s$ we choose $n \in \mathbb{N}$ such that $ns > t$. Then

$$T_u(s)^n = T_u(ns) = T(ns-t)T_u(t) = T_u(t)T_u(ns-t);$$

hence $T_u(t)$ is invertible, since $T_u(s)$ is bijective. Moreover, for the spectral radius we have $\mathrm{r}\big(T_u^{-1}(s)\big) < 1$, and again by Proposition 1.7.(d) this implies uniform exponential stability for the semigroup $\big(T_u(t)^{-1}\big)_{t\geq 0}$. ☐

The reader might be disturbed by the extra condition in Definition 1.14.(ii) requiring the operators $T_u(t)$ to be invertible on $X_u$. However, this is necessary in order to obtain the spectral characterization in Proposition 1.15.

**1.16 Example.** Take the rescaled (left) shift semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathrm{L}^1(\mathbb{R}_-)$ defined by

$$T(t)f(s) := \begin{cases} e^{\varepsilon t} f(s+t) & \text{for } s+t \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in \mathrm{L}^1(\mathbb{R}_-)$, $s \leq 0$, and some fixed $\varepsilon > 0$. Then

$$\|T(t)f\| = \mathrm{e}^{\varepsilon t}\|f\|$$

for all $f \in \mathrm{L}^1(\mathbb{R}_-)$, i.e., estimate (1.20) holds for all $f \in \mathrm{L}^1(\mathbb{R}_-)$. However, the operators $T(t)$ are not invertible and have spectrum

$$\sigma\big(T(t)\big) = \big\{\lambda \in \mathbb{C} : |\lambda| \leq \mathrm{e}^{\varepsilon t}\big\}$$

for all $t > 0$.

This phenomenon is due to the fact that an injective operator on an infinite-dimensional Banach space need not be surjective. We can exclude this by assuming $\dim X_u < \infty$. See also Exercise 1.19.(2).

Up to now, our definition and characterization of hyperbolic semigroups uses explicit knowledge of the semigroup itself. As in Section 1.b, we want to find a characterization in terms of the generator $A$ and its spectrum $\sigma(A)$. As we should expect from Lemma 1.9, we need some extra relation between $\sigma(A)$ and $\sigma\big(T(t)\big)$. Clearly, the spectral mapping theorem (SMT) or even the weak spectral mapping theorem (WSMT) from Section IV.3 are sufficient for this purpose. However, we show that an even weaker property does this job.

**1.17 Theorem.** *Let $A$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\ge0}$. Assume that the spectrum $\sigma(A)$ and the spectra $\sigma\big(T(t)\big)$ satisfy*

$$(1.21)\quad \sigma\big(T(t)\big) \subset \Gamma\cdot \mathrm{e}^{t\sigma(A)} := \big\{z\mathrm{e}^{t\lambda} : \lambda\in\sigma(A),\ |z|=1\big\} \qquad \textit{for all } t\ge 0.$$

*Then the following assertions are equivalent.*

(a) $\big(T(t)\big)_{t\ge0}$ *is hyperbolic.*

(b) $\sigma\big(T(t)\big)\cap\Gamma=\emptyset$ *for one/all* $t>0$.

(c) $\sigma(A)\cap i\mathbb{R}=\emptyset$.

Proof. The equivalence of (a) and (b) has been shown in Proposition 1.15. Property (b) always implies (c) (use Theorem IV.3.6), while (c) implies (b) if (1.21) holds. □

In Hilbert spaces we can use Theorem 1.11 to replace (1.21) by a growth estimate on the resolvent $R(\lambda, A)$ for $\lambda\in \mathrm{i}\mathbb{R}$.

**1.18 Theorem.** *A strongly continuous semigroup $\big(T(t)\big)_{t\ge0}$ with generator $A$ on a Hilbert space $H$ is hyperbolic if and only if*

$$\sigma(A)\cap \mathrm{i}\mathbb{R}=\emptyset \qquad \textit{and} \qquad \|R(\lambda,A)\|\le M \quad \textit{for all } \lambda\in\mathrm{i}\mathbb{R}.$$

**1.19 Exercises.** (1) Show, by rescaling the semigroup and the estimates in (1.19) and (1.20), that a decomposition analogous to Definition 1.14 holds whenever

$$\sigma\big(T(t)\big)\cap\alpha\Gamma=\emptyset$$

for some $\alpha>0$.

(2) Let $(T(t))_{t\ge0}$ satisfy (1.19) and (1.20) for a projection $P$ commuting with $T(t)$ for all $t\ge0$. Assume that for some $t_0>0$ the restriction $T_u(t_0)$ to $\ker P$ is compact. Show that $\dim\ker P<\infty$ and that $(T(t))_{t\ge0}$ is hyperbolic.

(3) Show that the generator $A$ of a hyperbolic strongly continuous semigroup $(T(t))_{t\ge0}$ is invertible and its inverse is given by

$$A^{-1}x=\int_0^\infty T_u(t)^{-1}(I-P)x\,dt-\int_0^\infty T_s(t)Px\,dt.$$

Derive an analogous representation of $R(\lambda,A)$ for $\operatorname{Re}\lambda<\varepsilon$, where $\varepsilon$ is the constant in (1.19) and (1.20).

(4*) Given a hyperbolic semigroup $(T(t))_{t\geq 0}$ and a corresponding decomposition $X = X_s \oplus X_u$, prove that

$$X_s = \bigl\{x \in X : \lim_{t\to\infty} T(t)x = 0\bigr\}.$$

Conclude from this that $X_s$ and $X_u$ are uniquely determined.

# 2. Compact Semigroups

"Stability" of a strongly continuous semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ as defined in Definition 1.1 means that the closure of $\{T(t) : t \geq 0\}$ (for the weak, strong, or uniform operator topology) is the compact set

$$\bigl\{T(t) : t \geq 0\bigr\} \cup \{0\}.$$

This set becomes a commutative semigroup if we extend the operator multiplication by

$$T(t) \cdot 0 = 0 \cdot T(t) := 0$$

for each $t \geq 0$. We will show that even more complicated behavior of the operators $T(t)$ as $t \to \infty$ can be described by the closure of $\{T(t) : t \geq 0\}$ as well as by certain algebraic properties of it. The theory of *(semitopological) compact semigroups* provides us with an elegant and powerful tool for this investigation. We therefore start with a preparatory subsection dealing with arbitrary semigroups instead of one-parameter semigroups only.

## *a. General Semigroups*

We start with a purely algebraic setup in which $(\mathcal{S}, \cdot)$ is a *semigroup*, i.e., $\mathcal{S}$ is a set with an associative multiplication

$$\mathcal{S} \times \mathcal{S} \ni (s,t) \mapsto s \cdot t \in \mathcal{S}.$$

These semigroups become interesting to us only if endowed with an additional topological structure. (See [Rup84] or [BJM89] for a systematic treatment.)

**2.1 Definition.** *A semigroup $\mathcal{S}$ is called a semitopological semigroup if $\mathcal{S}$ has a topology for which the multiplication is separately continuous on $\mathcal{S}$, i.e., such that the maps*

$$s \mapsto ts \qquad \text{and} \qquad s \mapsto st$$

*are continuous on $\mathcal{S}$ for each $t \in \mathcal{S}$. Compact semigroups* are semitopological semigroups that are compact.

In our application of this notion to operator semigroups we will use the weak operator topology. It is therefore important to require only *separate* continuity of the multiplication (see Example 2.11.(i) and Proposition A.6). Fortunately, this property is strong enough to yield a powerful structure theorem. In order to develop this theory, we recall that an *ideal* in a semigroup $\mathcal{S}$ is a nonempty subset $\mathcal{J}$ such that

$$\mathcal{S}\mathcal{J} \cup \mathcal{J}\mathcal{S} := \{st : s \in \mathcal{S}, t \in \mathcal{J}\} \cup \{tr : t \in \mathcal{J}, r \in \mathcal{S}\} \subset \mathcal{J}.$$

In addition, we state the following elementary properties.

**2.2 Lemma.** *Let $\mathcal{S}$ be a semitopological semigroup and consider a subsemigroup $\mathcal{H}$ in $\mathcal{S}$. Then $\overline{\mathcal{H}}$ is a subsemigroup in $\mathcal{S}$. If, in addition, $\mathcal{H}$ is commutative, then $\overline{\mathcal{H}}$ is commutative as well.*

The proof is left to the reader (cf. Exercise 2.6.(1)), and we now state the main structure theorem for commutative compact semigroups.

**2.3 Theorem.** *Every commutative compact semigroup $\mathcal{S}$ contains a unique minimal ideal $\mathcal{K}$ that is a compact group and is obtained as $\mathcal{K} = q\mathcal{S}$ for the unit element $q \in \mathcal{K}$.*

Proof. Let $\mathcal{S}$ be a commutative compact semigroup and choose finitely many closed ideals $\mathcal{J}_1, \ldots, \mathcal{J}_n$ in $\mathcal{S}$. Then

$$\bigcap_{i=1}^{n} \mathcal{J}_i \supset \mathcal{J}_1\mathcal{J}_2 \cdots \mathcal{J}_n \neq \emptyset,$$

which shows that the family of all closed ideals in $\mathcal{S}$ has the finite intersection property. By the compactness of $\mathcal{S}$, we conclude that

$$\mathcal{K} := \bigcap \{\mathcal{J} : \mathcal{J} \text{ closed ideal in } \mathcal{S}\} \tag{2.1}$$

is a nonempty, closed ideal.

If $\mathcal{J}$ is an arbitrary ideal in $\mathcal{S}$, we take $t \in \mathcal{J}$. The separate continuity of the multiplication implies that $t\mathcal{S}$ is a closed ideal. Since $t\mathcal{S} \subset \mathcal{J}$, we obtain $\mathcal{K} \subset t\mathcal{S} \subset \mathcal{J}$, showing $\mathcal{K}$ to be minimal and unique. In fact, it follows that

$$\mathcal{K} = \bigcap_{t \in \mathcal{S}} t\mathcal{S}. \tag{2.2}$$

We now show that $\mathcal{K}$ is a group. Observe first that $s\mathcal{K} = \mathcal{K}$ for each $s \in \mathcal{K}$, since $\mathcal{K}$ is minimal. Hence, for fixed $s \in \mathcal{K}$, there exists $q \in \mathcal{K}$ such that $sq = s$, and for every $r \in \mathcal{K}$, we find $r' \in \mathcal{K}$ such that $r's = r$. This implies

$$rq = r'sq = r's = r,$$

i.e., $q$ is the unit element in $\mathcal{K}$. Again from $r\mathcal{K} = \mathcal{K}$ we infer the existence of $t\,(= r^{-1})$ such that $rt = q$; hence $\mathcal{K}$ is a group. Finally, since $\mathcal{K}$ is a closed subset of a compact set, it is compact itself.

The same arguments yield the inclusions

$$\mathcal{K} = q\mathcal{K} \subset q\mathcal{S} \subset \mathcal{K}\mathcal{S} \subset \mathcal{K},$$

and hence

$$\mathcal{K} = q\mathcal{S} \tag{2.3}$$

for the (unique) unit element $q$ of the compact group $\mathcal{K}$. □

The attentive reader may have noticed an apparent inconsistency of the above theorem with the usual terminology, where topological groups are required to have jointly continuous multiplication (e.g., [HR63, Def. II.4.1]). In fact, up to now we only have proved that the group $\mathcal{K}$ has a topology making it a compact space such that the multiplication is separately continuous. We close this gap by quoting the following theorem by Ellis (see [Nam74]).

**2.4 Theorem.** *A compact semitopological group $\mathcal{K}$ is a topological group, i.e., the mappings*

$$\mathcal{K} \times \mathcal{K} \ni (s,t) \mapsto s \cdot t \in \mathcal{K} \qquad \textit{and} \qquad \mathcal{K} \ni s \mapsto s^{-1} \in \mathcal{K}$$

*are continuous.*

We now stop investigating general semigroups and will apply Theorem 2.3 to compact semigroups $(\mathcal{S}, \cdot)$ containing a dense one-parameter subsemigroup $(\alpha_t)_{t\geq 0}$, i.e., $\mathbb{R}_+ \ni t \mapsto \alpha_t \in \mathcal{S}$ is a continuous semigroup homomorphism from $(\mathbb{R}_+, +)$ into $(\mathcal{S}, \cdot)$. These semigroups can have quite complicated structure (e.g., they can contain uncountably many idempotents; cf. [BM71] or [Rup84, App. 4.2]). If, however, the multiplication is jointly continuous, it is easy to determine the minimal ideal and to visualize some examples.

**2.5 Example.** Let $(\mathcal{S}, \cdot)$ be a compact semigroup with jointly continuous multiplication containing a dense one-parameter subsemigroup $(\alpha_t)_{t\geq 0}$. Then the minimal ideal $\mathcal{K}$ can be obtained as

$$\mathcal{K} = \bigcap_{t\geq 0} \overline{\{\alpha_s : s \geq t\}}. \tag{2.4}$$

We leave the proof as Exercise 2.6.(2) and instead visualize some concrete examples in which the minimal ideal becomes

(i) $\mathcal{K} = \{\infty\}$ (Figure 4),
(ii) $\mathcal{K} = \Gamma$ (Figure 5), and
(iii) $\mathcal{K} = \Gamma^2$ (Figure 6).

In each case, the semigroup operation is inherited from the addition on $\mathbb{R}_+$ and should become clear from the picture.

**Figure 4**

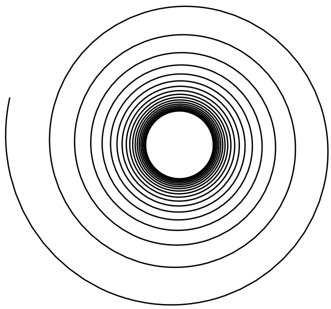


**Figure 5**

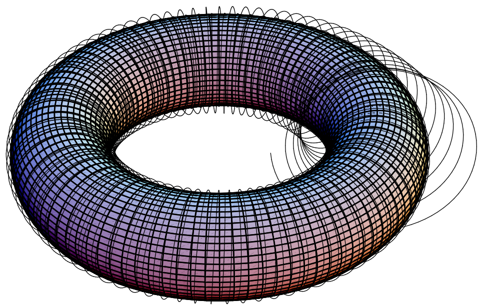


**Figure 6**

Before concluding this subsection, we point out a confusion possibly caused by the term "compact semigroup" of operators. It means a compactness property of the set of operators $\big(T(t)\big)_{t\geq 0}$ and not, as in Definition II.4.23 for "eventually compact semigroups," a compactness property of a single operator $T(t_0)$. See also the introduction to Section 3 below.

**2.6 Exercises.** (1) Prove Lemma 2.2. (Hint: To prove the commutativity of $\overline{\mathcal{H}}$ show first that $st = ts$ for $s \in \mathcal{H}$ and $t \in \overline{\mathcal{H}}$.)

(2) Prove identity (2.4) in Example 2.5. Then modify the construction from [Wes68] to show that (2.4) does not hold if the multiplication in $\mathcal{S}$ is only separately continuous.

## b. Weakly Compact Semigroups

From now on, $\big(T(t)\big)_{t\geq 0}$ is again a strongly continuous semigroup of bounded operators on a Banach space $X$. In order to apply the abstract semigroup theory above, we need compactness for $\{T(t) : t \geq 0\}$ as a subset of $\mathcal{L}(X)$. This can be achieved in various ways by considering the weak operator, strong operator, or uniform operator topology on $\mathcal{L}(X)$. We start with the weak operator topology, being the weakest among these three natural operator topologies, hence yielding more compact sets.

From Proposition A.4 and Corollary A.5 we recall that relative compactness of the set $\{T(t) : t \geq 0\}$ in the space $\mathcal{L}_\sigma(X)$ can be characterized as follows.

**2.7 Lemma.** *For $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$, $X$ a Banach space, the following assertions are equivalent.*

(a) *$\{T(t)x : t \geq 0\}$ is relatively weakly compact for all $x \in X$.*

(b) *$\big(T(t)\big)_{t\geq 0}$ is bounded, and $\{T(t)x : t \geq 0\}$ is relatively weakly compact for all $x$ in some dense subset of $X$.*

(c) *$\{T(t) : t \geq 0\}$ is relatively compact in $\mathcal{L}_\sigma(X)$.*

As another preparation for the application of Theorem 2.3 to operator semigroups, we observe that the multiplication in the algebra $\mathcal{L}_\sigma(X)$ is separately continuous (but not jointly continuous; see Example 2.11.(ii) below). Hence $(\mathcal{L}_\sigma(X), \cdot)$ is a semitopological semigroup in the sense of Definition 2.1. Therefore, $\big(T(t)\big)_{t\geq 0}$ as well as its closure in $\mathcal{L}_\sigma(X)$ (use Lemma 2.2) are commutative semitopological semigroups.

If now the semigroup $\big(T(t)\big)_{t\geq 0}$ satisfies the relative compactness conditions in Lemma 2.7, then its closure

$$\mathcal{S} := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)} \tag{2.5}$$

is a commutative compact semigroup in $\mathcal{L}_\sigma(X)$. Therefore, Theorem 2.3 can be applied, and the minimal ideal $\mathcal{K}$ in $\mathcal{S}$ is a commutative compact group. This opens the door for the application of powerful tools from harmonic analysis (see [HR63, Chap. VI]).

The unit $Q$ in $\mathcal{K}$ satisfies $Q = Q^2$, and hence is a projection commuting with all the operators in $\mathcal{S}$. It therefore induces a decomposition

$$X := \operatorname{rg} Q \oplus \ker Q$$

of $X$ into $\big(T(t)\big)_{t\geq 0}$-invariant subspaces $\operatorname{rg} Q$ and $\ker Q$. As the main feature of this approach we are now able to characterize these two spaces in terms of the action of the semigroup $\big(T(t)\big)_{t\geq 0}$.

**2.8 Theorem.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ on the Banach space $X$. If $\{T(t) : t \geq 0\}$ is relatively weakly compact, then there exists a projection $Q \in \mathcal{L}(X)$ commuting with $(T(t))_{t\geq 0}$ such that*

(i) $\ker Q = X_s := \left\{x \in X : 0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}\right\}$

(ii) $\operatorname{rg} Q = X_r := \overline{\operatorname{lin}}\left\{x \in D(A) : \exists\, \alpha \in \mathbb{R} \text{ such that } Ax = \mathrm{i}\alpha x\right\}$.

Proof. As already explained above, we take as projection $Q$ the unit element of the minimal ideal $\mathcal{K}$ in the commutative compact semigroup $\mathcal{S} := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)} \subset \mathcal{L}_\sigma(X)$. We have to prove the assertions on $\ker Q$ and $\operatorname{rg} Q$.

(i) Take $x \in X$ such that $Qx = 0$. Since the map

$$\mathcal{L}_\sigma(X) \ni T \mapsto Tx \in \big(X, \sigma(X, X')\big)$$

is continuous and since $Q \in \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$, we obtain that $0 = Qx \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}$. Conversely, assume that 0 is a weak accumulation point of the orbit $T(\cdot)x$, i.e., $0 \in \overline{\{T(t)x : t \geq 0\}}^{\sigma}$. By the compactness of $\mathcal{S}$, we find $R \in \mathcal{S}$ satisfying $Rx = 0$. This implies $R'QRx = 0$ for each $R' \in \mathcal{S}$. If we choose $R'$ to be the inverse of $QR$ in the group $\mathcal{K} = Q\mathcal{S}$, we obtain $Qx = 0$.

(ii) We first show that $\operatorname{rg} Q \subset X_r$. To that purpose consider the compact group $\mathcal{K} = Q\mathcal{S}$ and take its *character group* $\widehat{\mathcal{K}}$ consisting of all continuous group homomorphisms $\chi : \mathcal{K} \to \Gamma$ (cf. [HR63, §23]). For each character $\chi \in \widehat{\mathcal{K}}$ we define an operator $P_\chi \in \mathcal{L}(X)$ by

$$P_\chi x := \int_{\mathcal{K}} \overline{\chi(S)} Sx \, dm(S) \qquad \text{for } x \in X.$$

Here, $m$ is the normalized Haar measure on $\mathcal{K}$, and the integral is understood in the weak sense. It follows from the weak compactness of $\mathcal{S}x$ in $X$ (and of its closed convex hull, use Proposition A.1.(ii)) that $P_\chi$ is a well-defined operator from $X$ into $X$. (For the details we refer to [Rud73, Thm. 3.27].)

For $R \in \mathcal{K}$, we obtain

$$\begin{aligned}\langle RP_\chi x, x'\rangle &= \left\langle R\Big(\int_{\mathcal{K}} \overline{\chi(S)} Sx \, dm(S)\Big), x' \right\rangle = \int_{\mathcal{K}} \overline{\chi(S)} \, \langle RSx, x'\rangle \; dm(S) \\ &= \chi(R) \int_{\mathcal{K}} \overline{\chi(RS)} \, \langle RSx, x'\rangle \; dm(S) = \langle \chi(R) P_\chi x, x'\rangle\end{aligned}$$

for every $x \in X$ and $x' \in X'$. This implies

$$QP_\chi = P_\chi$$

and, taking $R := T(t)Q$,

$$T(t)P_\chi = T(t)QP_\chi = \chi\big(T(t)Q\big)P_\chi \qquad \text{for each } t \geq 0.$$

Since $\mathbb{R}_+ \ni t \mapsto \chi\big(T(t)Q\big) \in \Gamma$ is continuous and satisfies (FE) from Section I.1, we find $\alpha \in \mathbb{R}$ such that

$$T(t)P_\chi = \mathrm{e}^{\mathrm{i}\alpha t}P_\chi \qquad \text{for } t \geq 0.$$

By taking the derivative at $t = 0$, we obtain

$$AP_\chi = \mathrm{i}\alpha P_\chi, \qquad \text{hence} \qquad P_\chi X \subset X_r.$$

The assertion will now be proved if we show that

$$\operatorname{rg} Q \subset \overline{\operatorname{lin}} \bigcup_{\chi \in \widehat{\mathcal{K}}} P_\chi X.$$

Take $x' \in X'$ vanishing on $P_\chi X$ for each $\chi \in \widehat{\mathcal{K}}$, i.e., such that

$$\int_{\mathcal{K}} \overline{\chi(S)}\,\langle Sx, x'\rangle \; dm(S) = 0$$

for all $\chi \in \widehat{\mathcal{K}}$ and all $x \in X$. Since the mapping

$$\mathcal{K} \ni S \mapsto \langle Sx, x'\rangle \in \mathbb{C}$$

is continuous and since the characters form a total set in $\mathrm{L}^2(\mathcal{K}, m)$ (see [HR63, Thm. 22.17]), this implies $\langle Sx, x'\rangle = 0$ for all $S \in \mathcal{K}$. For $S = Q$ this shows that $x'$ vanishes on $\operatorname{rg} Q$, and the above inclusion is proved.

In a second step, we show that $X_r \subset \operatorname{rg} Q$. This is proved if $Q$, the unit element in $\mathcal{K}$, acts as the identity operator on $X_r$.

Take $x \in D(A)$ such that $Ax = \mathrm{i}\alpha x$ for some $\alpha \in \mathbb{R}$. By the Spectral Inclusion Theorem IV.3.6, it follows that $x$ is also an eigenvector for each $T(t)$ and hence for each $R \in \mathcal{S}$. In addition, the corresponding eigenvalues belong to $\Gamma$. Therefore, there exists $\lambda_Q \in \Gamma$ such that $Qx = \lambda_Q x$. Since $Q$ is a projection, this implies $\lambda_Q = 1$ and hence $Qx = x$. For the linear and continuous operator $Q$ we obtain $Qy = y$ for all $y \in X_r$, which completes the proof. ☐

Theorem 2.8, which is a special case of the *Jacobs–DeLeeuw–Glicksberg splitting theorem* (see [Kre85, Sec. II.4]), gives us the following description of the action of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ whenever the compactness condition from Lemma 2.7 holds.

If $\mathcal{S} := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$ is weakly compact, then one has a decomposition $X = X_s \oplus X_r$ such that for $x = x_s + x_r$

(i) the orbit $t \mapsto T(t)x_s$ is *stable* in the sense that it has 0 as a weak accumulation point, and

(ii) the orbit $t \mapsto T(t)x_r$ is *reversible* in the sense that for every $T(t)$ there exists $R \in \mathcal{S}$ such that $RT(t)x_1 = x_1$.

This means that each orbit approaches a reversible orbit as $t \to \infty$. In the following corollary we improve our information about this reversible part.

**2.9 Corollary.** *Let $(T(t))_{t \geq 0}$ be a bounded strongly continuous semigroup with generator $A$ on the Banach space $X$. The following assertions are equivalent.*

(a) *The eigenvectors of $A$ corresponding to imaginary eigenvalues span a dense subspace in $X$.*

(b) *The weak operator closure $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$ is a compact group with identity $I_X$.*

(c) *The strong operator closure $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_s(X)}$ is a compact group with identity $I_X$.*

*In particular, each of the above conditions implies that*

$$\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)} = \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_s(X)}.$$

Proof. The implication (c) $\Rightarrow$ (b) is obvious. Moreover, the compactness in (c) implies that the weak and the strong operator topologies coincide on $\{T(t) : t \geq 0\}$; hence the equality of $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$ and $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_s(X)}$ follows.

The equivalence (a) $\iff$ (b) follows from Theorem 2.8 for $X = X_r$ and from the second statement in Lemma 2.7. Finally, we observe that the sets $\{T(t)z : t \geq 0\} = \{e^{\lambda t}z : t \geq 0\}$ are norm-compact for each eigenvector $z$ of $A$ corresponding to an imaginary eigenvalue $\lambda$. Since the set of all $x \in X$ with relatively norm compact orbits is closed by Proposition A.4, we see that (b) in combination with Theorem 2.8.(ii) implies that $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$ is compact even for the strong operator topology. This shows (b) $\Rightarrow$ (c). □

The following corollary gives the most common situation to which the above results apply.

**2.10 Corollary.** *A bounded strongly continuous semigroup on a reflexive Banach space is relatively weakly compact, and therefore the decomposition from Theorem 2.8 is possible.*

**2.11 Examples.** (i) We consider the shift group $\big(T(t)\big)_{t\in\mathbb{R}}$ on $X := \mathrm{L}^p(\mathbb{R})$ for $1 \le p < \infty$. In the reflexive case, i.e., for $1 < p < \infty$, we can apply Corollary 2.10 and obtain $X_s = X$, $X_r = \{0\}$. In fact, one even has

$$\lim_{t\to\infty} T(t) = 0$$

for the weak operator topology (see Example 1.3.(ii)). So, we see that the weak operator closure of a group does not need to be a group anymore. In addition, since

$$I = \text{weak-}\lim_{t\to\infty} \big(T(t)T(-t)\big) \ne \text{weak-}\lim_{t\to\infty} T(t) \cdot \text{weak-}\lim_{t\to\infty} T(-t) = 0,$$

we note from this example that the multiplication on $\mathcal{L}_\sigma(X)$ is not jointly continuous.

Finally, the translation group on $X := \mathrm{L}^1(\mathbb{R})$ is not relatively weakly compact, since $X_r = \{0\}$ (use Theorem 2.8.(ii)), but $\langle T(t)f, \mathbb{1}\rangle = \langle f, \mathbb{1}\rangle \ne 0$ for each $0 < f \in \mathrm{L}^1(\mathbb{R})$. Therefore, 0 is not a weak accumulation point of $\{T(t)f : t \ge 0\}$ for $0 < f \in \mathrm{L}^1(\mathbb{R})$ and $X \ne X_s$.

(ii) In this example we show that a relatively weakly compact semigroup with $X_r = \{0\}$ need not be weakly stable in the sense of Definition 1.1.(d).

By [Rud62, Chap. 5], there exists a Cantor set $\Sigma$ in the unit circle $\Gamma$ that is a *Kronecker set*, i.e., for which there exists a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty} \sup_{s\in\Sigma} |s^{n_k} - 1| = 0$$

and that supports a diffuse probability measure $\mu$. On the Hilbert space $X := \mathrm{L}^2(\Gamma, \mu)$ we take the unitary group $\big(T(t)\big)_{t\in\mathbb{R}}$ given by

$$\big(T(t)f\big)(s) := s^t f(s)$$

for $f \in X$, $s \in \Gamma$. This group is relatively weakly compact, since it is bounded and $X$ is reflexive. Moreover, $X_r = \{0\}$, since $\mu$ is diffuse; hence $T(t)$ has no eigenvalues (see Exercise I.4.13.(7)). By Theorem 2.8 this implies $0 \in \overline{\{T(t) : t \ge 0\}}^{\mathcal{L}_\sigma(X)}$.

On the other hand, we have

$$\begin{aligned}
\big|\big((T(n_k) - I)f \,|\, g\big)\big| &= \left| \int_\Gamma (s^{n_k} - 1) f(s) \overline{g(s)}\, d\mu(s) \right| \\
&\le \int_\Sigma |s^{n_k} - 1| \cdot |f(s)\overline{g(s)}|\, d\mu(s) \\
&\le \sup_{s\in\Sigma} |s^{n_k} - 1| \cdot \|f\| \cdot \|g\|,
\end{aligned}$$

which tends to zero as $k \to \infty$ for every $f, g \in X$. This proves $I \in \overline{\{T(t) : t \ge 0\}}^{\mathcal{L}_\sigma(X)}$, and hence $\big(T(t)\big)_{t\ge 0}$ is not weakly stable.

**2.12 Exercises.** (1) Show that Corollary 2.9 does not hold without assuming $(T(t))_{t\geq 0}$ to be bounded. (Hint: For each $k \in \mathbb{N}$, consider the space $X_k := \mathrm{C}_{[0,k]}(\mathbb{R})$ of all $k$-periodic continuous functions on $\mathbb{R}$ with the norm

$$\|f\|_k := \sup_{0\leq s\leq k} (1+s)^{-1}|f(s)|,$$

which is equivalent to the usual sup-norm. On $X_k$, the (left) translation semigroup $(T_k(t))_{t\geq 0}$ is $k$-periodic, Theorem IV.2.26 holds, and $\|T_k(t)\| = 1+t$ for $0 \leq t \leq k$. If we define $X := \oplus^2_{k\in\mathbb{N}} X_k$ and $T(t) := \oplus_{k\in\mathbb{N}} T_k(t)$, we obtain an unbounded semigroup satisfying only condition (a) in Corollary 2.9.)

(2) Let $\mathcal{T}$ be one of the left translation (semi) groups from Section I.4.c.

(i) Show that on $X := \mathrm{C_{ub}}(\mathbb{R})$ the maximal subspace $Y$ of $X$ on which $\mathcal{T}$ becomes a relatively strongly compact group is the space of *almost periodic functions*, i.e.,

$$Y = \overline{\mathrm{lin}}\,\{\varepsilon_{\mathrm{i}\lambda} : \lambda \in \mathbb{R}\}$$

with $\varepsilon_{\mathrm{i}\lambda}(s) := \mathrm{e}^{\mathrm{i}\lambda s}$, $s \in \mathbb{R}$.

(ii) Characterize $Y := \overline{\mathrm{lin}}\,\{\varepsilon_{\mathrm{i}\lambda} : \lambda \in \mathbb{R}\} \oplus \mathrm{C}_0(\mathbb{R}_+)$ as the maximal subspace of $X := \mathrm{C_{ub}}(\mathbb{R}_+)$ on which $\mathcal{T}$ becomes a relatively strongly compact semigroup.

(iii*) Study the same problem for the weak compactness of $\mathcal{T}$. (Hint: See [RS89] and [RS92].)

## *c. Strongly Compact Semigroups*

We now turn our attention to semigroups that are relatively compact with respect to the strong operator topology. For the sake of completeness, we recall from Proposition A.4 and Corollary A.5 the analogue of Lemma 2.7 characterizing strong operator compactness.

**2.13 Lemma.** *For $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$, $X$ a Banach space, the following assertions are equivalent.*

(a) *$\{T(t)x : t \geq 0\}$ is relatively compact for all $x \in X$.*

(b) *$\big(T(t)\big)_{t\geq 0}$ is bounded and $\{T(t)x : t \geq 0\}$ is relatively compact for all $x$ in some dense subset of $X$.*

(c) *$\{T(t) : t \geq 0\}$ is relatively compact in $\mathcal{L}_s(X)$.*

Moreover, we observe that the multiplication is separately continuous in $\mathcal{L}_s(X)$ (and even jointly continuous on bounded subsets; see Proposition A.6). Therefore, and since weak and strong operator topologies coincide on strongly compact sets, we can again apply the theory of compact semigroups from Section 2.a and Theorem 2.8 from Section 2.b. This yields a decomposition $X = X_s \oplus X_r$ with a very simple and nice description of $X_s$ as the "stable" subspace. For a change, we formulate this result as an equivalence.

**2.14 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $\big(A, D(A)\big)$ on the Banach space $X$. Then the following assertions are equivalent.*

(a) *$\{T(t) : t \geq 0\}$ is relatively compact in $\mathcal{L}_s(X)$.*

(b) *There exists a projection $Q \in \mathcal{L}(X)$ commuting with $\big(T(t)\big)_{t\geq 0}$ such that*

  (i) $\ker Q = X_s := \{x \in X : \lim_{t\to\infty} T(t)x = 0\}$.

  (ii) $\operatorname{rg} Q = X_r := \overline{\operatorname{lin}}\{x \in D(A) : \exists\, \alpha \in \mathbb{R} \text{ such that } Ax = \mathrm{i}\alpha x\}$.

Proof. Implication (b) ⇒ (a) follows from condition (b) in Lemma 2.13. The converse implication (a) ⇒ (b) has been proved in Theorem 2.8 except for the description of $\ker Q$ as $X_s = \{x \in X : \lim_{t\to\infty} T(t)x = 0\}$. However, we know from Theorem 2.8 that for $x \in \ker Q$ one has 0 as a weak accumulation point of $\{T(t)x : t \geq 0\}$. Since by condition (a) the closures of $\{T(t) : t \geq 0\}$ coincide for the weak and strong operator topologies, there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} T(t_n)x = 0$. From the boundedness of $\big(T(t)\big)_{t\geq 0}$ we conclude that

$$\lim_{t\to\infty} T(t)x = 0,$$

i.e., $x \in X_s$. □

We recall that Theorem 2.8 holds for every bounded semigroup on every reflexive Banach space. In order to obtain the stronger properties from Theorem 2.14, we have to make more restrictive assumptions.

**2.15 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $\big(A, D(A)\big)$ on a Banach space $X$. Each of the conditions*

(i) *$A$ has compact resolvent,*

(ii) *$\big(T(t)\big)_{t\geq 0}$ is eventually compact*

*implies that $\big(T(t)\big)_{t\geq 0}$ is relatively compact in $\mathcal{L}_s(X)$ and hence that the decomposition from Theorem 2.14.(b) holds.*

Proof. Let $\|T(t)\| \leq M$ for all $t \geq 0$. In case (i), we have that

$$V := R(\lambda_0, A)U$$

is relatively compact for $\lambda_0 > 0$ and $U := \{x \in X : \|x\| \leq 1\}$. By Lemma 2.13.(b), it suffices to show that $\{T(t)x : t \geq 0\}$ is relatively compact for $x := R(\lambda_0, A)y \in D(A)$. This follows, since

$$T(t)x = R(\lambda_0, A)T(t)y \in M\,\|y\|\,V$$

for all $t \geq 0$. In case (ii) and for $T(t_0)$ compact, we observe that

$$\{T(t)x : t \geq 0\} = \{T(t)x : 0 \leq t \leq t_0\} \cup \{T(t_0)T(s)x : s \geq 0\}$$

is the union of two relatively compact sets, hence relatively compact itself. □

The examples in II.4.27 show that the conditions (i) and (ii) are indeed independent.

The theoretical and practical importance of Theorem 2.14, and of the space and semigroup decomposition it permits, is enormous. In particular, it indicates how we should proceed towards a complete understanding of the action of such semigroups.

**2.16 Problem.** *In order to study a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ that is relatively strongly compact, it suffices to restrict it to its "reversible" part $X_r$ and its "stable" part $X_s$, according to Theorem 2.14. Therefore, the following two questions arise.*

(i) *Characterize all strongly continuous semigroups $\big(T(t)\big)_{t\geq 0}$ on $X$ such that $\mathcal{S} := \overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_s(X)} \subset \mathcal{L}_s(X)$ becomes a compact group, i.e., for which, by Corollary 2.9 we have $X = X_r$.*

(ii) *Characterize all strongly continuous semigroups $\big(T(t)\big)_{t\geq 0}$ on $X$ that are strongly stable, i.e., for which $X = X_s$.*

There are many interesting answers to these questions, and we will present some of them. However, in each case we start with a series of examples.

**2.17 Examples.** (*Compact Operator Groups*). (i) Take $X$ to be one of the sequence spaces $\ell^p$, $1 \leq p < \infty$, or $c_0$. For every sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, the multiplication operator

$$A(x_n)_{n\in\mathbb{N}} := (\mathrm{i}\alpha_n x_n)_{n\in\mathbb{N}}$$

with maximal domain (compare Example I.4.7.(iii)) generates a group of isometries on $X$. Since each canonical basis vector is an eigenvector of $A$ with eigenvalue $\mathrm{i}\alpha_n$, it follows from Corollary 2.9 that the strong operator closure of the multiplication semigroup $\big(T(t)\big)_{t\geq 0}$ with

$$T(t)(x_n)_{n\in\mathbb{N}} := (\mathrm{e}^{\mathrm{i}\alpha_n t} x_n)_{n\in\mathbb{N}}, \quad t \geq 0,$$

is a strongly compact group. By Exercise I.4.8.(1), one has

$$P\sigma(A) = \{\mathrm{i}\alpha_n : n \in \mathbb{N}\} \qquad \text{and} \qquad \sigma(A) = \overline{P\sigma(A)}.$$

In particular, $\sigma(A)$ can be any nonempty closed subset of $\mathrm{i}\mathbb{R}$.

(ii) Our next example strongly relies on the theory of locally compact abelian groups (see [HR63]) and may be skipped by the reader not familiar with this theory.

Let $(G, \cdot)$ be a compact group (with multiplication $\cdot$) that is assumed to be *solenoidal*, i.e., there exists a dense one-parameter subgroup $H = (\alpha_t)_{t\in\mathbb{R}}$ in $G$ (see [HR63, Def. 9.2]). In other words, $G$ is a group compactification of (a continuous image of) the group $(\mathbb{R}, +)$ and therefore abelian. We then take the Banach space $X := \mathrm{C}(G)$ and define the corresponding *rotation group* $\big(T(t)\big)_{t\in\mathbb{R}}$ on $X$ by

$$T(t)f(\gamma) := f(\alpha_t \cdot \gamma) \qquad \text{for } \gamma \in G, t \in \mathbb{R} \text{ and } f \in X.$$

The operators $T(t)$ are isometries on $X$, satisfy

$$T(t+s)f(\gamma) = f(\alpha_{t+s} \cdot \gamma) = f(\alpha_t \cdot \alpha_s \cdot \gamma) = \big(T(t)(T(s)f)\big)(\gamma)$$

for $\gamma \in G$ and $s, t \in \mathbb{R}$, and therefore form a one-parameter operator group.

We prove the strong continuity of $\big(T(t)\big)_{t\in\mathbb{R}}$ using the *character group* $\widehat{G}$ consisting of all continuous homomorphisms $\chi$ from $G$ into the unit circle $\Gamma$. For each $\chi \in \widehat{G}$, one obtains

$$T(t)\chi(\gamma) = \chi(\alpha_t \cdot \gamma) = \chi(\alpha_t) \cdot \chi(\gamma) \qquad \text{for all } \gamma \in G,$$

i.e., $\chi$ is an eigenvector with eigenvalue $\chi(\alpha_t) \in \Gamma$ of $T(t)$. It follows that $t \mapsto T(t)\chi$ is continuous and that $\{T(t)\chi : t \geq 0\}$ is a bounded subset of a one-dimensional subspace, hence relatively compact in $X$. Since the character group $\widehat{G}$ spans a dense subspace in $\mathrm{C}(G)$ (see [HR63, Thm. 22.17]), we conclude from Proposition I.5.3, Corollary 2.9, and Lemma 2.13.(b) that $\big(T(t)\big)_{t\geq 0}$ is strongly continuous and that its strong operator closure $\mathcal{S}$ is a compact group.

In fact, the operator group $\mathcal{S} := \overline{\{T(t) : t \in \mathbb{R}\}}^{\mathcal{L}_s(X)} \subset \mathcal{L}_s(X)$ is isomorphic to the group $G$ we started with. This can be seen using the map

$$G \ni \gamma \mapsto T_\gamma \in \mathcal{L}(X)$$

with

$$T_\gamma f(\sigma) := f(\gamma \cdot \sigma) \qquad \text{for all } \sigma \in G, f \in X.$$

This is a continuous group isomorphism onto $\mathcal{S}$, since $\gamma_i \to \gamma$ implies $T_{\gamma_i}\chi \to T_\gamma\chi$ for all characters $\chi \in \widehat{G}$; hence $T_{\gamma_i} \to T_\gamma$ for the strong operator topology (use Proposition A.3).

In the next step we identify the character group $\widehat{G}$, which, by the general theory of locally compact abelian groups, must be a discrete group ([HR63, Thm. 23.17]) with the point spectrum $P\sigma(A)$ of the generator $A$ of $\big(T(t)\big)_{t\in\mathbb{R}}$. Recall that $P\sigma(A)$ is contained in $\mathrm{i}\mathbb{R}$, since $\big(T(t)\big)_{t\in\mathbb{R}}$ is an isometric group.

By continuity, every character $\chi$ is uniquely determined by its values on the dense subgroup $(\alpha_t : t \in \mathbb{R})$ and yields a continuous homomorphism

$$\mathbb{R} \ni t \mapsto \chi(\alpha_t) \in \Gamma,$$

which, by Theorem I.1.4, is of the form

$$t \mapsto \mathrm{e}^{\mathrm{i}\beta_\chi t}$$

for some $\beta_\chi \in \mathbb{R}$. Recalling the definition of $\big(T(t)\big)_{t\geq 0}$, we obtain for its generator $\big(A, D(A)\big)$ that $\chi$ is an eigenvector of $A$ such that

$$A\chi = \mathrm{i}\beta_\chi \chi,$$

i.e., $\mathrm{i}\beta_\chi \in P\sigma(A)$.

Conversely, for each $\mathrm{i}\beta \in P\sigma(A)$ with corresponding eigenfunction $\chi_\beta \in X$, we have

$$T(t)\chi_\beta = \mathrm{e}^{\mathrm{i}\beta t}\chi_\beta;$$

hence, by evaluation at the unit element $e \in G$,

$$\chi_\beta(\alpha_t) = \mathrm{e}^{\mathrm{i}\beta t}\chi_\beta(e).$$

This implies $\chi_\beta(e) \neq 0$, and we can choose the eigenfunction such that $\chi_\beta(e) = 1$. Therefore, we obtain

$$\chi_\beta(\alpha_t) = \mathrm{e}^{\mathrm{i}\beta t} \in \Gamma.$$

Since $\chi_\beta$ is continuous and $\{\alpha_t : t \in \mathbb{R}\}$ is dense in $G$, we conclude that (the continuous extension of) $\chi_\beta$ is a character on $G$, hence belongs to $\widehat{G}$.

It is now easy to show that this correspondence yields a group isomorphism from $\widehat{G}$ onto $P\sigma(A)$ as a subgroup of $(\mathrm{i}\mathbb{R}, +)$. Finally, we use the Pontryagin duality theorem ([HR63, Thm. 24.3]) to conclude that $\widehat{G}$ determines $G$ uniquely. Hence, this yields the following result.

**Proposition.** *For every subgroup $K$ of $(\mathrm{i}\mathbb{R}, +)$ endowed with the discrete topology, there exists a strongly continuous operator group $\big(T(t)\big)_{t\in\mathbb{R}}$ on a Banach space $X$ such that the strong operator closure*

$$\mathcal{S} := \overline{\{T(t) : t \in \mathbb{R}\}}^{\mathcal{L}_s(X)} \subset \mathcal{L}_s(X)$$

*is a solenoidal compact group isomorphic to the character group $\widehat{K}$ of $K$.*

Proof. Take the character group $G := \widehat{K}$ of $K$, which is compact, since $K$ is discrete. The characters

$$\alpha_t : K \ni \beta \mapsto \mathrm{e}^{t\beta} \in \Gamma$$

form a one-parameter subgroup $(\alpha_t)_{t\in\mathbb{R}}$ of $G$. Since the subgroup $(\alpha_t)_{t\in\mathbb{R}}$ separates $K$, it follows from [HR63, Lem. 24.4] that its closure is $G$; hence $G$ is solenoidal. Now take $X := \mathrm{C}(G)$ and the rotation group $\big(T(t)\big)_{t\in\mathbb{R}}$ induced by $(\alpha_t)_{t\in\mathbb{R}}$ as above in order to obtain the desired objects. □

The result is useful, since one has a complete classification of all discrete subgroups of $(\mathrm{i}\mathbb{R}, +)$ and, as a consequence, of all solenoidal compact groups (see [HR63, §25 and Thm. 25.12]).

As specific examples we mention

- the $n$-tori $\Gamma^n$ for $n \leq \operatorname{card} \mathbb{R}$ ([HR63, Cor. 25.15]),
- the *a-adic solenoid* $\sum_a$ ([HR63, Thm. 10.13]),
- the *Bohr compactification* $b\mathbb{R}$ ([HR63, Thm. 25.12]).

However, the production of examples was not the only reason for going through all these notions and results from harmonic analysis. In fact, the operator groups coming from rotations on solenoidal compact groups are, in a certain sense, typical among all relatively strongly compact semigroups for which the decomposition in Theorem 2.14 yields $X = X_r$. In fact, on certain function spaces they can be characterized in an abstract way.

**2.18 Theorem.** *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ on a Banach space $X := \mathrm{C}(\Omega)$, $\Omega$ compact, and assume the following.*

(i) *The eigenvectors of $A$ corresponding to imaginary eigenvalues form a total set in $X$, i.e., $X = X_r$.*

(ii) $\ker A = \langle \mathbb{1} \rangle$, *i.e., the fixed space of $(T(t))_{t\geq 0}$, consists of the constant functions only.*

(iii) *$T(t)f \geq 0$ for all $f \geq 0$, i.e., $(T(t))_{t\geq 0}$ is a positive semigroup.*

*Then $(T(t))_{t\geq 0}$ is isomorphic to a rotation (semi) group on some solenoidal compact group. More precisely, there exists a compact group $G$ with dense subgroup $(\alpha_t)_{t\in\mathbb{R}}$ and a homeomorphism $\varphi : \Omega \to G$ such that for each $t \geq 0$, the diagram*

$$\begin{array}{ccc} \mathrm{C}(\Omega) & \xrightarrow{\;T(t)\;} & \mathrm{C}(\Omega) \\ {\scriptstyle \Phi^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle \Phi} \\ \mathrm{C}(G) & \xrightarrow{\;U(t)\;} & \mathrm{C}(G) \end{array}$$

*commutes. Here, we define the operators*

$$\Phi f(w) := f\big(\varphi(w)\big) \qquad \text{and} \qquad U(t)f(\gamma) := f(\alpha_t \gamma)$$

*for $w \in \Omega$, $\gamma \in G$ and $f \in \mathrm{C}(G)$.*

The analogous result holds for $X := \mathrm{L}^p(\Omega, \mu)$, $1 \leq p < \infty$, and $\mu(\Omega) < \infty$, and is an operator-theoretic version of the classical *Halmos–von Neumann theorem* [HN42] (see also [Hal56] or [CFS82]). Proofs can be found in [Nag86, C-III, Cor. 3.9], [Gre82, Thm. 2.6], or for general groups of positive operators on arbitrary Banach lattices in [Sch74, Sec. III.10].

While for the above results we needed sophisticated tools from harmonic analysis, our subsequent discussion of Problem 2.16.(ii) uses only methods we have developed so far for strongly continuous semigroups.

In analogy to the case of uniform exponential stability and the various Liapunov stability Theorems I.2.10, I.3.14, and 1.10, it is now our main goal to find a spectral characterization of strong stability. However, the following examples show that the situation is quite complex.

**2.19 Examples.** (*Strongly Stable Semigroups*). (i) We take the left translation semigroup $\big(T_l(t)\big)_{t\geq 0}$ on $\mathrm{L}^p(\mathbb{R}_+)$, $1 \leq p < \infty$. As already observed in Paragraph I.4.16, its adjoint semigroup is given by the right translations $\big(T_r(t)\big)_{t\geq 0}$ on $\mathrm{L}^q(\mathbb{R}_+)$, $1/p + 1/q = 1$. From the definition of these semigroups it is immediate that

- $\big(T_l(t)\big)_{t\geq 0}$ is strongly stable for all $1 \leq p < \infty$, while
- $\big(T_r(t)\big)_{t\geq 0}$ consists of (nonsurjective) isometries for all $1 \leq q \leq \infty$.

In each case, the spectra of the corresponding generators coincide with the closed half-plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq 0\}$ (see Example IV.2.6.(i)) and hence do not distinguish between the very contrasting asymptotic behaviors of $\big(T_l(t)\big)_{t\geq 0}$ and $\big(T_r(t)\big)_{t\geq 0}$. To conclude this example, we observe that both semigroups $\big(T_l(t)\big)_{t\geq 0}$ and $\big(T_r(t)\big)_{t\geq 0}$ are weakly stable for $1 < p < \infty$.

(ii) Next, we look at multiplication semigroups on $X := \mathrm{C}_0(\Omega)$, $\Omega$ locally compact (see Section I.4.a). Let $q : \Omega \to \mathbb{C}$ be a continuous function inducing a multiplication semigroup $\big(T(t)\big)_{t\geq 0}$ by

$$T(t)f := \mathrm{e}^{tq} \cdot f, \quad f \in X,$$

with generator

$$Af = q \cdot f, \quad f \in D(A) = \{f \in X : qf \in X\}$$

(see Paragraph II.2.9). We assume $\big(T(t)\big)_{t\geq 0}$ to be bounded hence

$$\sup_{s\in\Omega} \mathrm{Re}\, q(s) \leq 0.$$

If there is a point $s_0 \in \Omega$ such that $\mathrm{Re}\, q(s_0) = 0$, then

$$|T(t)f(s_0)| = |f(s_0)| \qquad \text{for all } t \geq 0,$$

i.e., $\big(T(t)\big)_{t\geq 0}$ is not strongly stable. On the other hand, if $\mathrm{Re}\, q(s) < 0$ for all $s \in \Omega$, then for $f \in X$ with compact support $\Omega_0 \subset \Omega$, we obtain convergence

$$\lim_{t\to\infty} \|T(t)f\| \leq \lim_{t\to\infty} \mathrm{e}^{\sup_{s\in\Omega_0} \mathrm{Re}\, q(s)\cdot t}\, \|f\| = 0.$$

Since the continuous functions with compact support are dense in $X$, we conclude that $\big(T(t)\big)_{t\geq 0}$ is strongly stable.

From (iv) in Proposition I.4.2 we know that the spectrum $\sigma(A)$ of the generator is the closed range $\overline{q(\Omega)}$. One can therefore construct strongly stable multiplication semigroups on $\mathrm{C}_0(\Omega)$ such that the *boundary spectrum* $\sigma_+(A) = \sigma(A) \cap \mathrm{i}\mathbb{R}$ of its generator $A$ is a given closed subset of $\mathrm{i}\mathbb{R}$.

(iii) For multiplication semigroups on $\mathrm{L}^p$-spaces there is an analogous characterization of strong stability. Take $X := \mathrm{L}^p(\Omega, \mu)$, $1 \leq p < \infty$, for some $\sigma$-finite measure space $(\Omega, \mu)$ and consider a measurable function $q : \Omega \to \mathbb{C}$ satisfying

$$\operatorname*{ess\,sup}_{s \in \Omega} \operatorname{Re} q(s) \leq 0.$$

Then the associated multiplication semigroup $\big(T(t)\big)_{t \geq 0}$ (see Section I.4.b) is strongly stable if and only if

$$\mu(\Omega_0) = 0 \qquad \text{for} \qquad \Omega_0 := \{s \in \Omega : q(s) \in \mathrm{i}\mathbb{R}\}. \tag{2.6}$$

To prove this assertion, we consider

$$\Omega_n := \big\{s \in \Omega : \operatorname{Re} q(s) \leq -1/n\big\}$$

for $n \in \mathbb{N}$ and observe that

$$\lim_{t \to \infty} \|T(t) f\| = 0$$

for all $f \in \mathrm{L}^p(\Omega, \mu)$ vanishing outside some $\Omega_n$. If (2.6) holds, then these functions form a dense subspace of $X$, and the semigroup being bounded is strongly stable. Conversely, if $\mu(\Omega_0) > 0$, then

$$\|T(t) \mathbb{1}_{\Omega_0}\| = \|\mathbb{1}_{\Omega_0}\| > 0 \qquad \text{for all } t \geq 0,$$

and hence the semigroup $\big(T(t)\big)_{t \geq 0}$ is not strongly stable.

As an application of this characterization and of the spectral theorem for self-adjoint operators on Hilbert spaces (see Theorem I.4.9), we obtain the following result with the $n$-dimensional Laplace operator $\Delta$ on $\mathrm{L}^2(\mathbb{R}^n)$ as a typical example (see Paragraph II.2.13).

**Proposition.** *Let $\big(A, D(A)\big)$ be a self-adjoint operator on a Hilbert space $H$ such that $(Ax \,|\, x) \leq 0$ for all $x \in D(A)$. Then the following assertions are equivalent.*

(a) *The semigroup $\big(T(t)\big)_{t \geq 0}$ generated by $A$ is strongly stable.*

(b) *0 is not an eigenvalue of $A$.*

Proof. By the Spectral Theorem I.4.9, $A$ is isomorphic to a multiplication operator on $\mathrm{L}^2(\Omega, \mu)$ such that the essential range of the corresponding function satisfies $q_{\mathrm{ess}}(\Omega) \subset (-\infty, 0]$. Moreover, $\mu([q = 0]) > 0$ if and only if 0 is an eigenvalue of $A$. $\square$

All these examples do not suggest a characterization of strong stability through spectral properties of the generator. However, there are some immediate necessary properties. In particular, every strongly stable semigroup must be

- *bounded*,

and its generator $A$ has to satisfy

- $P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$, and
- $P\sigma(A') \cap \mathrm{i}\mathbb{R} = \emptyset$.

The first property follows from the uniform boundedness principle. For the other two observe that

$$Ax = \mathrm{i}\lambda x \qquad \text{for } x \in X \quad (\text{or, } A'x' = \mathrm{i}\lambda x' \text{ for } x' \in X')$$

implies

$$T(t)x = \mathrm{e}^{\mathrm{i}\lambda t}x \quad (\text{or, } T(t)'x' = \mathrm{e}^{\mathrm{i}\lambda t}x')$$

for all $t \geq 0$ (use Theorem IV.3.7). Therefore, if $P\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \emptyset$ or $P\sigma(A') \cap \mathrm{i}\mathbb{R} \neq \emptyset$, the semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ is not even weakly stable.

The following lemma shows that these properties are not unrelated.

**2.20 Lemma.** *For a bounded strongly continuous semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ with generator $A$ on a Banach space $X$, the following properties hold.*

(i) $P\sigma(A) \subset P\sigma(A')$.

(ii) $P\sigma(A) = P\sigma(A')$ *if $X$ is reflexive.*

Proof. (i) Due to the rescaling technique from Paragraph II.2.2, it suffices to show that $0 \in P\sigma(A)$ implies $0 \in P\sigma(A')$. Assume that $Ax_0 = 0$ and hence $T(t)x_0 = x_0$ for all $t \geq 0$ and some $0 \neq x_0 \in D(A)$. Choose $x' \in X'$ such that $\langle x_0, x' \rangle = 1$ and define

$$y'_n := \frac{1}{n}\int_0^n T(s)'x'\,ds, \qquad n \in \mathbb{N}.$$

Since $\bigl(T(t)\bigr)_{t\geq 0}$ is bounded, the sequence $(y'_n)_{n\in\mathbb{N}}$ is bounded as well and has a weak* accumulation point $y'_0$. This accumulation point satisfies

$$\langle x_0, y'_0 \rangle = 1$$

and

$$\langle y, T(t)'y'_0 - y'_0 \rangle = 0$$

for all $y \in X$ and $t > 0$. Therefore, $y'_0$ is a nontrivial fixed vector of $\bigl(T(t)'\bigr)_{t\geq 0}$, and 0 is an eigenvalue of $A'$.

(ii) If $X$ reflexive, then the adjoint semigroup $\bigl(T(t)'\bigr)_{t\geq 0}$ is strongly continuous (see Paragraph I.5.14), and it follows from assertion (i) that $P\sigma(A') \subset P\sigma(A'') = P\sigma(A)$. □

We are now ready to give a partial answer to question (ii) in Problem 2.16. The subsequent sufficient conditions for strong stability were found independently by W. Arendt–C.J.K. Batty [AB88] and Yu.I. Lyubich–Q.Ph. Vũ [LV88] and confirm in a beautiful way what could be taken as a leitmotif for these investigations:

*"Small boundary spectrum $\sigma(A) \cap \mathrm{i}\mathbb{R}$"*
implies
*"good stability properties for $(T(t))_{t\geq 0}$;"*
see also Exercise 2.25.(4).

**2.21 Theorem.** (Arendt, Batty, Lyubich, Vũ, 1988). *Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $A$ on a Banach space $X$. If*

(i) *$P\sigma(A') \cap \mathrm{i}\mathbb{R} = \emptyset$ and*
(ii) *$\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable,*

*then $(T(t))_{t\geq 0}$ is strongly stable, i.e.,*

$$\lim_{t\to\infty} T(t)x = 0 \qquad \text{for all } x \in X. \tag{2.7}$$

Proof. As a preparatory step, we renorm $X$ as in Lemma II.3.10 to make $(T(t))_{t\geq 0}$ a contraction semigroup. Assume now that $(T(t))_{t\geq 0}$ is not strongly stable. In this case, the construction of the isometric limit semigroup $(S(t))_{t\geq 0}$ performed in Proposition IV.2.20 yields a nontrivial Banach space

$$Z := \left( {}^{X}/_{Y}, |\!|\!|\cdot|\!|\!| \right)^{\sim} \neq \{0\}$$

for the norm

$$|\!|\!|x + Y|\!|\!| := \lim_{t\to\infty} \|T(t)x\|, \quad x \in X,$$

and the subspace

$$Y := \Big\{ x \in X : \lim_{t\to\infty} \|T(t)x\| = 0 \Big\}.$$

As shown in Proposition IV.2.20, the generator $B$ of $(S(t))_{t\geq 0}$ satisfies $\sigma(B) \subset \sigma(A)$, and assumption (ii) implies that case (ii) of Lemma IV.2.19 holds. Hence, $\emptyset \neq \sigma(B) \subset \mathrm{i}\mathbb{R}$ (use Corollary IV.3.21), and $(S(t))_{t\geq 0}$ extends to a group of isometries on $Z$.

It is now that we use the full strength of assumption (ii). In fact, $\sigma(A) \cap \mathrm{i}\mathbb{R}$; hence $\sigma(B)$ is a countable, locally compact space. By Baire's theorem, there exists an isolated point $\lambda_0 \in \sigma(B)$. We perform the spectral decomposition with spectral projection $P_0$ corresponding to the spectral set $\{\lambda_0\}$ (see Proposition IV.1.16). This yields a strongly continuous group $(S_0(t))_{t\geq 0}$ of isometries on $Z_0 := P_0 Z$ such that its generator $B_0$ has spectrum $\sigma(B_0) = \{\lambda_0\}$. By the Weak Spectral Mapping Theorem for Bounded Groups IV.3.16, we conclude that

$$\sigma\big(S_0(t)\big) = \{\mathrm{e}^{\lambda_0 t}\} \quad \text{for all } t \in \mathbb{R}.$$

Gelfand's $T = I$ theorem (see Theorem B.17) then implies that

$$S_0(t) = \mathrm{e}^{\lambda_0 t} \cdot I \quad \text{and} \quad B_0 = \lambda_0 \cdot I.$$

From this we can show that $\lambda_0 \in P\sigma(A')$, contradicting assumption (i). In fact, take $0 \neq z' \in Z_0'$ and define $0 \neq x' \in X'$ by

$$\langle x, x' \rangle := \langle P_0(x+Y), z' \rangle \quad \text{for all } x \in X.$$

Then

$$\begin{aligned} \langle x, T(t)'x' \rangle &= \langle T(t)x, x' \rangle = \big\langle P_0\big(T(t)x + Y\big), z' \big\rangle \\ &= \langle S_0(t)P_0(x+Y), z' \rangle = \big\langle \mathrm{e}^{\lambda_0 t} P_0(x+Y), z' \big\rangle \\ &= \big\langle x, \mathrm{e}^{\lambda_0 t} x' \big\rangle \quad \text{for all } x \in X. \end{aligned}$$

This proves that $\mathrm{e}^{\lambda_0 t} \in P\sigma\big(T(t)'\big)$ for all $t \geq 0$; hence $\lambda_0 \in P\sigma(A')$. □

If we assume our Banach space to be reflexive, we obtain from Lemma 2.20 the following slightly simpler version of the above stability theorem.

**2.22 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $A$ on a reflexive Banach space $X$. If*

(i) *$P\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$ and*

(ii) *$\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable,*

*then $\big(T(t)\big)_{t\geq 0}$ is strongly stable.*

We have already seen in Example 2.19.(i) that condition (ii) is not necessary for strong stability. Still, Theorem 2.21 and its Corollary 2.22 are quite useful, and the examples below (in particular, Example 2.23.(iii)) show that the result is, in a certain sense, optimal.

**2.23 Examples.** (i) Let $\big(T(t)\big)_{t\geq 0}$ be a bounded analytic semigroup in a sector $\Sigma_\delta$. Then the resolvent set $\rho(A)$ of its generator $A$ contains the sector $\Sigma_{\pi/2+\delta}$ (see Theorem II.4.6); hence $\sigma(A) \cap \mathrm{i}\mathbb{R} \subset \{0\}$. From Theorem 2.21 we deduce the following equivalence extending the proposition in Example 2.19.(iii).

*A bounded analytic semigroup $\big(T(t)\big)_{t\geq 0}$ is strongly stable if and only if $0$ is not an eigenvalue of the adjoint $A'$ of its generator $A$.*

(ii) Using multiplication operators as in Examples 2.19.(ii) and 2.19.(iii), it is easy to produce strongly stable semigroups such that $\sigma(A) \cap \mathrm{i}\mathbb{R}$ becomes a given countable and closed subset of $\mathrm{i}\mathbb{R}$.

(iii) For the following example, we start with an arbitrary uncountable and closed subset $\Omega$ of $\mathrm{i}\mathbb{R}$. It is known from measure theory (e.g., see [Sem71, 8.5.5 and 19.7.6]) that there exists a nontrivial diffuse probability measure

$\mu$ whose support is contained in $\Omega$. On the space $X := \mathrm{L}^2(\Omega, \mu)$ we then take the semigroup $\big(T(t)\big)_{t\ge 0}$ generated by the multiplication operator

$$Af(s) := sf(s), \quad s \in \Omega \text{ and } f \in \mathrm{L}^2(\Omega, \mu).$$

Then $\sigma(A) = \operatorname{supp}\mu$ (by Proposition I.4.10.(iv)) is uncountable and the point spectrum $P\sigma(A)$ is empty (since $\mu$ is diffuse). However, $\big(T(t)\big)_{t\ge 0}$ is not strongly stable (since each $T(t)$ is an isometry) and, in general, not even weakly stable (use Example 2.11.(ii)). This shows that the countability condition in Theorem 2.21, while not being necessary, cannot be weakened in general.

However, by adding an appropriate assumption, we can characterize strong stability by a spectral property.

**2.24 Corollary.** *Let $\big(T(t)\big)_{t\ge 0}$ be a strongly continuous semigroup with generator $A$ on a Banach space $X$. If $\big(T(t)\big)_{t\ge 0}$ is relatively compact for the strong operator topology, then the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\ge 0}$ is strongly stable.*

(b) *$P\sigma(A) \cap i\mathbb{R} = \emptyset$.*

Proof. By Theorem 2.14, strong stability is equivalent to $X_r = \{0\}$, which by definition is condition (b). ☐

The conditions (i) and (ii) in Corollary 2.15 provide examples in which the above compactness assumption is satisfied. See also Exercise 2.25.(2).

**2.25 Exercises.** (1) Let $(T(t))_{t\ge 0}$ be a bounded strongly continuous semigroup on the Banach space $X$ with generator $A$. Show that for each $\lambda \in \mathbb{C}$,

$$\ker(\lambda - A') \quad \text{separates} \quad \ker(\lambda - A)$$

and, if $X$ is reflexive,

$$\ker(\lambda - A) \quad \text{separates} \quad \ker(\lambda - A').$$

(Hint: Use the idea in the proof of Lemma 2.20.)

(2) Let $(T(t))_{t\ge 0}$ be a bounded strongly continuous semigroup and assume that there exists a compact operator $K \in \mathcal{L}(X)$ such that $\overline{\operatorname{rg} K} = X$ and $T(t_0)K = KT(t_0)$ for some $t_0 > 0$. Show that $(T(t))_{t\ge 0}$ is relatively compact for the strong operator topology.

(3) Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup on the Banach space $X$ that is relatively compact for the weak operator topology and consider the splitting $X = X_s \oplus X_r$ according to Theorem 2.8. Construct examples satisfying various combinations of the following properties.

(i) $X_r = \{0\}$ or $X_r = \operatorname{fix}(T(t))_{t\ge 0}$.

(ii) $X_s = \big\{x \in X : \lim_{t\to\infty} e^{\varepsilon t}\|T(t)x\| = 0\big\}$ for some $\varepsilon > 0$,
$X_s = \{x \in X : \lim_{t\to\infty} \|T(t)x\| = 0\}$, or
$X_s = \{x \in X : \lim_{t\to\infty} \langle T(t)x, x'\rangle = 0 \text{ for all } x' \in X'\}$.

(4*) Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $A$ on a Banach space $X$ such that $\sigma(A) \cap \mathrm{i}\mathbb{R} \subseteq \{0\}$.

(i) Show that $\lim_{t\to\infty} \|T(t)(T(s) - I)R(\lambda, A)\| = 0$ for all $s > 0$, $\lambda > 0$. (Hint: See [Vũ92].)

(ii) If $X$ is reflexive, this implies that $\lim_{t\to\infty} T(t)x$ exists for every $x \in X$. (Hint: Use the property stated in Example 4.7 below.)

# 3. Eventually Compact and Quasi-compact Semigroups

The structure theory for compact semigroups, developed in Section 2.a, was the clue for the systematic treatment of asymptotic properties with respect to the weak or the strong operator topology carried out in Section 2.b and 2.c. This theory will not be needed if we are interested in convergence properties for the uniform operator topology. Instead, we will assume that some or all semigroup operators, and not the semigroup or the semigroup orbits, satisfy some compactness condition. However, we continue to call such semigroups “compact,” even if the two concepts are logically independent. For instance, if $T(t) = I$ for all $t \geq 0$, then the semigroup is (norm) compact, while no $T(t)$ is a compact operator (if $\dim X = \infty$). On the other hand, a multiplication semigroup on $c_0$ satisfying property (ii) in the proposition in Paragraph II.4.32 consists of compact operators, but may be unbounded, hence is not compact. We will distinguish between the two compactness concepts by using a prefix such as “strongly” or “weakly” in the first case and “eventually” or “quasi” in the second case.

The analysis of semigroups $\bigl(T(t)\bigr)_{t\geq 0}$ containing some compact operator $T(t_0)$ is based on the description of its spectrum. From this, we then deduce important consequences for its asymptotic behavior. We start with a more general situation and use the essential spectrum from Paragraph IV.1.20 and the essential growth bound $\omega_{\mathrm{ess}}$ from Definition IV.2.9.

**3.1 Theorem.** *Let $\mathcal{T} := \bigl(T(t)\bigr)_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ and take $\lambda_1, \dots, \lambda_m \in \sigma(A)$ satisfying $\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_m > \omega_{\mathrm{ess}}(\mathcal{T})$. Then $\lambda_1, \dots, \lambda_m$ are isolated spectral values of $A$ with finite algebraic multiplicity. If $P_1, \dots, P_m$ denote the corresponding spectral projections and $k_1, \dots, k_m$ the corresponding orders of poles of $R(\cdot, A)$, then*

$$T(t) = T_1(t) + \cdots + T_m(t) + R_m(t),$$

*where*

$$T_n(t) = \mathrm{e}^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!}(A - \lambda_n)^j P_n \qquad \text{for } n = 1, \dots, m.$$

*Moreover, for every $w > \sup\bigl\{\omega_{\mathrm{ess}}(\mathcal{T})\bigr\} \cup \bigl\{\operatorname{Re}\lambda : \lambda \in \sigma(A) \setminus \{\lambda_1, \dots, \lambda_m\}\bigr\}$ there exists $M > 0$ such that*

$$\|R_m(t)\| \leq M\mathrm{e}^{wt}$$

*for all $t \geq 0$.*

Proof. By Corollary IV.2.11, the spectral values $\lambda_1, \dots, \lambda_m$ are isolated with finite algebraic multiplicity. Now let $P := \sum_{n=1}^m P_n$ be the spectral projection of $A$ corresponding to the spectral set $\{\lambda_1, \dots, \lambda_m\}$; cf. Proposition IV.1.16. We then obtain $T(t) = T(t)P_1 + \cdots + T(t)P_m + T(t)(I - P)$, and by Paragraph II.2.3, the restricted semigroup $\big(T(t)_{| \operatorname{rg} P_n}\big)_{t\geq 0}$ has generator $A_{| \operatorname{rg} P_n}$. Since $\operatorname{rg} P_n$ is finite-dimensional and $(A - \lambda_n)^{k_n}_{| \operatorname{rg} P_n} = 0$ (cf. Example I.2.5), we obtain $T(t)_{| \operatorname{rg} P_n} = \mathrm{e}^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!}(A - \lambda_n)^j_{| \operatorname{rg} P_n}$, i.e.,

$$T_n(t) = T(t)P_n = \mathrm{e}^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!}(A - \lambda_n)^j P_n \quad \text{for all } t \geq 0.$$

Consider now the semigroup $\big(T(t)_{| \ker P}\big)_{t\geq 0}$. By Corollary IV.2.11 its growth bound is given by

$$\omega_0(\mathcal{T}_{| \ker P}) = \max\big\{\omega_{\mathrm{ess}}(\mathcal{T}_{| \ker P}), s(A_{| \ker P})\big\}.$$

Since $\omega_{\mathrm{ess}}(\mathcal{T}_{| \ker P}) = \omega_{\mathrm{ess}}(\mathcal{T})$ (use Proposition IV.2.12) and

$$s(A_{| \ker P}) = \sup\big\{ \operatorname{Re}\lambda : \lambda \in \sigma(A) \setminus \{\lambda_1, \dots, \lambda_m\}\big\},$$

this implies

$$\omega_0(\mathcal{T}_{| \ker P}) = \sup\big\{ \omega_{\mathrm{ess}}(\mathcal{T})\big\} \cup \big\{ \operatorname{Re}\lambda : \lambda \in \sigma(A) \setminus \{\lambda_1, \dots, \lambda_m\}\big\}.$$

Hence, for every $w \in \mathbb{R}$ larger than this number there exists $M > 0$ such that

$$\|R_m(t)\| = \|T(t)(I - P)\| \leq \|T(t)_{| \ker P}\| \cdot \|I - P\| \leq M\mathrm{e}^{wt}$$

for all $t \geq 0$. □

We now consider semigroups $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ containing some compact operator, i.e., that are eventually compact semigroups in the sense of Section II.4.d. In this case, we have $\omega_{\mathrm{ess}}(\mathcal{T}) = -\infty$, and Theorem 3.1 combined with Corollary IV.2.11 gives the following result.

**3.2 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ be an eventually compact semigroup with generator $A$ on a Banach space $X$. Then the following properties hold.*

(i) *The spectrum $\sigma(A)$ is countable (or finite or empty) and consists of poles of $R(\cdot, A)$ of finite algebraic multiplicity only.*

(ii) *The set $\{\mu \in \sigma(A) : \operatorname{Re}\mu \geq r\}$ is finite for every $r \in \mathbb{R}$.*

*Therefore, we can write* $\sigma(A) = \{\lambda_1, \lambda_2, \dots\}$ *with* $\operatorname{Re}\lambda_{n+1} \le \operatorname{Re}\lambda_n$ *for all* $n \in \mathbb{N}$ *and* $\operatorname{Re}\lambda_n \downarrow -\infty$ *if* $\sigma(A)$ *is infinite. Denote by* $k_n$ *the order of the pole* $\lambda_n$ *of* $R(\cdot, A)$ *and by* $P_n$ *the corresponding residue. Then one has for every* $m \in \mathbb{N}$ *that*

(iii) $T(t) = T_1(t) + T_2(t) + \cdots + T_m(t) + R_m(t)$, *where*

$$T_n(t) = e^{\lambda_n t} \sum_{j=0}^{k_n - 1} \frac{t^j}{j!}(A - \lambda_n)^j P_n, \quad t \ge 0 \text{ and } 1 \le n \le m, \tag{3.1}$$

*and for every* $\varepsilon > 0$ *there exists* $M > 0$ *such that*

$$\|R_m(t)\| \le M e^{(\varepsilon + \operatorname{Re}\lambda_{m+1})t} \qquad \text{for all } t \ge 0. \tag{3.2}$$

From (3.1) and (3.2) it is now clear which additional hypotheses imply norm convergence of $T(t)$ as $t \to \infty$. First, we assume the existence of a *dominant eigenvalue* $\lambda_1$, i.e.,

$$\operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_n \qquad \text{for } n = 2, 3, \dots. \tag{3.3}$$

Moreover, $\lambda_1$ has to be a pole of order 1; hence $T_1(t)$ simply becomes $e^{\lambda_1 t} P_1$. Using estimate (3.2), we obtain

$$\left\| e^{-\lambda_1 t} T(t) - P_1 \right\| \le e^{-\lambda_1 t} \|T(t) - T_1(t)\| = e^{-\lambda_1 t} \|R_1(t)\| \le M e^{-\varepsilon t}$$

for some $\varepsilon > 0$ and $M \ge 1$.

**3.3 Corollary.** *Let* $(T(t))_{t \ge 0}$ *be a strongly continuous semigroup that is eventually compact. If* $\lambda_1$ *is a dominant eigenvalue of the generator and a first-order pole of the resolvent with residue* $P$, *then there exist constants* $\varepsilon > 0$ *and* $M \ge 1$ *such that*

$$\left\| e^{-\lambda_1 t} T(t) - P_1 \right\| \le M e^{-\varepsilon t}$$

*for all* $t \ge 0$.

It should be evident that the most interesting case occurs if $\lambda_1 = 0$ in the above corollary, and we refer to [Nag86, B-IV, Prop. 2.4 and Expl. 2.6] for an important class of examples.

The above results, and the asymptotic behavior they describe, are quite satisfying. However, the assumption of eventual compactness is not needed in order to obtain a conclusion as in Corollary 3.3. In fact, we used only the existence of a dominated eigenvalue $\lambda_1$ and therefore the representation from Corollary 3.2.(iii) for $m = 1$ only. This leads to the following property.

**3.4 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called quasi-compact if*

$$\lim_{t\to\infty} \inf\big\{ \|T(t)-K\| : K \in \mathcal{L}(X),\ K \text{ compact}\big\} = 0.$$

So, the operators in a quasi-compact semigroup $\big(T(t)\big)_{t\geq 0}$ need not be compact, but have to approach the subspace $\mathcal{K}(X)$ of all compact operators on $X$. Quasi-compactness can be characterized in various ways, e.g., using the essential growth bound $\omega_{\text{ess}}$ from Definition IV.2.9.

**3.5 Proposition.** *For a strongly continuous semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\geq 0}$ is quasi-compact.*

(b) *$\omega_{\text{ess}}(\mathcal{T}) < 0$.*

(c) *$\|T(t_0) - K\| < 1$ for some $t_0 > 0$ and $K \in \mathcal{K}(X)$.*

Proof. Property (a) implies (c) by definition, and (c) implies $r_{\text{ess}}\big(T(t_0)\big) \leq \|T(t_0)\|_{\text{ess}} < 1$, hence $\omega_{\text{ess}}(\mathcal{T}) < 0$ (use Proposition IV.2.10).

We now show that (b) $\Rightarrow$ (a). Again by Proposition IV.2.10, we have that $r_{\text{ess}}\big(T(1)\big) < 1$; hence $\lim_{n\to\infty} \|T(n)\|_{\text{ess}}^{1/n} < 1$ for $\|S\|_{\text{ess}} := \operatorname{dist}\big(S, \mathcal{K}(X)\big)$. Thus, we find $n_0 \in \mathbb{N}$ and $a < 1$ such that

$$\|T(n)\|_{\text{ess}} < a^n \qquad \text{for all } n \geq n_0.$$

Now choose compact operators $K_n \in \mathcal{K}(X)$ such that $\|T(n) - K_n\| < a^n$ for $n \geq n_0$ and define $M := \sup_{0\leq s\leq 1} \|T(s)\|$. We then obtain

$$\|T(t) - T(t-n)K_n\| \leq \|T(t-n)\| \cdot \|T(n) - K_n\| \leq M a^n$$

for $t \in [n, n+1]$ and $n \geq n_0$. This implies $\lim_{t\to\infty} \operatorname{dist}\big(T(t), \mathcal{K}(X)\big) = 0$. □

A natural name for these semigroups could also be *essentially uniformly exponentially stable* semigroups. In fact, condition (b) above means that the semigroup of quotient operators in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is uniformly exponentially stable.

The easiest examples of quasi-compact semigroups, which are not eventually compact, are uniformly exponentially stable semigroups. The generators of such semigroups can now be perturbed by an arbitrary compact operator destroying the uniform exponential stability but without losing the quasi-compactness. In fact, from Proposition IV.2.12 and Proposition 3.5.(c) we deduce the following result.

**3.6 Proposition.** *Let $\big(T(t)\big)_{t\geq 0}$ be a quasi-compact strongly continuous semigroup with generator $A$ on the Banach space $X$ and take a compact operator $K \in \mathcal{L}(X)$. Then $A + K$ generates a quasi-compact semigroup.*

It is now possible to obtain for quasi-compact semigroups the (essential part of the) assertions in Corollary 3.2. In fact, the following result follows immediately from Theorem 3.1 and the characterization of quasi-compactness in Proposition 3.5.(b).

**3.7 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be a quasi-compact strongly continuous semigroup with generator $A$ on a Banach space $X$. Then the following holds.*

(i) *The set $\{\lambda \in \sigma(A) : \operatorname{Re}\lambda \geq 0\}$ is finite (or empty) and consists of poles of $R(\cdot, A)$ of finite algebraic multiplicity.*

*If we denote these poles by $\lambda_1, \dots, \lambda_m$, their residues by $P_1, \dots, P_m$ with poles of order $k_1, \dots, k_m$, we have*

(ii) $T(t) = T_1(t) + T_2(t) + \cdots + T_m(t) + R(t)$, *where*

$$T_n(t) = \mathrm{e}^{\lambda_n t} \sum_{j=0}^{k_n-1} \frac{t^j}{j!}(A-\lambda_n)^j P_n, \quad t \geq 0 \text{ and } 1 \leq n \leq m, \tag{3.4}$$

*and*

$$\|R(t)\| \leq M\mathrm{e}^{-\varepsilon t} \qquad \text{for some } \varepsilon > 0,\ M \geq 1 \text{ and all } t \geq 0. \tag{3.5}$$

Clearly, assumptions as in Corollary 3.3, i.e., the existence of a dominant eigenvalue being a first-order pole, imply norm convergence of the (rescaled) semigroup. We do not restate this explicitly, but apply it to a concrete example.

**3.8 Example.** On the Banach space $X := \mathrm{C}(\mathbb{R}_- \cup \{-\infty\})$ we consider the first-order differential operator

$$Af := f' + mf \tag{3.6}$$

with domain

$$D(A) := \big\{ f \in X : f \text{ is differentiable, } f' \in X \text{ and } f'(0) = Lf \big\}, \tag{3.7}$$

where $m \in X$ is real-valued and $L$ is a continuous linear form on $X$. As in Paragraph II.3.29 we can show that the operator $\big(A, D(A)\big)$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$.

**Lemma 1.** *The semigroup $\big(T(t)\big)_{t\geq 0}$ satisfies*

$$T(t)f(s) = \begin{cases} \mathrm{e}^{\int_s^0 m(\sigma)\,d\sigma}\Big[\mathrm{e}^{(s+t)m(0)}f(0) \\ \qquad + \displaystyle\int_0^{s+t} \mathrm{e}^{\tau\, m(0)} L\,T(s+t-\tau)f\,d\tau\Big] & \text{for } s+t > 0, \\ \mathrm{e}^{\int_s^{s+t} m(\sigma)\,d\sigma} f(s+t) & \text{for } s+t \leq 0. \end{cases}$$

Proof. For $f \in D(A)$ we have

$$\frac{d}{dr}\Big( e^{rm(0)} \big(T(t-r)f\big)(0) + \int_0^r e^{\tau m(0)} L\,T(t-\tau)f\,d\tau \Big) = 0.$$

This implies

$$\big(T(t)f\big)(0) = e^{tm(0)} f(0) + \int_0^t e^{\tau m(0)} L\,T(t-\tau)f\,d\tau.$$

On the other hand, we have

$$\frac{d}{dr}\Big( e^{\int_s^{s+r} m(\sigma)\,d\sigma} \big(T(t-r)f\big)(s+r) \Big) = 0.$$

Therefore, we obtain

$$\big(T(t)f\big)(s) = \begin{cases} e^{\int_s^0 m(\sigma)\,d\sigma} \big(T(s+t)f\big)(0) & \text{for } s+t>0, \\ e^{\int_s^{s+t} m(\sigma)\,d\sigma} f(s+t) & \text{for } s+t \le 0. \end{cases}$$

□

This lemma allows us to give a condition that forces the semigroup $\big(T(t)\big)_{t\ge0}$ to be quasi-compact.

**Lemma 2.** *If $m(-\infty) < 0$, then the semigroup $\big(T(t)\big)_{t\ge0}$ is quasi-compact.*

Proof. We define operators $K(t) \in \mathcal{L}(X)$ by

$$K(t)f(s) := \begin{cases} e^{\int_s^0 m(\sigma)\,d\sigma}\Big[ e^{(s+t)\,m(0)} f(0) & \\ \quad + \displaystyle\int_0^{s+t} e^{(s+t-\tau)\,m(0)} L\,T(\tau)f\,d\tau \Big] & \text{for } 0 < s+t, \\ (t+s+1)\cdot e^{\int_s^0 m(\sigma)\,d\sigma} f(0) & \text{for } -1 < s+t \le 0, \\ 0 & \text{for } s+t \le -1. \end{cases}$$

These operators are compact by the Arzelà–Ascoli theorem. On the other hand, since $m(-\infty) < 0$, we have

$$\lim_{t\to\infty} \|T(t) - K(t)\| = 0.$$

Therefore, the semigroup $\big(T(t)\big)_{t\ge0}$ is quasi-compact. □

Assume in the following that $m(-\infty) < 0$. In order to apply the above result, we have to find the eigenvalues $\lambda$ of $A$ with $\operatorname{Re}\lambda \geq 0$. An eigenfunction $f \in D(A)$ with eigenvalue $\lambda$ satisfies

$$f' = \lambda f - mf,$$

hence is of the form $f = cg_\lambda$, where

$$g_\lambda(s) := \mathrm{e}^{\int_s^0 m(\sigma)\,d\sigma}\mathrm{e}^{\lambda s}$$

for all $s \in \mathbb{R}_-$. Since $\operatorname{Re}\lambda \geq 0$, the functions $g_\lambda$ and $g_\lambda'$ vanish at $-\infty$, hence belong to $X$. Consequently, $g_\lambda \in D(A)$ if and only if

$$\lambda - Lg_\lambda - m(0) = 0.$$

This shows that $\lambda$ is an eigenvalue of $A$ if and only if the *characteristic equation*

$$\xi(\lambda) := \lambda - Lg_\lambda - m(0) = 0 \tag{3.8}$$

holds.

Now, suppose that $\lambda$ with $\operatorname{Re}\lambda \geq 0$ is not an eigenvalue of $A$. For each $g \in X$ we want to find a function $f \in D(A)$ such that

$$f' = \lambda f - mf - g.$$

This equation is solved by

$$f = cg_\lambda + h_\lambda,$$

where

$$h_\lambda(s) := \int_s^0 \mathrm{e}^{\int_s^\tau m(\sigma)\,d\sigma}\mathrm{e}^{\lambda(s-\tau)}g(\tau)\,d\tau$$

for all $s \in \mathbb{R}_-$. If the constant $c$ is chosen as

$$c := \frac{g(0) + Lh_\lambda}{\lambda - Lg_\lambda - m(0)}, \tag{3.9}$$

we then obtain the unique $f \in D(A)$ satisfying $(\lambda - A)f = g$. This yields an explicit representation of the resolvent of $A$ in $\lambda$.

In the remaining part of this paragraph we assume that $L$ is of the form

$$L = L_0 + a\delta_0, \tag{3.10}$$

where $a$ is a real number and $L_0$ is a *positive* linear form on $X$. We then have the following lemma proving the existence of a dominant eigenvalue.

**Lemma 3.** *Suppose that $m(-\infty) < 0$. If $\xi(0) \leq 0$, i.e., $Lg_0 \geq -m(0)$, then the characteristic function $\xi$ has a unique zero $\lambda_0 \geq 0$ that is a dominant eigenvalue of the operator $A$.*

PROOF. *The function* $\xi : \mathbb{R}_+ \ni \lambda \mapsto \lambda - L_0 g_\lambda - a - m(0)$ *is strictly increasing from* $\xi(0)$ *to* $\infty$. *Consequently, if* $\xi(0) \le 0$, *it has a unique zero* $\lambda_0$ *that is an eigenvalue of* $A$. *Now take an arbitrary eigenvalue* $\lambda$ *of* $A$ *with* $\operatorname{Re}\lambda \ge \lambda_0$. *Then, we have*

$$|\lambda - a - m(0)| = |L_0 g_\lambda| \le L_0 g_{\lambda_0} = \lambda_0 - a - m(0).$$

*This implies* $\lambda = \lambda_0$, *and therefore* $\lambda_0$ *is a dominant eigenvalue of* $A$. □

The eigenspace corresponding to the dominant eigenvalue $\lambda_0$ is spanned by the function $g_{\lambda_0}$, hence is one-dimensional. Moreover, it is a first-order pole, as can be seen from (3.9).

After these preparations, we can give a precise description of the asymptotic behavior of the semigroup $\big(T(t)\big)_{t\ge0}$. In particular, it follows that the rescaled semigroup $\big(\mathrm{e}^{-\lambda_0 t}T(t)\big)_{t\ge0}$ converges in norm to a one-dimensional projection (see also Exercise 3.9.(3)).

**Proposition 4.** *Assume that* $m(-\infty) < 0$, $L = L_0 + a\delta_0$ *as in* (3.10), *and* $L_0 g_0 + a \ge -m(0)$. *Then there is a dominant eigenvalue* $\lambda_0 \ge 0$ *of* $A$, *a continuous linear form* $\varphi$ *on* $X$, *and constants* $\varepsilon$, $M > 0$ *such that*

$$\left\|\mathrm{e}^{-\lambda_0 t}T(t)f - (g_{\lambda_0} \otimes \varphi)f\right\| \le M\mathrm{e}^{-\varepsilon t}\|f\| \qquad \text{for all } f \in X,\ t \ge 0,$$

*where* $(g_{\lambda_0} \otimes \varphi)f := \varphi(f) \cdot g_{\lambda_0}$.

**3.9 Exercises.** (1) Let $(T(t))_{t\ge0}$ be an eventually compact semigroup such that the spectrum $\sigma(A)$ of its generator $A$ is infinite. Show that there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\sigma(A) = P\sigma(A) = \{\mu_n : n \in \mathbb{N}\}$ and $\lim_{n\to\infty} \operatorname{Re}\mu_n = -\infty$. (Hint: Use Corollary 3.2 and Theorem II.4.18.)

(2) Call a strongly continuous semigroup $\mathcal{T} = (T(t))_{t\ge0}$ with generator $A$ *essentially compact* if $\omega_{\mathrm{ess}}(\mathcal{T}) < \mathrm{s}(A)$ and prove an analogue of Theorem 3.7 for these semigroups. (Hint: Rescale the semigroup to make it quasi-compact.)

(3) A strongly continuous semigroup $(T(t))_{t\ge0}$ with generator $A$ on a Banach space $X$ satisfies *balanced exponential growth* if there exists a projection $P \in \mathcal{L}(X)$ such that

$$\lim_{t\to\infty} \left\|\mathrm{e}^{-\mathrm{s}(A)t}T(t) - P\right\| = 0.$$

(i) A semigroup with balanced exponential growth is essentially compact if and only if $P$ has finite-dimensional range.

(ii) An essentially compact semigroup has balanced exponential growth if and only if $\mathrm{s}(A)$ is a dominant eigenvalue (cf. Corollary 3.3). (Hint: See [Web87, Sec. 2].)

# 4. Mean Ergodic Semigroups

Up to now the asymptotic behavior of a semigroup $\big(T(t)\big)_{t\ge 0}$ has been described by looking at the existence of the limit

$$\text{“}\lim_{t\to\infty} T(t)\text{”}$$

in some appropriate topology. However, it is also interesting to study the convergence of certain "mean values" of the semigroup. The most natural means for this purpose are the *Cesàro means* to be introduced now for a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ on a Banach space $X$.

**4.1 Definition.** *For each $r>0$, the operators*

$$C(r) := \frac{1}{r}\int_0^r T(s)\,ds,$$

*defined pointwise as*

$$C(r)x := \frac{1}{r}\int_0^r T(s)x\,ds \qquad \text{for each } x\in X,$$

*will be called the Cesàro means of the semigroup $\big(T(t)\big)_{t\ge 0}$.*

These means have some simple algebraic and analytic properties that we collect in the following lemma. To that purpose we use the notation $\overline{\mathrm{co}}\,K$ for the closed convex hull of a subset $K\in X$ and

$$\operatorname{fix}\big(T(t)\big)_{t\ge 0} := \bigcap_{t\ge 0}\operatorname{fix}\big(T(t)\big) = \big\{x\in X : T(t)x = x \text{ for all } t\ge 0\big\}$$

for the *fixed space* of a semigroup $\big(T(t)\big)_{t\ge 0}$. Recall that if $\big(T(t)\big)_{t\ge 0}$ is strongly continuous and has generator $A$, then

$$\operatorname{fix}\big(T(t)\big)_{t\ge 0} = \ker A$$

by Corollary IV.3.8.(i).

**4.2 Lemma.** *The Cesàro means $\big(C(r)\big)_{r\ge 0}$ of a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ with generator $\big(A,D(A)\big)$ satisfy the following properties.*

(i) *$C(r)x\in\overline{\mathrm{co}}\{T(t)x : t\ge 0\}$ for each $x\in X$.*

(ii) *$\big(I-T(t)\big)C(r) = C(r)\big(I-T(t)\big) = {}^1\!/\!_r\big(I-T(r)\big)\int_0^t T(s)\,ds$ for each $t,r>0$.*

(iii) *If $y:=\lim_{r\to\infty} C(r)x$ exists for some $x\in X$ and $\lim_{r\to\infty} {}^1\!/\!_r\,\|T(r)\| = 0$, then*

$$y\in\operatorname{fix}\big(T(t)\big)_{t\ge 0} = \ker A.$$

Proof. Assertion (i) follows from the definition of the Riemann integral, while (ii) is an immediate consequence of the semigroup law (FE).

To prove (iii), we take $y := \lim_{r\to\infty} C(r)x$ and conclude from (ii) that

$$\big(I - T(t)\big)y = \lim_{r\to\infty} \frac{1}{r}\big(I - T(r)\big)\int_0^t T(s)y\,ds = 0$$

for each fixed $t > 0$. This proves $y \in \operatorname{fix}\big(T(t)\big)_{t\geq 0}$. $\square$

Under the assumption $\lim_{r\to\infty} {}^1\!/_r\,\|T(r)\| = 0$, it follows from identity (ii) in Lemma 4.2 that $\lim_{r\to\infty} C(r)x$ exists for each $x$ of the form $x := y - T(t)y$ for some $y \in X$, $t > 0$.

In our new concept of asymptotic behavior we require convergence of the Cesàro means $C(r)$ for *all* $x \in X$.

**4.3 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called mean ergodic if*

$$\lim_{r\to\infty} C(r)x$$

*exists for all $x \in X$. For such semigroups, the operator*

$$P : x \mapsto Px := \lim_{r\to\infty} C(r)x$$

*will be called the (mean ergodic) projection associated to $\big(T(t)\big)_{t\geq 0}$.*

The reader interested in the roots of this notion in nineteenth century thermodynamics and in its manifold applications throughout mathematics and theoretical physics is referred to books on ergodic theory (e.g., [Kre85]). We discuss only the basic mathematical consequences of the above definition and explain first why we call the limit operator $P$ a "projection."

**4.4 Lemma.** *Let $\big(T(t)\big)_{t\geq 0}$ be a mean ergodic semigroup on a Banach space $X$ satisfying $\lim_{r\to\infty} {}^1\!/_r\,\|T(r)\| = 0$. For the associated projection $P$ we have*

$$P = T(t)P = PT(t) = P^2 \qquad \text{for all } t \geq 0.$$

*Therefore, $P$ is a projection decomposing $X$ into*

$$X = \operatorname{rg} P \oplus \ker P$$

*such that*

(i) $\operatorname{rg} P = \operatorname{fix}\big(T(t)\big)_{t\geq 0} = \ker A$ *and*

(ii) $\ker P = \overline{\operatorname{lin}}\{x - T(t)x : x \in X, t \geq 0\} = \overline{\operatorname{rg} A}$.

PROOF. The identity $P = T(t)P = PT(t)$ follows from Lemma 4.2.(ii) and implies

$$P = C(r)P = \Bigl(\lim_{r\to\infty} C(r)\Bigr) P = P^2.$$

Therefore, $P$ is a bounded projection and its range is $\operatorname{fix}\bigl(T(t)\bigr)_{t\geq 0}$ by Lemma 4.2.(iii). Its kernel $\ker P$ is closed and contains $\overline{\operatorname{lin}}\{x - T(t)x : x \in X, t \geq 0\}$ by Lemma 4.2.(ii). To show the converse inclusion, assume that a linear form $x' \in X'$ vanishes on each $x - T(t)x$. Then $T(t)'x' = x'$ for each $t \geq 0$ and therefore

$$\langle x, x'\rangle = \langle C(r)x, x'\rangle$$

for each $x \in X$ and $r > 0$. If we choose $x \in \ker P$, i.e., such that $\lim_{r\to\infty} C(r)x = 0$, then this implies $\langle x, x'\rangle = 0$. This shows that each continuous linear form vanishing on $\{x - T(t)x : x \in X, t \geq 0\}$ vanishes on $\ker P$. The Hahn–Banach theorem now yields the desired inclusion. $\square$

Bounded semigroups $\bigl(T(t)\bigr)_{t\geq 0}$ clearly satisfy $\lim_{r\to\infty} {}^1\!/_r \,\|T(r)\| = 0$, and their mean ergodicity can be characterized by the following series of quite different, but equivalent, properties.

**4.5 Theorem.** *Let $\bigl(T(t)\bigr)_{t\geq 0}$ be a strongly continuous semigroup with generator $\bigl(A, D(A)\bigr)$ on a Banach space $X$. If $\|T(t)\| \leq M$ for all $t \geq 0$, then the following assertions are equivalent.*

(a) *$\bigl(T(t)\bigr)_{t\geq 0}$ is mean ergodic.*

(b) *The Cesàro means $\bigl(C(r)\bigr)_{r>0}$ converge in the weak operator topology as $r \to \infty$.*

(c) *For each $x \in X$ there exists a monotone, unbounded sequence $(r_n)_{n\in\mathbb{N}}$ such that $(C(r_n)x)_{n\in\mathbb{N}}$ has a weak accumulation point in $X$.*

(d) *For each $x \in X$ one has $\overline{\operatorname{co}}\{T(t)x : t \geq 0\} \cap \operatorname{fix}\bigl(T(t)\bigr)_{t\geq 0} \neq \emptyset$.*

(e) *The fixed space $\operatorname{fix}\bigl(T(t)\bigr)_{t\geq 0} = \ker A$ separates the dual fixed space $\operatorname{fix}\bigl(T(t)'\bigr)_{t\geq 0} = \ker A'$.*

PROOF. The implications (a) ⇒ (b) ⇒ (c) are trivial.

To show that (c) ⇒ (d), assume that $y$ is contained in the weak closure of $\{C(r_n)x : n \geq m\}$ for each $m \in \mathbb{N}$ and some sequence $r_n \uparrow \infty$. Then, since each operator $T(t)$ is weakly continuous, $y - T(t)y$ is in the weak closure of

$$\bigl\{\bigl(I - T(t)\bigr)C(r_n)x : n \geq m\bigr\},$$

which again, by Lemma 4.2.(ii) and the fact that the unit ball $U := \{x \in X : \|x\| \leq 1\}$ is weakly closed, is contained in

$$\frac{1}{r_m}\left(tM + tM^2\right)\|x\|\, U.$$

Since this holds for each $m \in \mathbb{N}$, we obtain $y - T(t)y = 0$; hence $y \in \operatorname{fix}\big(T(t)\big)_{t\geq 0}$. Finally, since $C(r_n)x$ is contained in the weakly closed set $\overline{\text{co}}\{T(t)x : t \geq 0\}$ (use Proposition A.1.(i)), we conclude that $y \in \overline{\text{co}}\{T(t)x : t \geq 0\} \cap \operatorname{fix}\big(T(t)\big)_{t\geq 0}$.

(d) $\Rightarrow$ (e). Take $x', y' \in \ker A' = \operatorname{fix}\big(T(t)'\big)_{t\geq 0}$ such that $x' \neq y'$ and choose $x_0 \in X$ such that $\langle x_0, x'\rangle \neq \langle x_0, y'\rangle$.

By assumption (d) there exists $\bar{x} \in \overline{\text{co}}\{T(t)x_0 : t \geq 0\} \cap \operatorname{fix}\big(T(t)\big)_{t\geq 0}$. The linear forms $x'$ and $y'$, both belonging to $\operatorname{fix}\big(T(t)'\big)_{t\geq 0}$, remain constant on $\overline{\text{co}}\{T(t)x_0 : t \geq 0\}$. Therefore, we obtain

$$\langle \bar{x}, x'\rangle = \langle x_0, x'\rangle \neq \langle x_0, y'\rangle = \langle \bar{x}, y'\rangle ,$$

i.e., $\operatorname{fix}\big(T(t)\big)_{t\geq 0}$ separates $\operatorname{fix}\big(T(t)'\big)_{t\geq 0}$.

(e) $\Rightarrow$ (a). Consider the subspace

$$G := \operatorname{fix}\big(T(t)\big)_{t\geq 0} \oplus \operatorname{lin}\{x - T(t)x : x \in X, t \geq 0\}$$

of $X$ and take a linear form $x' \in X'$ vanishing on $G$. Since $x'$ vanishes on each element of the form $x - T(t)x$, this implies that $x' \in \operatorname{fix}\big(T(t)'\big)_{t\geq 0}$. However, $x'$ also vanishes on $\operatorname{fix}\big(T(t)\big)_{t\geq 0}$, which is assumed to separate $\operatorname{fix}\big(T(t)'\big)_{t\geq 0}$. As a conclusion, we obtain that $x' = 0$ and therefore $\overline{G} = X$. Since the Cesàro means converge for each $x \in G$ (use Lemma 4.2.(ii)) and since they form a bounded family, we have proved (by Proposition A.3) that they converge for each $x \in X$, i.e., $\big(T(t)\big)_{t\geq 0}$ is mean ergodic. □

The above equivalences are powerful tools to decide whether a given semigroup is mean ergodic or not. For example, property (c) immediately implies that the relatively compact semigroups studied in Section 2.b are always mean ergodic.

**4.6 Corollary.** *If a strongly continuous semigroup is relatively compact for the weak operator topology, then it is mean ergodic.*

The main examples are provided, as in Corollary 2.10, by bounded semigroups on reflexive Banach spaces.

**4.7 Example.** All bounded strongly continuous semigroups on reflexive Banach spaces are mean ergodic. The corresponding mean ergodic projection is nonzero if and only if 0 is an eigenvalue of the generator.

As the next step, we show that compactness of the resolvent, as in Corollary 2.15.(i), improves the type of convergence of the Cesàro means.

**4.8 Corollary.** *Let $(T(t))_{t\geq 0}$ be a bounded strongly continuous semigroup whose generator has compact resolvent. Then $(T(t))_{t\geq 0}$ is mean ergodic with projection $P$, and the Cesàro means converge in norm, i.e.,*

$$\lim_{r\to\infty} \|C(r) - P\| = 0.$$

PROOF. If the resolvent of $A$ is compact, we know from Proposition II.4.25 that the canonical injection $i : X_1 \to X$ is compact. (Recall that $X_1 := (D(A), \|\cdot\|_A)$, as in Definition II.5.1.) Moreover, the operator

$$V := \int_0^1 T(\tau)\, d\tau$$

is continuous from $X$ into $X_1$ (use (1.6) in Chapter II). Composing $V$ with the injection $i$, we conclude that $V$ is compact in $X$.

After this preparation, we observe that boundedness of the semigroup and compactness of the resolvent imply relative compactness of $(T(t))_{t\geq 0}$ in the strong, hence weak, operator topology (use Corollary 2.15). Therefore, $(T(t))_{t\geq 0}$ is mean ergodic by Corollary 4.6, and we denote its mean ergodic projection by $P$. Since $P$ is a projection onto $\operatorname{fix}(T(t))_{t\geq 0}$, one has $PV = P$ and

$$\big(C(r) - P\big)V = C(r)V - P \qquad \text{for all } r > 0.$$

We now use that $V$ is compact and that $\big(C(r) - V\big)$ converges pointwise, hence uniformly, on compact sets to zero as $r \to \infty$ (use Proposition A.3). Therefore, we obtain

$$\lim_{r\to\infty} \|C(r)V - P\| = 0.$$

On the other hand, one has

$$\begin{aligned} C(r)V - C(r) &= \frac{1}{r}\int_0^r \int_0^1 T(s)T(\tau)\, d\tau\, ds - \frac{1}{r}\int_0^r T(s)\, ds \\ &= \frac{1}{r}\int_0^1 \Big(\int_0^r T(s+\tau)\, ds - \int_0^r T(s)\, ds\Big)\, d\tau \\ &= \frac{1}{r}\int_0^1 \Big(\int_r^{r+\tau} T(s)\, ds - \int_0^\tau T(s)\, ds\Big)\, d\tau \end{aligned}$$

for all $r > 0$. It is now an easy consequence of the boundedness of $(T(t))_{t\geq 0}$ to show that

$$\lim_{r\to\infty} \|C(r)V - C(r)\| = 0.$$

By adding both limits, we obtain the desired conclusion. □

This corollary motivates us to have a closer look at the type of convergence appearing there.

**4.9 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space $X$ is called uniformly mean ergodic if $\lim_{r\to\infty} C(r)$ exists in the operator norm.*

Clearly, a uniformly mean ergodic semigroup is mean ergodic, and one has

$$\lim_{r\to\infty} \|C(r) - P\| = 0,$$

where $P$ is the associated mean ergodic projection. In analogy to Theorem 4.5, we now try to characterize uniform mean ergodicity by different properties. Here, spectral properties turn out to be particularly adequate.

**4.10 Theorem.** *For a bounded strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with generator $A$ on a Banach space $X$, the following conditions are equivalent.*

(a) *$\big(T(t)\big)_{t\geq 0}$ is uniformly mean ergodic.*

(b) *$\lim_{\lambda\downarrow 0} \lambda R(\lambda, A)$ exists in the operator norm.*

(c) *$\operatorname{rg} A$ is closed in $X$.*

(d) *$0 \in \rho(A)$ or $0$ is a first-order pole of the resolvent of $A$.*

Proof. (a) $\Rightarrow$ (d). Let $\big(T(t)\big)_{t\geq 0}$ be uniformly mean ergodic with corresponding projection $P$. Since $\operatorname{rg} P = \ker A$ and $\ker P = \overline{\operatorname{rg} A}$ (use Lemma 4.4 and (1.6) in Chapter II), we obtain the resolvent of $A$ as

$$R(\lambda, A) = \frac{1}{\lambda} P + R(\lambda, A_|)(I - P)$$

for $0 \neq \lambda \in \rho(A_|)$, where $A_|$ is the restriction of $A$ to $\ker P$ (use Proposition IV.2.15). For the proof of (d), it suffices to consider the case $0 \in \sigma(A)$. Then the above representation shows that $0$ is a first-order pole if $0 \notin \sigma(A_|)$. Assume the contrary, i.e., $0 \in \sigma(A_|) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$. By Proposition IV.1.10, there exist $x_n \in D(A_|) = D(A) \cap \ker P$, $\|x_n\| = 1$ such that $\|Ax_n\| \to 0$. This implies that

$$\big(T(s) - I\big)x_n = \int_0^s T(\tau)Ax_n\, d\tau \to 0$$

uniformly in $s \in [0, r]$ for every $r > 0$ (use (1.7) in Chapter II). In particular,

$$\big(C(r) - I\big)x_n \to 0$$

for every $r > 0$. This shows that $\big\|C(r)_{|\ker P}\big\| \geq 1$, while the uniform mean ergodicity implies

$$\big\|C(r)_{|\ker P}\big\| \to 0.$$

(d) ⇒ (b). If $0 \in \rho(A)$, we have $\lim_{\lambda \to 0} \lambda R(\lambda, A) = 0$. If 0 is a first-order pole, we can write the resolvent as

$$R(\lambda, A) = \frac{1}{\lambda} P + H(\lambda)$$

for $0 \neq \lambda$ in a suitable neighborhood of 0. Here, $\lambda \mapsto H(\lambda) \in \mathcal{L}(X)$ is analytic and $P$ is the residue of $R(\cdot, A)$ at 0. This implies

$$\lim_{\lambda \downarrow 0} \lambda R(\lambda, A) = P.$$

(b) ⇒ (c). Consider $Y := \overline{\operatorname{rg} A}$, which is a $\big(T(t)\big)_{t \geq 0}$-invariant subspace of $X$. Therefore, it is also $R(\lambda, A)$-invariant for all $\lambda > 0$ (use the integral representation (1.14) of the resolvent in Chapter II). Take $x \in D(A)$ and $y := Ax$. Then

$$\lambda R(\lambda, A) y = \lambda[\lambda R(\lambda, A)x - x] \to 0$$

as $\lambda \downarrow 0$, i.e., the operators $\lambda R(\lambda, A)$ converge to zero pointwise on $Y$. Assumption (b) now implies

$$\lim_{\lambda \downarrow 0} \left\| \lambda R(\lambda, A)_{|_Y} \right\| = 0.$$

From the identity $AR(\lambda, A) = \lambda R(\lambda, A) - I$, we conclude that $AR(\lambda, A)_{|_Y}$ must be invertible on $Y$; hence

$$\overline{\operatorname{rg} A} = AR(\lambda, A)Y \subset \operatorname{rg} A.$$

(c) ⇒ (a). For $y = Ax \in \operatorname{rg} A$, we have

$$\|rC(r)y\| = \left\| \int_0^r T(s)y \, ds \right\| = \left\| \int_0^r T(s)Ax \, ds \right\| = \|T(r)x - x\|;$$

hence

$$\overline{\lim_{r \to \infty}} \, \|rC(r)y\| < \infty.$$

By the uniform boundedness principle, this implies

$$\overline{\lim_{r \to \infty}} \, \|rC(r)_{|\operatorname{rg} A}\| < \infty,$$

and therefore

$$\lim_{r \to \infty} \|C(r)_{|\operatorname{rg} A}\| = 0. \tag{4.1}$$

Since the assumption $0 \in \sigma(A_{|_{\operatorname{rg} A}})$ implies, as in the proof of the implication (a) ⇒ (d), that $1 \in \sigma\big(C(r)_{|\operatorname{rg} A}\big)$, it follows from (4.1) that $A_{|\operatorname{rg} A}$ is invertible. Now, for $x \in D(A)$, choose $y \in \operatorname{rg} A$ with $Ax = Ay$. Then $x = (x - y) + y$ and $D(A) \subset \ker A \oplus \operatorname{rg} A$. Since $D(A)$ is dense, this proves $X = \ker A \oplus \operatorname{rg} A$, and the uniform mean ergodicity of $\big(T(t)\big)_{t \geq 0}$ follows. □

The spectral condition (d) in the above theorem allows us to improve Corollary 4.8.

**4.11 Corollary.** *Let $\big(T(t)\big)_{t\geq 0}$ be a bounded strongly continuous semigroup with generator $A$ on a Banach space $X$ satisfying one of the following conditions.*

(i) *The semigroup $\big(T(t)\big)_{t\geq 0}$ is quasi-compact.*

(ii) *The generator $A$ has compact resolvent.*

*Then, for each $\lambda \in \mathbb{R}$, the semigroups $\big(\mathrm{e}^{\mathrm{i}\lambda t}T(t)\big)_{t\geq 0}$ are all uniformly mean ergodic, and the associated mean ergodic projections have finite rank.*

Proof. Observe first that the assumptions (i) and (ii) also hold for the rescaled semigroups $\big(\mathrm{e}^{\mathrm{i}\lambda t}T(t)\big)_{t\geq 0}$. Therefore, the boundedness of $\big(T(t)\big)_{t\geq 0}$ and Theorem 3.7 or Corollary IV.1.19 imply that $\sigma(A) \cap \mathrm{i}\mathbb{R}$ consists of finitely many first-order poles having residue of finite rank. Since the generator of $\big(\mathrm{e}^{\mathrm{i}\lambda t}T(t)\big)_{t\geq 0}$ is $A+\mathrm{i}\lambda$, the assertions follow from Theorem 4.10.(d). □

We conclude this section by a series of examples by which the reader should realize how much the mean ergodicity of the "same" semigroup depends on the choice of the underlying Banach space.

**4.12 Examples.** (i) In the case of multiplication semigroups (cf. Definition I.4.3) on $\mathrm{C}_0(\Omega)$ given by

$$T(t)f := \mathrm{e}^{tq} \cdot f, \quad t \geq 0,$$

for some continuous function $q$, it is convenient to test mean ergodicity by using Theorem 4.5.(e). To guarantee boundedness of the semigroup, we assume $\operatorname{Re} q \leq 0$, and then identify the fixed space of $\big(T(t)\big)_{t\geq 0}$ as

$$\operatorname{fix}\big(T(t)\big)_{t\geq 0} = \big\{f \in \mathrm{C}_0(\Omega) : \operatorname{supp} f \subset [q=0]\big\},$$

where $[q=0] := \{s \in \Omega : q(s) = 0\}$. Similarly, the dual fixed space consists of all $\mu \in \mathrm{C}_0(\Omega)'$ such that $\operatorname{supp}\mu \subset [q=0]$. Therefore, the separation property from Theorem 4.5.(e) is satisfied if and only if $F$ separates all point measures $\delta_s$ with $s \in [q=0]$. This is the case if and only if $[q=0]$ is open (and closed) in $\Omega$.

If we consider the same multiplication semigroup on $\mathrm{L}^1(\Omega,\mu)$, we have

$$\operatorname{fix}\big(T(t)\big)_{t\geq 0} = \mathrm{L}^1([q=0],\mu) \qquad \text{and} \qquad \operatorname{fix}\big(T(t)'\big)'_{t\geq 0} = \mathrm{L}^\infty([q=0],\mu);$$

hence Theorem 4.5.(e) is satisfied, and $\big(T(t)\big)_{t\geq 0}$ is always mean ergodic on $\mathrm{L}^1(\Omega,\mu)$ and, by Example 4.7, on $\mathrm{L}^p(\Omega,\mu)$ for $1 < p < \infty$.

(ii) The left translation (semi) group $\big(T_l(t)\big)_{t\geq 0}$ (cf. Paragraph I.4.16) is always mean ergodic on the reflexive spaces $\mathrm{L}^p(\mathbb{R})$ and $\mathrm{L}^p(\mathbb{R}_+)$, $1 < p < \infty$. On $\mathrm{L}^1(\mathbb{R})$, however, it has trivial fixed space $\operatorname{fix}\big(T(t)\big)_{t\geq 0} = \{0\}$, while the adjoint group $\big(T_r(t)\big)_{t\in\mathbb{R}}$ (see Example (i) in Paragraph II.2.6) has one-dimensional fixed space $\operatorname{lin}\{\mathbb{1}\}$. By Theorem 4.5.(e), $\big(T_l(t)\big)_{t\in\mathbb{R}}$ is not mean ergodic on $\mathrm{L}^1(\mathbb{R})$.

This changes again for the semigroup $\big(T_l(t)\big)_{t\geq 0}$ on $\mathrm{L}^1(\mathbb{R}_+)$. There it becomes strongly stable (see Example 2.19.(i)), hence mean ergodic with projection $P = 0$.

The same argument applies on the space $\mathrm{C}_0(\mathbb{R}_+)$, while the translation semigroup is not mean ergodic on the larger space $\mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+)$ (see Paragraph I.4.16). This can be seen directly by calculating the Cesàro means for a function $f \in \mathrm{C}_{\mathrm{ub}}(\mathbb{R}_+)$ satisfying

$$f(s) := \begin{cases} 1 & \text{for } s \in \big[10^{2n}, 10^{2n+1} - 1\big], \\ -1 & \text{for } s \in \big[10^{2n+1}, 10^{2(n+1)} - 1\big], \end{cases}$$

for $n \in \mathbb{N}_0$.

**4.13 Exercises.** (1) Show that every bounded semigroup $(T(t))_{t\geq 0}$ on a reflexive Banach space $X$ is *totally ergodic*, i.e.,

$$P_\lambda x := \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathrm{e}^{\mathrm{i}\lambda s} T(s)x\, ds$$

exists for every $x \in X$ and $\lambda \in \mathbb{R}$. Characterize the subspaces $\operatorname{rg} P_\lambda$ as eigenspaces of the generator $A$ of $(T(t))_{t\geq 0}$.

(2) A continuous semiflow $\Phi : \mathbb{R}_+ \times \Omega \to \Omega$ (compare the definition in Paragraph II.3.28) on a compact space $\Omega$ is called *uniquely ergodic* if there exists a unique $\Phi$-invariant probability measure on $\Omega$ (see [Kre85, §5.1]). Show that for such flows the limit as $t \to \infty$ of

$$\frac{1}{t} \int_0^t f\big(\Phi(\tau, s)\big)\, d\tau$$

exists uniformly in $s \in \Omega$ and for every $f \in \mathrm{C}(\Omega)$. (Hint: Apply Theorem 4.5.(e).)

(3) Show the following properties for the left translation semigroup $(T(t))_{t\geq 0}$ on the Banach space $X := \{f \in \mathrm{L}^1(\mathbb{R}) : \int_{-\infty}^{\infty} f(s)\, ds = 0\}$.

- (i) $(T(t))_{t\geq 0}$ is mean ergodic with projection $P = 0$.
- (ii) The "discrete" Cesàro means $1/n \sum_{k=0}^{n-1} T(kt_0)$ do not converge as $n \to \infty$ whenever $t_0 > 0$.

# Notes to Chapter V

**Section 1.** Most of the results on the *asymptotic behaviour of semigroups of linear operators* presented in this chapter are contained and even extended in van Neerven's monograph [Nee96] (see also the short survey on the interplay between spectral theory and asymptotics in [Nag93]). A different approach to the asymptotic behavior of solutions of Cauchy problems, essentially based on the Laplace transform, is pursued in [ABHN99].

**Section 1.a.** Starting with [Sle76] and followed, e.g., by [Wei90] and [NSW95], many more growth bounds for semigroups $\mathcal{T} = (T(t))_{t\geq 0}$ have been introduced

and then characterized by spectral properties of the generator $A$. As the Banach space analogue of Theorem 1.11, it is shown in [WW96] that

$$\omega_1(\mathcal{T}) := \inf \left\{ w \in \mathbb{R} : \begin{array}{l} \text{for all } x \in D(A) \text{ exists } M_w(x) \geq 1 \text{ such} \\ \text{that } \|T(t)x\| \leq M_w(x)\, \mathrm{e}^{wt} \text{ for all } t \geq 0 \end{array} \right\}$$

is dominated by

$$\mathrm{s}_0(A) := \inf \big\{ w \in \mathbb{R} : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > w\} \subseteq \rho(A) \text{ and } \sup_{\operatorname{Re}\lambda > w} \|R(\lambda, A)\| < \infty \big\}$$

(see also [Nee96, Cor. 4.2.7]). This result shows that $\mathrm{s}_0(A) < 0$ implies exponential stability. Therefore, this can be considered as the most general Banach space version of Liapunov's stability theorem.

**Section** 1.b**.** Theorem 1.8 is due to R. Datko [Dat70] and A. Pazy [Paz72], while the short proof is taken from [PZ81]. It allows the following qualitative version: If

$$\int_0^\infty \|T(t)x\|^p \, dt \leq C^p \|x\|^p \qquad \text{for all } x \in X,$$

then $\omega_0 \leq -{}^1\!/_{(pC^p)}$, which is the best possible constant up to renorming of $X$ (see [Nee96, Thm. 3.1.8]).

Theorem 1.10 is due to Gearhart [Gea78], Prüss [Prü84], and Greiner [Nag86, A-III.7] and is of great practical importance (see Section VI.3 and Section VI.8). Our proof, however, uses ideas of M. Blake and S. Huang.

**Section** 1.c**.** Hyperbolicity is one of the basic concepts in the qualitative study of differential equations. We refer to [Cop78] and [DK74] for the classical theory and to [DGLW95], [Hen81], [HVL93], and [Lun95] for applications to nonlinear equations. The results presented follow from the previous spectral and stability theorems and are well known.

**Section** 2.a**.** The theory of semitopological semigroups contained in [Rup84] and [BJM89] has found applications in quite different branches of mathematics (see, e.g., [HLP90]). A systematic application to groups and semigroups of operators on Banach spaces is given in [Lyu88].

**Section** 2.b**.** More on the Jacobs–DeLeeuw–Glicksberg splitting theorem (i.e., Theorem 2.8) can be found in [Lyu88] and in [Kre85] with emphasis on applications to ergodic theory. Example 2.11.(ii) and a more detailed analysis of the complicated structure of weakly compact operator semigroups $\overline{\{T(t) : t \geq 0\}}^{\mathcal{L}_\sigma(X)}$ can be found in [BM71] and [Wes71].

**Section** 2.c**.** Operators generating compact groups as in Corollary 2.9 and Example 2.17 are said to have *discrete* (or *pure point*) *spectrum*. The classification of solenoidal groups is well known in harmonic analysis and is taken from [HR63]. The papers by Arendt–Batty [AB88] and Lyubich–Vũ [LV88] started the research on strongly stable semigroups and their spectral properties. The state of the art is surveyed in [Bat94] and [Vũ97].

**Section** 3**.** Our presentation closely follows [Nag86, B-IV, Sec. 2]. Some typical applications are treated in Sections VI.1 and VI.2. For a detailed study of semigroups with balanced exponential growth (see Exercise 3.9.(3)) we refer to [Thi98a], [Thi98b].

**Section** 4**.** The standard reference for ergodic theorems is the monograph by Krengel [Kre85], but the basic results are also contained in [Dav80]. We point out that ergodic theorems can be proved for much more general semigroups (see [Sch74, Sec. III.7]).

# Chapter VI

# Semigroups Everywhere

*I hail a semigroup when I see one and I seem to see them everywhere!*[1]

(Einar Hille [Hil48, Foreword])

It is only now that *evolution equations* or, more precisely, *initial value problems* will become the focus of our investigation. We will establish "well-posedness" for such equations and, in addition, investigate the qualitative properties of their solutions. To this end, we use *one-parameter semigroups* and the theory developed so far. As a general rule, we propose the following steps.

(i) Take an evolution equation (i.e., the initial value problem) and try to understand its physical, biological... significance.

(ii) Find a Banach space $X$ and a linear operator $A : D(A) \subset X \to X$ such that the original equation can be rewritten as an abstract Cauchy problem (see Definition II.6.1)

$$\text{(ACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0, \\ u(0) = x. \end{cases}$$

(iii) Show that $A$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$ and discuss how the solutions $t \mapsto u(t) := T(t)x$ of (ACP) yield solutions of the original problem.

(iv) Study the spectrum of $A$ and then the qualitative and, in particular, asymptotic behavior of $\big(T(t)\big)_{t\geq 0}$.

[1] *Friends have observed, however, that there are mathematical objects which are not semigroups* (Einar Hille, continued).

In the following sections we will try to perform, in a more or less complete way, this program. In each case, however, it is not our intention to give an exhaustive treatment of the particular equation. For this we will refer to the specialized literature. Instead, it is our aim to show how semigroups can be "seen everywhere" and that they yield a flexible and unifying tool for the study of many and quite different equations.

# 1. Semigroups for Population Equations

Semigroup methods have been applied with great success to equations arising from biomathematical models describing the growth (and/or properties like diffusion or convection) of certain populations. We refer to the monographs by Metz–Diekmann [MD86] and Webb [Web85] for a systematic treatment and concentrate here on a simple, but typical, type of equation.

We consider a population of cells that are distinguished by their individual size. Therefore, we can describe the population at time $t$ by the number $n(t,s)$ of cells having size $s$. More precisely,

$$\int_{s_1}^{s_2} n(t,s)\,ds$$

is the number of cells that at time $t$ have size $s$ between $s_1$ and $s_2$. As time passes, the following processes are supposed to take place in this population.

- Each cell grows linearly in time.
- Each cell dies with a probability depending on its size.
- Each cell divides into 2 daughter cells of equal size with a probability depending on its size.

Moreover, we assume that

- there exists a maximal cell size (normalized to $s = 1$) and
- there exists a minimal cell size $s = \alpha > 0$ after which division can occur.

As a consequence, we have that the size $s$ of each cell in our population must satisfy $s \geq {}^{\alpha}\!/_2$. From these assumptions the following evolution equation can be derived (see [MD86, Part A-I.4]):

$$\text{(CE)}\qquad \begin{aligned} \tfrac{\partial}{\partial t}n(t,s) = &- \tfrac{\partial}{\partial s}n(t,s) - \mu(s)n(t,s) - b(s)n(t,s) \\ &+ \begin{cases} 4b(2s)n(t,2s) & \text{for } {}^{\alpha}\!/_2 \leq s \leq {}^{1}\!/_2, \\ 0 & \text{for } {}^{1}\!/_2 < s \leq 1, \end{cases} \end{aligned}$$

with the boundary condition

$$n(t, {}^{\alpha}\!/_2) = 0 \qquad \text{for } 0 \leq t$$

and the initial condition

$$n(0,s) = n_0(s) \qquad \text{for } {}^{\alpha}\!/\!{}_2 \le s \le 1.$$

Moreover, we assume that the *death rate* $\mu$ is a positive, continuous function on $[{}^{\alpha}\!/\!{}_2, 1]$, while the *division rate* $b$ should be continuous with

$$b(s) > 0 \quad \text{for } s \in (\alpha, 1) \qquad \text{and} \qquad b(s) = 0 \quad \text{otherwise.}$$

Using one-parameter semigroups we will show "well-posedness" of this equation and discuss the qualitative properties of its solutions.

## a. Semigroup Method for the Cell Equation

We will, in a more or less complete way, perform the steps (ii)–(iv) outlined in the introduction to this chapter. Therefore, we start with the necessary definitions in order to rewrite (CE) as an abstract Cauchy problem. As a natural Banach space we choose $L^1[{}^{\alpha}\!/\!{}_2, 1]$, in which the norm $\|f\|$ of a positive function is the size of the total cell population represented by $f$.

**1.1 Definition.** *On the Banach space* $X := L^1[{}^{\alpha}\!/\!{}_2, 1]$ *define the operators*

$$A_0 f := -f' - (\mu + b) f \quad \text{with } D(A_0) := \bigl\{ f \in W^{1,1}[{}^{\alpha}\!/\!{}_2, 1] : f({}^{\alpha}\!/\!{}_2) = 0 \bigr\},$$

$$Bf(s) := \begin{cases} 4b(2s)f(2s) & \text{for } {}^{\alpha}\!/\!{}_2 \le s \le {}^{1}\!/\!{}_2, \\ 0 & \text{for } {}^{1}\!/\!{}_2 < s \le 1, \end{cases} \qquad \text{for all } f \in X,$$

$$A := A_0 + B \qquad \text{with } D(A) := D(A_0).$$

With these definitions our partial differential equation (CE) becomes the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \dot u(t) = A_0 u(t) + Bu(t) & \text{for } t \ge 0, \\ u(0) = n_0, \end{cases}$$

for the vector-valued function $u : \mathbb{R}_+ \to L^1[{}^{\alpha}\!/\!{}_2, 1]$. In order to show that $A = A_0 + B$ generates a strongly continuous semigroup on $X$, and hence that (ACP) is well-posed by Corollary II.6.9, we use perturbation methods. In fact, for the operator $A_0$ everything can be computed explicitly.

**1.2 Lemma.** (i) *The operator* $\bigl(A_0, D(A_0)\bigr)$ *generates a strongly continuous semigroup* $\bigl(T_0(t)\bigr)_{t\ge0}$ *on* $X$ *given by*

$$\text{(1.1)} \qquad T_0(t)f(s) := \begin{cases} e^{-\int_{s-t}^{s} (\mu(\tau)+b(\tau))\, d\tau} \cdot f(s-t) & \text{for } s-t > {}^{\alpha}\!/\!{}_2, \\ 0 & \text{elsewhere.} \end{cases}$$

(ii) *The spectrum of* $A_0$ *is empty. Moreover, the resolvent* $R(\lambda, A_0)$ *is a compact operator that is given by*

$$\text{(1.2)} \qquad R(\lambda, A_0)g(s) := \int_{{}^{\alpha}\!/\!{}_2}^{s} e^{-\int_{\tau}^{s} (\lambda+\mu(\sigma)+b(\sigma))\, d\sigma} \cdot g(\tau)\, d\tau$$

*for all* $g \in X$, ${}^{\alpha}\!/\!{}_2 \le s \le 1$, *and* $\lambda \in \mathbb{C}$.

PROOF. (i) The operator $A_0$ is the sum of the generator of the nilpotent right translation semigroup on $L^1[{}^{\alpha}\!/_2, 1]$ (cf. Paragraph II.2.11) and the bounded multiplication operator given by the function $m := -\mu - b$. The formula (1.1) now follows as in Exercise III.1.17. (5).

(ii) Since $\big(T_0(t)\big)_{t\geq 0}$ is a nilpotent semigroup, we have $\sigma(A_0) = \emptyset$. The resolvent $R(\lambda, A_0)$ is compact, since $D(A_0) \subset W^{1,1}[{}^{\alpha}\!/_2, 1] \overset{i}{\hookrightarrow} L^1[{}^{\alpha}\!/_2, 1]$ with compact injection $i$ (use Exercise II.4.30.(4) and Proposition II.4.25). Finally, the explicit formula (1.2) is obtained as the unique solution $f \in D(A_0)$, i.e., satisfying $f({}^{\alpha}\!/_2) = 0$, of the differential equation $(\lambda - A_0)f = \lambda f + f' + (\mu + b)f = g$. □

It now suffices to observe that $B$ is a bounded operator on $X$. By Theorem III.1.3, we conclude that $A_0 + B$ is again a generator.

**1.3 Proposition.** *The operator $\big(A, D(A)\big)$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$, and the above abstract Cauchy problem* (ACP) *is well-posed.*

This result yields solutions of the original cell equation (CE) (see Exercise 1.6), but in the following we concentrate on the qualitative properties of our semigroup.

While it was easy to obtain the generation property of $A$ from that of $A_0$ established in Lemma 1.2, it is not so clear which qualitative properties of the nilpotent semigroup $\big(T_0(t)\big)_{t\geq 0}$ are inherited by $\big(T(t)\big)_{t\geq 0}$ (compare Example III.1.15). We are particularly interested in compactness properties and note first that the resolvent of $A$ remains compact by Proposition III.1.12.(ii). It is much less evident that the semigroup $\big(T(t)\big)_{t\geq 0}$ remains eventually compact.

**1.4 Proposition.** *The semigroup $\big(T(t)\big)_{t\geq 0}$ is eventually norm continuous and even eventually compact for $t > 1 - {}^{\alpha}\!/_2$.*

PROOF. Using the Volterra operator $V$ introduced in Definition III.1.8 (with $A$ and $T(t)$ replaced by $A_0$ and $T_0(t)$, respectively), we first verify that the map $t \mapsto VT_0(t)$ is norm continuous for $t > 0$.

In order to simplify the notation we set[2]

$$E(s) := \mathrm{e}^{-\int_{\alpha/2}^{s} (\mu(\tau)+b(\tau))\, d\tau}$$

and observe that

$$T_0(t)f(s) = \frac{E(s)}{E(s-t)} \cdot f(s-t),$$

[2] In fact, ${}^{E(s)}\!/_{E(\sigma)}$ for $\sigma < s$ is the probability that a cell of size $\sigma$ reaches size $s$.

where we extend the functions $\mu$, $b$, and $f$ for arguments outside the interval $[{}^{\alpha}\!/_2, 1]$ by zero. Using this representation for $T_0(t)$ we obtain

$$\begin{aligned}\big(VT_0(t)f\big)(s) &= \left(\int_0^t T_0(t-r)BT_0(r)f\,dr\right)(s)\\ &= \int_{2s-2t}^{2s-t} \frac{E(s)}{E(t+\tau-s)}\cdot 4b\big(2(t+\tau-s)\big)\cdot\frac{E\big(2(t+\tau-s)\big)}{E(\tau)}\cdot f(\tau)\,d\tau\\ &=: \int_{2s-2t}^{2s-t} G(t,\tau,s)\cdot f(\tau)\,d\tau.\end{aligned}$$

Hence, for fixed $t > 0$ and $t > h > 0$, we can estimate

$$\begin{aligned}&\big\|(VT_0)(t+h)f - (VT_0)(t)f\big\|\\ &= \int_{\alpha/2}^1 \left|\int_{2s-2(t+h)}^{2s-(t+h)} G(t+h,\tau,s)\cdot f(\tau)\,d\tau - \int_{2s-2t}^{2s-t} G(t,\tau,s)\cdot f(\tau)\,d\tau\right|\,ds\\ &\le \int_{\alpha/2}^1\int_{2s-2t-2h}^{2s-2t} \big|G(t+h,\tau,s)\big|\cdot|f(\tau)|\,d\tau\,ds\\ &\quad+\int_{\alpha/2}^1\int_{2s-2t}^{2s-t-h} \big|G(t+h,\tau,s)-G(t,\tau,s)\big|\cdot|f(\tau)|\,d\tau\,ds\\ &\quad+\int_{\alpha/2}^1\int_{2s-t-h}^{2s-t} \big|G(t,\tau,s)\big|\cdot|f(\tau)|\,d\tau\,ds.\end{aligned}$$

Observe next that in the above integrals the arguments of $G = G(\cdot,\cdot,\cdot)$ run over a compact subset $\Omega$ of $\mathbb{R}^3$. Hence, since $G$ is continuous, it is uniformly continuous and bounded by some constant $K \ge 0$. Using these facts we obtain

$$\big\|(VT_0)(t+h)f-(VT_0)(t)f\big\| \le 3hK\,\|f\|+t\sup_{\tau,s}\big\|G(t+h,\tau,s)-G(t,\tau,s)\big\|\cdot\|f\|,$$

which converges to zero as $h \downarrow 0$ uniformly for $\|f\| \le 1$. Similarly, it follows that the difference $\big\|(VT_0)(t+h) - (VT_0)(t)\big\|$ converges to zero for $h \uparrow 0$, and hence $t \mapsto VT_0(t)$ is norm continuous for $t > 0$.

Thus, by Theorem III.1.16.(ii) we conclude that the perturbed semigroup $\big(T(t)\big)_{t\ge0}$ generated by $A_0 + B$ is norm continuous for $t > 1 - {}^{\alpha}\!/_2$. Moreover, we recall from Lemma 1.2.(ii) that $R(\lambda, A_0)$ is compact, and hence Proposition III.1.12.(ii) implies that $R(\lambda, A) = R(\lambda, A_0 + B)$ is compact as well. This implies that $R(\lambda, A)T(t)$ is compact and therefore $T(t)$ is compact for $t > 1 - {}^{\alpha}\!/_2$ by Lemma II.4.28. □

Once we have obtained the eventual compactness of $\big(T(t)\big)_{t\ge0}$, we know that the spectral mapping theorem holds (see Corollary IV.3.12) and that the behavior of $\big(T(t)\big)_{t\ge0}$ is described by the eigenvalues of $A$ (see Corollary V.3.2). So it remains to determine the spectrum $\sigma(A)$.

**1.5 Proposition.** *The spectrum $\sigma(A)$ of $A$ consists of eigenvalues only and is determined by a characteristic equation, more precisely,*

$$\lambda \in \sigma(A) \iff \xi(\lambda) = 0, \tag{1.3}$$

*where $\xi(\cdot)$ is the characteristic function*

$$\xi(\lambda) := -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma) \mathrm{e}^{-\int_\sigma^{2\sigma} (\mu(\tau)+b(\tau)+\lambda)\, d\tau}\, d\sigma \qquad \text{for } \lambda \in \mathbb{C}. \tag{1.4}$$

Proof. The first assertion is clear, since $A$ has compact resolvent (cf. Corollary IV.1.19). For the second statement we have to determine $\lambda \in \mathbb{C}$ for which there exists $0 \neq g \in D(A)$ such that

$$\lambda g - Ag = 0. \tag{1.5}$$

This equation means that

$$\lambda g(s) + g'(s) + \big(\mu(s) + b(s)\big) g(s) = 0 \qquad \text{for } 1/2 \le s \le 1,$$

and, by normalizing to $g(1) = 1$, that

$$g(s) = \mathrm{e}^{\int_s^1 (\mu(\sigma)+b(\sigma)+\lambda)\, d\sigma} \qquad \text{for } 1/2 \le s \le 1.$$

On the interval $[\alpha/2, 1/2]$, (1.5) means that

$$\lambda g(s) + g'(s) + \big(\mu(s) + b(s)\big) g(s) - 4b(2s) g(2s) = 0 \qquad \text{for } \alpha/2 \le s \le 1/2.$$

Since $g$ must be continuous at $s = 1/2$, we then obtain

$$g(s) = \mathrm{e}^{\int_s^1 (\mu(\sigma)+b(\sigma)+\lambda)\, d\sigma} \Big[ 1 - \int_s^{1/2} 4b(2\sigma)\, \mathrm{e}^{-\int_\sigma^{2\sigma} (\mu(\tau)+b(\tau)+\lambda)\, d\tau}\, d\sigma \Big]$$

for $\alpha/2 \le s \le 1/2$. In order to become an eigenfunction of $A$, the function $g \in \mathrm{W}^{1,1}[\alpha/2, 1]$ must also belong to $D(A)$; hence we need in addition

$$g(\alpha/2) = 0.$$

This yields (1.3) with the characteristic function $\xi$ as in (1.4). □

At this point it may seem that we have accomplished our task in a satisfactory way. However, a look at the characteristic function $\xi$ in (1.4) tells us that it will be quite difficult to determine all the zeros of $\xi$ or even to obtain information on the location of these zeros, as needed in stability theorems like Theorem V.1.10. In order to tackle this problem and to present a very elegant answer, we interrupt the discussion of the cell equation and introduce some additional abstract tools.

**1.6 Exercise.** Discuss in which sense the semigroup solutions of (ACP) above yield solutions of (CE).

## b. Intermezzo on Positive Semigroups

In the above model, as in many others arising from biology or physics (see also Section 2 and Section 6), there is a natural notion of "positivity," and only "positive" solutions of the equation make sense. In terms of the corresponding semigroup $\big(T(t)\big)_{t\geq 0}$ this means that the operators $T(t)$ should be "positive."

The rich theory of such "one-parameter semigroups of positive operators" can be found in [Nag86]. In the following we present only a few fundamental results in a concrete context.

For our purposes it suffices to restrict our attention to Banach spaces of type $X := \mathrm{L}^p(\Omega,\mu)$ or $\mathrm{C}_0(\Omega)$. On these spaces we call a function $f \in X$ *positive* (in symbols: $0 \leq f$) if

$$0 \leq f(s) \qquad \text{for (almost) all } s \in \Omega.$$

For real-valued functions $f, g \in X$ we then write $f \leq g$ if $0 \leq g - f$ and obtain an ordering making (the real part of) $X$ into a *vector lattice*; cf. [Sch74, Sec. II.1]. Moreover, for an arbitrary (complex-valued) function $f \in X$ we define its absolute value $|f|$ as

$$|f|(s) := |f(s)| \qquad \text{for } s \in \Omega.$$

Recalling the definition of the norm on $X$, we see that

$$(1.6) \qquad |f| \leq |g| \quad \text{implies} \quad \|f\| \leq \|g\| \qquad \text{for all } f, g \in X.$$

These properties make the space $X$ a *Banach lattice*, and we refer to [Sch74] or [AB85] for precise definitions. It will be convenient to use this general terminology and to state the results for general Banach lattices. However, the reader not accustomed to this terminology may always think of the space $X$ as one of the concrete function spaces $\mathrm{L}^p(\Omega,\mu)$ or $\mathrm{C}_0(\Omega)$ with the canonical ordering.

**1.7 Definition.** *A strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach lattice $X$ is called positive if each operator $T(t)$ is positive, i.e., if*

$$0 \leq f \in X \quad \textit{implies} \quad 0 \leq T(t)f \qquad \textit{for each } t \geq 0.$$

There are many ways to characterize positivity of a semigroup (mainly by properties of its generator; see [Nag86, C-II]). We give only a very elementary characterization.

**1.8 Characterization Theorem.** *A strongly continuous semigroup $\mathcal{T} := \big(T(t)\big)_{t\geq 0}$ on a Banach lattice $X$ is positive if and only if the resolvent $R(\lambda, A)$ of its generator $A$ is positive for all sufficiently large $\lambda$.*

Proof. The positivity of $\mathcal{T}$ implies the positivity of $R(\lambda, A)$ by the integral representation (1.13) in Section II.1. Conversely, the positivity of $T(t) = \lim_{n\to\infty}\bigl[{}^n\!/\!_t R({}^n\!/\!_t, A)\bigr]^n$ (see Corollary III.5.5) follows from that of $R(\lambda, A)$ for $\lambda$ large. ☐

In the years 1907–1912, O. Perron and G. Frobenius discovered very peculiar properties of the spectrum of positive matrices. Many of these properties still hold for the spectra of positive operators on arbitrary Banach lattices (cf. [Sch74, Secs. V.4&5]), and even carry over to generators of positive semigroups (cf. [Nag86]).

In order to prove the basic results of this theory, we need the following lemma. It shows that for positive semigroups the integral representation of the resolvent holds even for $\operatorname{Re}\lambda > \mathrm{s}(A)$ and not only for $\operatorname{Re}\lambda > \omega_0(A)$ as shown in Theorem II.1.10.

**1.9 Lemma.** *For a positive strongly continuous semigroup $(T(t))_{t\ge 0}$ with generator $A$ on a Banach lattice $X$ we have*

$$(1.7) \qquad R(\lambda, A)f = \int_0^\infty \mathrm{e}^{-\lambda s} T(s) f \, ds, \qquad f \in X,$$

*for all $\operatorname{Re}\lambda > \mathrm{s}(A)$. Moreover, the following properties are equivalent for $\lambda_0 \in \rho(A)$.*

(a) $0 \le R(\lambda_0, A)$.

(b) $\mathrm{s}(A) < \lambda_0$.

Proof. Using the rescaling techniques from Paragraph I.5.11 it suffices to prove the representation (1.7) for $\operatorname{Re}\lambda > 0$ whenever $\mathrm{s}(A) < 0$.

Since the integral representation (1.7) certainly holds for $\operatorname{Re}\lambda > \omega_0(A)$, we obtain from the positivity of $(T(t))_{t\ge0}$ the positivity of $R(\lambda, A)$ for $\lambda > \omega_0(A)$. The power series expansion (1.3) in Proposition IV.1.3 of the resolvent yields $0 \le R(\lambda, A)$ for all $\lambda > \mathrm{s}(A)$.

The assumption $\mathrm{s}(A) < 0$ and Lemma II.1.3.(iv) then imply

$$0 \le V(t) := \int_0^t T(s)\, ds = R(0, A) - R(0, A)T(t) \le R(0, A),$$

hence $\|V(t)\| \le M$ for all $t \ge 0$ and some constant $M$. From this estimate we deduce that

$$\int_0^\infty \mathrm{e}^{-\lambda s} V(s)\, ds, \qquad \operatorname{Re}\lambda > 0,$$

exists in operator norm. An integration by parts yields

$$\int_0^t \mathrm{e}^{-\lambda s} T(s)\, ds = \mathrm{e}^{-\lambda t} V(t) + \lambda \int_0^t \mathrm{e}^{-\lambda s} V(s)\, ds,$$

which converges to $\lambda \int_0^\infty \mathrm{e}^{-\lambda s} V(s)\, ds$ as $t \to \infty$. This proves (1.7) by Theorem II.1.10.(i) and then the implication (b) $\Rightarrow$ (a).

Moreover, as shown in Theorem 1.10 below, the integral representation (1.7) implies that $\mathrm{s}(A) \in \sigma(A)$. Therefore, by Corollary IV.1.14, we obtain for the spectral radius of the resolvent

$$\mathrm{r}\big(R(\lambda, A)\big) = \frac{1}{\lambda - \mathrm{s}(A)} \tag{1.8}$$

for all $\lambda > \mathrm{s}(A)$.

In order to prove (a) $\Rightarrow$ (b) we now assume that $R(\lambda_0, A) \geq 0$ and observe that this can be true only for $\lambda_0$ real. As we have shown above, $R(\lambda, A)$ is positive for $\lambda > \max\{\lambda_0, \mathrm{s}(A)\}$. Hence, an application of the resolvent equation yields

$$R(\lambda_0, A) = R(\lambda, A) + (\lambda - \lambda_0)R(\lambda, A)R(\lambda_0, A) \geq R(\lambda, A) \geq 0$$

for $\lambda > \max\{\lambda_0, \mathrm{s}(A)\}$. It follows from (1.8) and (1.6) that

$$\frac{1}{\lambda - \mathrm{s}(A)} = \mathrm{r}\big((R(\lambda, A))\big) \leq \|R(\lambda, A)\| \leq \|R(\lambda_0, A)\|$$

for all $\lambda > \max\{\lambda_0, \mathrm{s}(A)\}$. This implies that $\lambda_0$ is greater than $\mathrm{s}(A)$. ☐

The semigroup version of Perron's result from 1907 assuring that the spectral radius of a positive matrix is always an eigenvalue reads as follows.

**1.10 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be a positive strongly continuous semigroup with generator $A$ on a Banach lattice $X$. If $\mathrm{s}(A) > -\infty$, then*

$$\mathrm{s}(A) \in \sigma(A).$$

Proof. The positivity of the operators $T(t)$ means that

$$|T(t)f| \leq T(t)|f| \qquad \text{for all } f \in X,\ t \geq 0.$$

We therefore obtain from the integral representation (1.7) that

$$|R(\lambda, A)f| \leq \int_0^\infty \mathrm{e}^{-\operatorname{Re}\lambda\cdot s} T(s)|f|\, ds$$

for all $\operatorname{Re}\lambda > \mathrm{s}(A)$ and $f \in X$. Using the inequality in (1.6) we deduce that

$$\|R(\lambda, A)\| \leq \|R(\operatorname{Re}\lambda, A)\| \qquad \text{for all } \operatorname{Re}\lambda > \mathrm{s}(A). \tag{1.9}$$

By Corollary IV.1.14, there exist $\lambda_n \in \rho(A)$ such that $\operatorname{Re}\lambda_n \downarrow \mathrm{s}(A)$ and $\|R(\lambda_n, A)\| \uparrow \infty$. The estimate (1.9) then implies $\|R(\operatorname{Re}\lambda_n, A)\| \uparrow \infty$ and therefore $\mathrm{s}(A) \in \sigma(A)$ by Proposition IV.1.3.(iii). ☐

The arguments above now lead to the monotonicity of the spectral bound under positive perturbations.

**1.11 Corollary.** *Let $A$ be the generator of a positive strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and let $B \in \mathcal{L}(X)$ be a positive operator on the Banach lattice $X$. Then the following hold.*

(i) *$A+B$ generates a positive semigroup $\big(S(t)\big)_{t\geq 0}$ satisfying $0 \leq T(t) \leq S(t)$ for all $t \geq 0$.*

(ii) *$\mathrm{s}(A) \leq \mathrm{s}(A+B)$ and $R(\lambda, A) \leq R(\lambda, A+B)$ for all $\lambda > \mathrm{s}(A+B)$.*

Proof. Since $B$ is bounded, we obtain the generation property of $A+B$ from Theorem III.1.3. Moreover, the perturbed resolvent is

$$R(\lambda, A+B) = R(\lambda, A) + R(\lambda, A)\sum_{n=1}^{\infty}\big(BR(\lambda, A)\big)^n \qquad \text{for } \lambda \text{ large}$$

(see Section III.1, (1.3)). Since $B$ and $R(\lambda, A)$ are positive for $\lambda > \mathrm{s}(A)$, this implies

$$0 \leq R(\lambda, A) \leq R(\lambda, A+B) \tag{1.10}$$

for $\lambda$ large. The inequality in (i) then follows from the Post–Widder inversion formula in Corollary III.5.5. Next, we use the representation (1.7) for the resolvents of $A$ and $A+B$, respectively, and infer that (1.10) and hence

$$\|R(\lambda, A)\| \leq \|R(\lambda, A+B)\|$$

holds for all $\lambda > \max\{\mathrm{s}(A), \mathrm{s}(A+B)\}$. The inequality in (ii) for the spectral bounds then follows, since $\mathrm{s}(A) \in \sigma(A)$ by Theorem 1.10 and therefore $\overline{\lim}_{\lambda\downarrow \mathrm{s}(A)} \|R(\lambda, A)\| = \infty$. $\square$

The above elementary properties will be applied in Section 1.c below, in Section 2.b, and in Section 6 to positive semigroups arising from concrete evolution equations and will enormously facilitate the discussion of their qualitative behavior. Moreover, we will make use the following *Perron–Frobenius* results concerning the *boundary spectrum*

$$\sigma_+(A) := \sigma(A) \cap (\mathrm{s}(A) + \mathrm{i}\mathbb{R})$$

see [Nag86, C-III, Cor. 2.12, Thm. 3.12].

**1.12 Theorem.** *Let $\big(T(t)\big)_{t\ge 0}$ be a positive strongly continuous semigroup with generator $A$ on a Banach lattice $X$ such that $\sigma_+(A)$ consists of poles of the resolvent. Then the following assertions hold.*

(i) *The boundary spectrum $\sigma_+(A)$ is cyclic, i.e.,*

$$\left.\begin{array}{l}\mathrm{s}(A)+\mathrm{i}\alpha\in\sigma(A)\\ \text{for some } \alpha\in\mathbb{R}\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\mathrm{s}(A)+\mathrm{i}k\alpha\in\sigma(A)\\ \text{for all } k\in\mathbb{Z}.\end{array}\right.$$

(ii) *Assume, for simplicity, that $X := \mathrm{L}^1(\Omega,\mu)$ and let $\big(T(t)\big)_{t\ge 0}$ be irreducible, i.e.,*

$$0 \lneqq f\in X \;\Rightarrow\; \left\{\begin{array}{l}\big(R(\lambda,A)f\big)(s)>0 \text{ for almost}\\ \text{all } s\in\Omega \text{ and some } \lambda>\mathrm{s}(A).\end{array}\right.$$

*Then*

- *$\mathrm{s}(A)$ is a first-order pole of $R(\lambda,A)$ with one-dimensional residue $P$ such that $0 < Pf$ whenever $0 < f$, and*
- *$\sigma_+(A) = \mathrm{s}(A) + \mathrm{i}\alpha\mathbb{Z}$ for some $\alpha\in\mathbb{R}$.*

For many generalizations of this result we refer to [Nag86, C-III] and only state an immediate, but important, consequence.

**1.13 Corollary.** *If the positive strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ is eventually norm continuous and its generator has compact resolvent, then the boundary spectrum $\sigma_+(A)$ of its generator $A$ is equal to $\{\mathrm{s}(A)\}$.*

PROOF. It suffices to recall from Theorem II.4.18 that $\sigma(A)$ is bounded along imaginary lines. Since $\sigma_+(A)$ is cyclic by the theorem above, we must have $\sigma_+(A) = \{\mathrm{s}(A)\}$. □

Before concluding this short intermezzo, we look at stability properties of positive semigroups.

Using Theorem 1.10 it is often quite simple to determine the spectral bound $\mathrm{s}(A)$ of a positive semigroup $\big(T(t)\big)_{t\ge 0}$ with generator $A$. If $\mathrm{s}(A) < 0$, we can, in general, not conclude that $\omega_0 < 0$, i.e., the semigroup $\big(T(t)\big)_{t\ge 0}$ need not be uniformly exponentially stable (cf. Counterexample IV.2.7, which is a positive semigroup on a Banach lattice). However, it is one of the nice features of positive semigroups that exponential stability (see Definition V.1.5) always holds.

**1.14 Proposition.** *Let $\big(T(t)\big)_{t\ge 0}$ be a positive strongly continuous semigroup with generator $A$ on a Banach lattice $X$. Then the spectral bound $\mathrm{s}(A)$ satisfies $\mathrm{s}(A) < 0$ if and only if $\big(T(t)\big)_{t\ge 0}$ is exponentially stable.*

Proof. Let $\mathrm{s}(A) < -\varepsilon < 0$ and $f \in D(A)$. By the identity (1.11) in Lemma II.1.9, we have

$$\mathrm{e}^{\varepsilon t}T(t)t = f + \int_0^t \mathrm{e}^{\varepsilon s}T(s)(A+\varepsilon)f\, ds \qquad \text{for } t \geq 0.$$

The integral representation of the resolvent shown in Lemma 1.9 then implies that

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t}T(t)f = f + \int_0^\infty \mathrm{e}^{\varepsilon s}T(s)(A+\varepsilon)f\, ds$$

exists; hence

$$\lim_{t\to\infty} \mathrm{e}^{\varepsilon t/2}T(t)f = 0 \qquad \text{for all } f \in D(A).$$

The proof of the converse implication is left as Exercise 1.16.(3). □

However, if we restrict ourselves to Banach lattices $X := \mathrm{L}^p(\Omega,\mu)$, $1 \leq p < \infty$, then the following much stronger result has been proved by Derndinger [Der80] for $p = 1$, by Greiner–Nagel [GN83] for $p = 2$, and by Weis [Wei95], [Wei98] for arbitrary $p$ (see also [Nag86, C-IV.1] or [Nee96, Sec. 3.5]).

**1.15 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be a positive strongly continuous semigroup with generator $A$ on a Banach lattice $\mathrm{L}^p(\Omega,\mu)$, $1 \leq p < \infty$. Then*

$$\mathrm{s}(A) = \omega_0$$

*holds.*

**1.16 Exercises.** (1) Let $(T(t))_{t\geq 0}$ be a positive strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$ on a Banach lattice $X$.

(i) Show that $\mathrm{s}(A) = \inf\{\lambda > \mathrm{s}(A) : \sup_{\mu\in\mathbb{R}} \|R(\lambda+\mathrm{i}\mu, A)\| < \infty\}$. (Hint: Use Lemma 1.9.)

(ii) If $X$ is a $\mathrm{L}^2$-space, then $\mathrm{s}(A) = \omega_0(A)$. (Hint: Use Theorem V.1.11.)

(2) Let $(T(t))_{t\geq 0}$ satisfy all the assumptions in Theorem 1.12.(ii). Show that $\mathrm{s}(A)$ is a dominant eigenvalue and that $(T(t))_{t\geq 0}$ satisfies balanced exponential growth (see Exercise V.3.9.(3)). (Hint: Use Corollary V.3.3.)

(3) Show that a positive, exponentially stable strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator $A$ on a Banach lattice $X$ satisfies $\mathrm{s}(A) < 0$. (Hint: Use Theorem II.1.10.(i).)

## c. Asymptotics for the Cell Equation

We now return to the cell equation (CE) and the corresponding semigroup $\big(T(t)\big)_{t\geq 0}$ generated by the operator $A$ on the Banach lattice $X := \mathrm{L}^1[\alpha/2, 1]$ (see Definition 1.1 and Proposition 1.3). In order to apply the results from Section 1.b, we note from the explicit formulas in Lemma 1.2 and Definition 1.1 that the unperturbed semigroup $\big(T_0(t)\big)_{t\geq 0}$ and the perturbing operator $B$ are all positive. Therefore, Corollary 1.11.(i) guarantees the positivity of $\big(T(t)\big)_{t\geq 0}$.

**1.17 Corollary.** *The semigroup* $\big(T(t)\big)_{t\geq 0}$ *corresponding to the cell equation* (CE) *is positive on the Banach lattice* $X := \mathrm{L}^1[{}^{\alpha}\!/\!_2, 1]$.

The positivity of $\big(T(t)\big)_{t\geq 0}$ and Theorem 1.10 imply that the spectral bound $\mathrm{s}(A)$ is a spectral value of $A$. However, the spectrum $\sigma(A)$ is obtained as the zeros of the characteristic function $\xi$ (see (1.3) in Proposition 1.5). This function, restricted to $\mathbb{R}$, is continuous, strictly decreasing, with $\lim_{\lambda\to-\infty}\xi(\lambda) = +\infty$ and $\lim_{\lambda\to+\infty}\xi(\lambda) = -1$. Therefore, $\xi$ has a unique real zero $\lambda_0$, which by the above must be the spectral bound $\mathrm{s}(A)$.

**1.18 Lemma.** *The spectral bound* $\mathrm{s}(A)$ *is the unique* $\lambda_0 \in \mathbb{R}$ *such that* $\xi(\lambda_0) = 0$.

While it is already much easier to determine the real instead of all complex zeros of $\xi$, we are mainly interested in the sign of $\mathrm{s}(A)$. In fact, since $\big(T(t)\big)_{t\geq 0}$ is eventually norm continuous by Proposition 1.4, we know by Theorem V.1.10 (or directly by Theorem 1.15) that it is uniformly exponentially stable if and only if

$$\mathrm{s}(A) < 0.$$

A look at the behavior of $\xi$ as a function on $\mathbb{R}$ (see Figure 7)

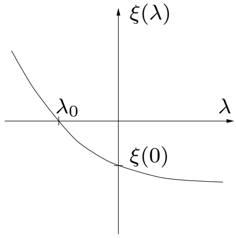



**Figure 7**

immediately yields the following criterion.

**1.19 Theorem.** *The semigroup* $\big(T(t)\big)_{t\geq 0}$ *corresponding to the cell equation* (CE) *is uniformly exponentially stable if and only if*

$$\xi(0) = -1 + \int_{\alpha/2}^{1/2} 4b(2\sigma)\mathrm{e}^{-\int_\sigma^{2\sigma}(\mu(\tau)+b(\tau))\,d\tau}\,d\sigma < 0. \tag{1.11}$$

This is a simple stability criterion for (CE) involving in a direct and computable manner only the given parameters $\mu$ and $b$. Exactly the same arguments apply to many more equations and yield similar stability criteria (see, e.g., Corollary 6.17).

We conclude our discussion of (CE) with a more precise description of the asymptotic behavior of the corresponding semigroup. This will be based on Theorem 1.12.(ii) and is valid for irreducible semigroups.

**1.20 Lemma.** *The semigroup* $(T(t))_{t\geq 0}$ *corresponding to the cell equation* (CE) *is irreducible on* $L^1[\alpha/2, 1]$.

PROOF. We verify the property of $R(\lambda, A)$ stated in Theorem 1.12.(ii) and use the formula for the perturbed resolvent

$$R(\lambda, A) = R(\lambda, A_0 + B) = R(\lambda, A_0) + R(\lambda, A_0)BR(\lambda, A_0) + \cdots$$

for $\lambda > s(A_0+B)$ (see (1.3) in the proof of Theorem III.1.3). By assumption, all the above summands are positive, and for $0 \leq f \in L^1[\alpha/2, 1]$ and $s_0 := \sup\{s \geq \alpha/2 : \operatorname{supp} f \subset [s, 1]\}$, the explicit formula (1.2) and the definition of $B$ yield

$$\begin{aligned} \big(R(\lambda, A_0)f\big)(s) &> 0 \qquad \text{for } s \in [s_0, 1], \\ \big(BR(\lambda, A_0)f\big)(s) &> 0 \qquad \text{for } s \in [s_0/2, 1/2]\,, \\ \big(R(\lambda, A_0)BR(\lambda, A_0)f\big)(s) &> 0 \qquad \text{for } s \in [s_0/2, 1]\,, \\ \big(BR(\lambda, A_0)BR(\lambda, A_0)f\big)(s) &> 0 \qquad \text{for } s \in [s_0/4, 1/2]\,. \end{aligned}$$

Continuing in this way, we obtain

$$R\big((\lambda, A)f\big)(s) > 0 \qquad \text{for all } s \in [\alpha/2, 1]\,.$$

□

We now collect the information we have gained so far for the semigroup $(T(t))_{t\geq 0}$ and apply the theoretical results from Section V.3.

- $(T(t))_{t\geq 0}$ is positive and eventually norm continuous (see Proposition 1.4); hence
$$\sigma_+(A) = \{s(A)\}$$
by Corollary 1.13.
- $(T(t))_{t\geq 0}$ is irreducible (Lemma 1.20), and its generator has compact resolvent (see Lemma 1.2.(ii) and Proposition III.1.12.(ii)); hence $s(A)$ is an eigenvalue that is a first-order pole with one-dimensional residue $P$ by Theorem 1.12.
- $(T(t))_{t\geq 0}$ is eventually compact (see Proposition 1.4); hence $s(A)$ is a *dominant eigenvalue* by Corollary V.3.2, i.e.,
$$\operatorname{Re}\lambda < s(A) \qquad \text{for all } \lambda \in \sigma(A) \setminus \{s(A)\}.$$

Combining these properties as in Corollary V.3.3, we obtain our final result.

**1.21 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be the strongly continuous semigroup on $X := \mathrm{L}^1[\alpha/2, 1]$ corresponding to the cell equation* (CE). *Then there exist a one-dimensional projection $P \in \mathcal{L}(X)$ and constants $\varepsilon > 0$, $M \geq 1$ such that*

$$\left\| \mathrm{e}^{-\,\mathrm{s}(A)\cdot t} T(t) - P \right\| \leq M \mathrm{e}^{-\varepsilon t} \qquad \text{for all } t \geq 0.$$

**1.22* Exercise.** Replace the equation (CE) by

$$\tfrac{\partial}{\partial t} n(t,s) = -\tfrac{\partial}{\partial s}\big(g(s)n(t,s)\big) - \mu(s)n(t,s) - b(s)n(t,s) + 4b(2s)n(t,2s)$$

for $\alpha/2 \leq s \leq 1$, $0 \leq t$, where $g$ is a differentiable "growth function" satisfying $0 < \varepsilon \leq g(s) \leq \delta$ for $s \in [\alpha/2, 1]$ and $2g(s) > g(2s)$ for $s \in [\alpha/2, 1/2]$. Prove for the corresponding semigroup the same results as above. (Hint: See [GN88].)

## *Notes and Further Reading to Section* 1

The two monographs [MD86] and [Web85] by Metz–Diekmann and Webb are the main references for the semigroup approach to population equations. The systematic use of positivity and the Perron–Frobenius theory can be found in [GN88] and [Hei86]. Greiner applied in [Gre84c] the same methods to a different population equation. More sophisticated equations involving several populations and their interactions are treated in [GW87], [Gra94], and [Ulm96].

# 2. Semigroups for the Transport Equation

A class of equations where semigroups and in particular positive semigroups have been applied with great success are linear transport (or Boltzmann) equations. In the following Section 2.a we discuss one particular transport equation describing the flow of neutrons in a reactor. The spectral and asymptotic behavior of the corresponding semigroup is then treated in Section 2.b.

## *a. Solution Semigroup for the Reactor Problem*

As a typical, but simple, example of a linear transport (or Boltzmann) equation we consider the so-called *reactor problem.*

**2.1 A Transport Equation.** We assume that $n(s,v,t)$ describes the density distribution of particles at position $s \in S$ with speed $v \in V$ at time $t \geq 0$. The *configuration space* $S$ is assumed to be a compact convex subset of $\mathbb{R}^3$ with nonempty interior, and the *velocity space* $V$ is

$$V := \big\{ v \in \mathbb{R}^3 : v_{\min} \leq \|v\|_2 \leq v_{\max} \big\}$$

for some minimal speed $v_{\min} > 0$ and some maximal speed $v_{\max} < \infty$. The particles are assumed

- to move according to their speed $v$,

- to be absorbed with probability $\sigma$ depending on the position $s$ and the speed $v$,
- to be scattered according to a scattering kernel $\kappa$ depending on the position $s$, the incoming speed $v'$, and the outgoing speed $v$.

These assumptions lead to the equation

$$\begin{aligned} \tfrac{\partial}{\partial t} n(s,v,t) = &- \sum_{i=1}^{3} v_i \tfrac{\partial}{\partial s_i} n(s,v,t) - \sigma(s,v) n(s,v,t) \\ &+ \int_V \kappa(s,v,v') n(s,v',t)\, dv' \end{aligned} \tag{2.1}$$

with initial value

$$n(s,v,0) = n_0(s,v)$$

and boundary conditions given by the domain of the operator $A_0$ below.

We now rewrite this equation as an abstract Cauchy problem in an appropriate Banach space.

**2.2 The Abstract Cauchy Problem.** As Banach space we take $X := \mathrm{L}^1(S \times V)$ with Lebesgue measure on $S \times V \subset \mathbb{R}^6$ and then define the *collisionless transport operator* $A_0$ by

$$\big(A_0 f\big)(s,v) := -\sum_{i=1}^{3} v_i \tfrac{\partial}{\partial s_i} f(s,v)$$

with suitable domain $D(A_0)$ (see the following paragraph), the *absorption operator*

$$\big(M_\sigma f\big)(s,v) := \sigma(s,v) \cdot f(s,v)$$

for some $0 \le \sigma \in \mathrm{C}(S \times V)$, and the *scattering operator*

$$\big(K_\kappa f\big)(s,v) := \int_V \kappa(s,v,v') f(s,v')\, dv'$$

for some $0 \le \kappa \in \mathrm{C}(S \times V \times V)$. The *transport*, or *Boltzmann, operator* is then

$$B := A_0 - M_\sigma + K_\kappa$$

with

$$D(B) := D(A_0).$$

With these definitions, equation (2.1) corresponds to the abstract Cauchy problem

$$\dot{u}(t) = Bu(t), \qquad u(0) = u_0 \tag{2.2}$$

in $\mathrm{L}^1(S \times V)$. Next we show that it can be solved by semigroup methods.

**2.3 The Streaming Semigroup.** We start from the strongly continuous semigroups $\big(T_0(t)\big)_{t\geq 0}$ and $\big(T(t)\big)_{t\geq 0}$ on $X$ given by

$$(2.3)\qquad \big(T_0(t)f\big)(s,v) := \mathbb{1}_S(s-vt)f(s-vt,v), \qquad \text{and}$$

$$(2.4)\qquad \big(T(t)f\big)(s,v) := \mathrm{e}^{-\int_0^t \sigma(s-v\tau,v)\,d\tau}\mathbb{1}_S(s-vt)f(s-vt,v)$$

for all $s \in S$, $v \in V$ and $f \in X$ (cf. Exercise 2.9.(1.i)). From these formulas and the assumption on $\sigma$ we immediately obtain that both semigroups consist of positive operators on $X$ and satisfy

$$(2.5)\qquad 0 \leq T(t) \leq T_0(t) \qquad \text{for all } t \geq 0.$$

The generator of $\big(T_0(t)\big)_{t\geq 0}$ is the above collisionless transport operator $A_0$, thereby defining its domain $D(A_0)$ (see Exercise 2.9.(1.ii)). Moreover, since $M_\sigma$ is a bounded perturbation, one obtains the generator of the *streaming semigroup* $\big(T(t)\big)_{t\geq 0}$ as

$$A := A_0 - M_\sigma, \qquad D(A) = D(A_0).$$

We finally note that due to the minimal speed $v_{\min} > 0$ and the compactness of $S$, both semigroups $\big(T_0(t)\big)_{t\geq 0}$ and $\big(T(t)\big)_{t\geq 0}$ are nilpotent.

**2.4 The Transport Semigroup.** We now perturb $A$ by the bounded scattering operator $K_\kappa$ and obtain the solution semigroup $\big(S(t)\big)_{t\geq 0}$ corresponding to the Cauchy problem (2.2).

**Theorem.** *The transport operator $B := A_0 - M_\sigma + K_\kappa$ on $\mathrm{L}^1(S \times V)$ generates the transport semigroup $\big(S(t)\big)_{t\geq 0}$ given by*

$$(2.6)\qquad S(t) = \sum_{n=0}^{\infty} S_n(t),$$

*where $S_0(t) := T(t)$ and*

$$S_{n+1}(t)f := \int_0^t S_n(t-\tau)K_\kappa T(\tau)f\,d\tau \qquad \text{for } f \in \mathrm{L}^1(S\times V),\ n \in \mathbb{N}.$$

For the proof it is enough to quote the Bounded Perturbation Theorem III.1.3 and to refer to the Dyson–Phillips expansion in Theorem III.1.10 (see also Exercise III.1.17.(3)). Moreover, since all terms in (2.6) are positive operators on the Banach lattice $\mathrm{L}^1(S \times V)$, we obtain preliminary information on the qualitative behavior of the transport semigroup.

**Corollary.** *For the streaming semigroup $\big(T(t)\big)_{t\geq 0}$ and the transport semigroup $\big(S(t)\big)_{t\geq 0}$ one has*

$$(2.7)\qquad 0 \leq T(t) \leq S(t) \qquad \text{for all } t \geq 0.$$

*Moreover, the growth bound $\omega_0$ of the transport semigroup $\big(S(t)\big)_{t\geq 0}$ coincides with the spectral bound $\mathrm{s}(B)$ of the transport operator $B$.*

Proof. Since $X$ is an $\mathrm{L}^1$-space and $\big(S(t)\big)_{t\geq 0}$ is a positive semigroup, the second assertion follows from Theorem 1.15. □

While this result already allows us to characterize the stability of the transport semigroup by spectral properties of the transport operator, it is our goal to obtain more precise information on the asymptotic behavior of $\big(S(t)\big)_{t\geq 0}$. In particular, we are interested in properties like balanced exponential growth (see Exercise V.3.9.(3)).

## *b. Spectral and Asymptotic Behavior*

The strategy is to compute the spectra $\sigma\big(S(t)\big)$ and then use the results from Sections IV.4 and V.3. For the streaming semigroup $\big(T(t)\big)_{t\geq 0}$ the situation is quite simple. In fact, it is a nilpotent semigroup, and hence we have $\sigma\big(T(t)\big) = \{0\}$ and $\sigma(A) = \emptyset$. In order to obtain information on $\sigma\big(S(t)\big)$ we will use Theorem IV.4.4 and therefore need certain compactness properties of the terms $S_n(t)$ in the Dyson–Phillips series (2.6).

It was G. Greiner in [Gre84b] who discovered that the order properties of the operators involved are helpful.

**2.5 Order Properties.** We first introduce some notation and then list some elementary properties.

The characteristic function $\mathbb{1}$ of $S \times V$ belongs to the Banach lattice $X = \mathrm{L}^1(S \times V)$ and to its dual space $X' = \mathrm{L}^\infty(S \times V)$. Therefore, we can define a one-dimensional operator $\mathbb{1} \otimes \mathbb{1} \in \mathcal{L}(X)$ by

$$(\mathbb{1} \otimes \mathbb{1})f := \Big(\int_S \int_V f(s,v)\, dv\, ds\Big) \cdot \mathbb{1} \qquad \text{for } f \in X.$$

This positive operator "dominates" the streaming semigroup $\big(T(t)\big)_{t\geq 0}$ in the following way.

(i) $T(t)\mathbb{1} \leq T_0(t)\mathbb{1} \leq \mathbb{1}$ for the order in $\mathrm{L}^1(S \times V)$.

(ii) $T(t)'\mathbb{1} \leq T_0'(t)\mathbb{1} \leq \mathbb{1}$ for the order in $\mathrm{L}^\infty(S \times V)$.

(iii) $T(t) \circ (\mathbb{1} \otimes \mathbb{1}) = \mathbb{1} \otimes \big(T(t)\mathbb{1}\big) \leq \mathbb{1} \otimes \mathbb{1}$ for the order in $\mathcal{L}(X)$.

(iv) $(\mathbb{1} \otimes \mathbb{1}) \circ T(t) = \big(T(t)'\mathbb{1}\big) \otimes \mathbb{1} \leq \mathbb{1} \otimes \mathbb{1}$ for the order in $\mathcal{L}(X)$.

These properties allow the decisive estimate for operator products involving the scattering operator.

**Lemma.** *With the above notation we have*

$$K_\kappa T(t) K_\kappa \leq \frac{\|\kappa\|_\infty^2}{t^3}\, \mathbb{1} \otimes \mathbb{1} \qquad \textit{for all } t > 0. \tag{2.8}$$

Proof. We first recall that $T(t) \le T_0(t)$ by (2.5). Moreover, we have

$$\big(K_\kappa f\big)(s,v) = \int_V \kappa(s,v,v')f(s,v')\,dv' \le \|\kappa\|_\infty \int_V f(s,v')\,dv'$$

for all $0 \le f \in X$ and $s \in S$, $v \in V$. This and the substitution $s' := s - v''t$ imply the estimate

$$\begin{aligned}
\big(K_\kappa T(t) K_\kappa f\big)(s,v) &\le \big(K_\kappa T_0(t) K_\kappa f\big)(s,v) \\
&\le \|\kappa\|_\infty^2 \int_V \int_V \mathbb{1}_S(s - v''t) f(s - v''t, v')\,dv''\,dv' \\
&\le \|\kappa\|_\infty^2 \int_V \int_S \mathbb{1}_S(s') f(s',v') t^{-3}\,ds'\,dv' \\
&\le \|\kappa\|_\infty^2 t^{-3} \int_V \int_S f(s',v')\,ds'\,dv' \\
&\le \|\kappa\|_\infty^2 t^{-3} \langle f, \mathbb{1}\rangle .
\end{aligned}$$

□

The estimate (2.8) shows that each operator $K_\kappa T(t) K_\kappa$ is dominated by a one-dimensional, hence by a compact, operator. The theorem of Aliprantis–Burkinshaw [AB80] (see also [AB85, Cor. 16.16]) then implies that its square is a compact operator. We now show that the same property holds for the operators $S_2(t)$ in the Phillips–Dyson series (2.6).

**Proposition.** *The square of the operators $S_2(t)$, defined by*

$$S_2(t)f := \int_0^t \int_0^\tau T(t-\tau)K_\kappa T(\tau - r) K_\kappa T(r) f\,dr\,d\tau$$

*for $f \in X$, is compact for every $t \ge 0$.*

Proof. For $\varepsilon > 0$ consider

$$S_2^\varepsilon(t) := \int_\varepsilon^t \int_0^{\tau-\varepsilon} T(t-\tau)K_\kappa T(\tau - r) K_\kappa T(r)\,dr\,d\tau$$

and note that

$$\|S_2(t) - S_2^\varepsilon(t)\| \le m^3 \|K_\kappa\|_\infty^2 \varepsilon t$$

for $m := \sup\{\|T(\tau)\| : 0 \le \tau \le t\}$. Since the square compact operators form a closed subspace in $\mathcal{L}(X)$, it suffices to show that the square of $S_2^\varepsilon(t)$ is compact for each $\varepsilon > 0$. Using the lemma and properties (iii) and (iv) above we obtain

$$T(t-\tau)K_\kappa T(\tau - r) K_\kappa T(r) \le \|\kappa\|_\infty^2 (\tau - r)^{-3} \mathbb{1} \otimes \mathbb{1},$$

hence

$$S_2^\varepsilon(t) \le \Big(\|\kappa\|_\infty^2 \int_\varepsilon^t \int_0^{\tau-\varepsilon} (\tau - r)^{-3}\,dr\,d\tau\Big)\, \mathbb{1} \otimes \mathbb{1}.$$

Again by the Aliprantis–Burkinshaw theorem quoted above, we obtain that the square of $S_2^\varepsilon(t)$ is compact. □

This proposition ensures that the assumptions of Theorem IV.4.4 are satisfied for $n = 2$, and we obtain

$$\mathrm{r}_{\mathrm{ess}}\big(S(t)\big) \le \mathrm{r}_{\mathrm{ess}}\big(T(t)\big). \tag{2.9}$$

Since we have $\mathrm{r}_{\mathrm{ess}}\big(T(t)\big) = \mathrm{r}\big(T(t)\big) = 0$ for all $t > 0$, we can describe the spectrum of $S(t)$ in the following way.

**2.6 Theorem.** (Greiner, 1984). *Let $B$ be the transport operator generating the transport semigroup $\big(S(t)\big)_{t\ge0}$ on $X = \mathrm{L}^1(S \times D)$. Then the following hold.*

(i) *The spectral mapping theorem* (SMT) *holds for $\big(S(t)\big)_{t\ge0}$, and for all $t > 0$ each $0 \ne \lambda \in \sigma\big(S(t)\big)$ is a pole of the resolvent and the corresponding residue is of finite rank.*

(ii) *For every $\lambda_0 \in \mathbb{R}$, the set $\{\lambda \in \sigma(B) : \operatorname{Re}\lambda \ge \lambda_0\}$ contains only finitely many elements, each of which is a pole of the resolvent with finite rank residue.*

(iii) *If $\mathrm{s}(B) > -\infty$, then $\mathrm{s}(B)$ is a dominant eigenvalue of $B$, i.e.,*

$$\sigma(B) \cap \big\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \mathrm{s}(B) - \varepsilon\big\} = \{\mathrm{s}(B)\}$$

*for some $\varepsilon > 0$.*

Proof. Assertion (i) follows from the inequality (2.9) and the definition of the essential spectrum (see Paragraph IV.1.20), while Corollary IV.2.11 implies (ii).

Finally, in case (iii) we have $\mathrm{s}(B) \in \sigma(B)$ by Theorem 1.10. Moreover, the boundary spectrum $\sigma_+(B) := \sigma(B) \cap (\mathrm{s}(B) + i\mathbb{R})$ is cyclic by Theorem 1.12.(i). Hence, if $\sigma_+(B) \ne \{\mathrm{s}(B)\}$, we must have infinitely many elements in $\sigma_+(B)$, contradicting (ii). Combined with (ii) this implies that $\mathrm{s}(B)$ is a dominant eigenvalue. □

If we now assume $\big(S(t)\big)_{t\ge0}$ to be irreducible, then we obtain from Theorem 1.12 our final result.

**2.7 Corollary.** *Assume that $\mathrm{s}(B) > -\infty$ and that $\big(S(t)\big)_{t\ge0}$ is irreducible. Then the transport semigroup $\big(S(t)\big)_{t\ge0}$ has balanced exponential growth. More precisely, there exists a one-dimensional projection $P$ satisfying $0 < Pf$ whenever $0 < f$ such that*

$$\big\|\mathrm{e}^{-\mathrm{s}(B)t}S(t) - P\big\| \le M\mathrm{e}^{-\varepsilon t}$$

*for all $t \ge 0$ and appropriate constants $M \ge 1$ and $\varepsilon > 0$.*

**2.8 Example.** If the scattering kernel $\kappa$ is strictly positive, i.e., $\kappa(s, v, v') > 0$ for all $s \in S$, $v, v' \in V$, then the transport semigroup $\big(S(t)\big)_{t\ge0}$ is irreducible; hence Corollary 2.7 applies. Moreover, it follows from [Pag86] that $\mathrm{r}\big(S_2(t)\big) > 0$, and therefore $\mathrm{s}(B) > -\infty$. See also Exercise 2.9.(2.iii).

**2.9 Exercises.** (1) Let $T_0(t)$ and $T(t)$, $t \geq 0$, be defined by (2.3) and (2.4) in Paragraph 2.3, respectively.

(i) Show that $(T_0(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are strongly continuous semigroups on $\mathrm{L}^1(S \times V)$.

(ii) Show that

$$D := \left\{ f \in \mathrm{W}^{1,1}(S \times V) : \begin{array}{l} f(s,v) = 0 \text{ if } s \in \partial S \text{ and} \\ s - vt \notin S \text{ for all } t > 0 \end{array} \right\}$$

is a core for both semigroups.

(2*) Discuss the transport equation (2.1) for $v_{\min} = 0$.

(i) Show that the spectral bound of the streaming operator is

$$\mathrm{s}(A) = -\inf\bigl\{\sigma(s,0) : s \in S\bigr\}.$$

(ii) Assume $\mathrm{s}(B) > \mathrm{s}(A)$ and prove that the transport semigroup has balanced exponential growth (see Exercise V.3.9.(3)).

(iii) Find conditions on the scattering kernel $\kappa$ implying irreducibility of the transport semigroup. In particular, show that it suffices that $\kappa$ not vanish in a neighborhood of the boundary of $S$. (Hint: See [Gre84a].)

(3*) Discuss the transport equation (2.1) for $S = \mathbb{R}^3$ and assume $\sigma$ and $\kappa$ to vanish outside a compact convex set in the configuration space $\mathbb{R}^3$. (Hint: See [Gre84b, Sec. 3].)

## *Notes and Further Reading to Section* 2

The use of spectral and order-theoretic methods for the transport equation has a long history. We mention [Bir59], [Jör58], and [Vid70]. Our presentation is based on the work of Greiner [Gre84b], whose results were improved by [Voi84] and [Tak85]. For more information on the underlying physics we refer to [KLH82], while more recent developments are presented in [BMM98] and [MK97].

# 3. Semigroups for Second-Order Cauchy Problems

In this section we will study the *abstract second-order Cauchy problem*

$$\text{(ACP}_2\text{)} \qquad \begin{cases} \ddot{u}(t) = B\dot{u}(t) + Au(t) & \text{for } t \geq 0, \\ u(0) = x, \ \dot{u}(0) = y \end{cases}$$

with closed linear operators $\bigl(A, D(A)\bigr)$ and $\bigl(B, D(B)\bigr)$ on a Banach space $X$. Problems of this type arise frequently in mathematical physics. For example, in the study of wave equations, $B$ can be interpreted as a *damping* (or *dissipation*) *operator* for the "undamped" (or *conservative*) abstract wave equation $\ddot{u}(t) = Au(t)$ governed by the *elastic operator* $A$.

In order to treat the second-order problem ($\mathrm{ACP}_2$) within our semigroup framework we will reduce it to a first-order Cauchy problem (ACP) on a bigger Banach space $\mathcal{X}$. To this end, we introduce the variable

$$v := \dot{u} \tag{3.1}$$

and obtain the (formally) equivalent system

$$\text{(ACP)} \qquad \begin{cases} \dot{\mathfrak{u}}(t) = \mathcal{A}\mathfrak{u}(t) & \text{for } t \geq 0, \\ \mathfrak{u}(0) = \mathfrak{x} \end{cases}$$

for $\mathfrak{u} := \binom{u}{v}$, the initial value $\mathfrak{x} := \binom{x}{y}$, and the operator

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}.$$

Before justifying this reformulation in the following subsections, we explain what is meant by a solution of ($\mathrm{ACP}_2$).

**3.1 Definition.** *A function $u : \mathbb{R}_+ \to X$ is called a (classical) solution of* ($\mathrm{ACP}_2$) *if*

(i) *$u$ is twice continuously differentiable,*
(ii) *$u(t) \in D(A)$ for all $t \geq 0$ and $Au : \mathbb{R}_+ \to X$ is continuous,*
(iii) *$\dot{u}(t) \in D(B)$ for all $t \geq 0$ and $B\dot{u} : \mathbb{R}_+ \to X$ is continuous, and*
(iv) *$u$ satisfies* ($\mathrm{ACP}_2$).

The problem is now to find an appropriate *state space* $\mathcal{X}$ and a domain for the reduction matrix $\mathcal{A}$ such that (ACP) corresponding to the matrix $\mathcal{A}$ is solvable and its solution yields a (unique) solution of the second-order problem ($\mathrm{ACP}_2$).

As we know from Theorem II.6.7, (ACP) is well-posed if and only if the operator $\mathcal{A}$ generates a strongly continuous semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ on $\mathcal{X}$. In this case, (ACP) is uniquely solvable for all initial values $\mathfrak{x} \in D(\mathcal{A})$, and the solution is given by the function $\mathfrak{u} : t \mapsto \mathcal{T}(t)\mathfrak{x}$. By Definition II.6.1 this means that $\mathfrak{u} \in \mathrm{C}^1(\mathbb{R}_+, \mathcal{X}) \cap \mathrm{C}(\mathbb{R}_+, \mathcal{X}_1^{\mathcal{A}})$ and that $\mathfrak{u}$ satisfies (ACP). In order to obtain a solution of the problem ($\mathrm{ACP}_2$) from the semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ we need the following condition.

**Condition** (S). If $\mathfrak{u} = \binom{u}{v}$ is a solution of (ACP) for the operator $\mathcal{A}$ and the initial value $\mathfrak{x} = \binom{x}{y}$ belonging to $D(\mathcal{A})$, then the first coordinate $u$ of $\mathfrak{u}$ is a solution of ($\mathrm{ACP}_2$) for the initial values $u(0) = x$ and $\dot{u}(0) = y$.

Our goal is now to construct, depending on the properties of $A$ and $B$, a state space $\mathcal{X}$ such that Condition (S) is satisfied. In this way we obtain a solution of ($\mathrm{ACP}_2$) whenever $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{X}$. First, however, we will show the uniqueness of the solution of ($\mathrm{ACP}_2$) obtained in this way.

**3.2 Proposition.** *Let $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}$ be the generator of a strongly continuous semigroup on the Banach space $\mathfrak{X} \subseteq X \times X$. If $\big(D(A) \cap D(B)\big) \times \big(D(A) \cap D(B)\big) \subseteq D(\mathcal{A})$ and $(X_1^A \cap X_1^B) \times X_1^B \hookrightarrow \mathfrak{X}$ with continuous injection, then* (ACP$_2$) *has at most one solution for each pair of initial values $\binom{x}{y} \in D(\mathcal{A})$.*

Proof. It suffices to show the uniqueness of the zero solution of (ACP$_2$) for the initial values $x = y = 0$. To this end, we take a solution $u$ of (ACP$_2$) with $u(0) = \dot{u}(0) = 0$. Then from Definition 3.1 it follows that

$$\dot{u} \in \mathrm{C}(\mathbb{R}_+, X_1^B) \qquad \text{and} \qquad u \in \mathrm{C}(\mathbb{R}_+, X_1^A) \cap \mathrm{C}(\mathbb{R}_+, X_1^B);$$

hence the map $\mathfrak{u} : \mathbb{R}_+ \to \mathfrak{X}$ with $\mathfrak{u}(t) := \binom{u(t)}{\dot{u}(t)}$ is well-defined and continuous. Moreover,

$$\int_0^t \mathfrak{u}(s)\, ds = \begin{pmatrix} \int_0^t u(s)\, ds \\ u(t) \end{pmatrix} \in \big(D(A) \cap D(B)\big)^2 \subseteq D(\mathcal{A})$$

and

$$\mathcal{A} \int_0^t \mathfrak{u}(s)\, ds = \begin{pmatrix} u(t) \\ A \int_0^t u(s)\, ds + Bu(t) \end{pmatrix} = \mathfrak{u}(t),$$

where the last equality follows by integrating (ACP$_2$). This shows that $\mathfrak{u}$ is a mild solution of (ACP) for $\mathcal{A}$ with zero initial value, and therefore $\mathfrak{u} = 0$ by Proposition II.6.4. □

## *a. The State Space $\mathfrak{X} = X_1^B \times X$*

The continuity condition $B\dot{u} \in \mathrm{C}(\mathbb{R}_+, X)$ in Definition 3.1.(iii) is in particular satisfied if $u \in \mathrm{C}^1(\mathbb{R}_+, X_1^B)$. Therefore, it is quite natural to choose the state space

$$\mathfrak{X} := X_1^B \times X.$$

In this way the first coordinate $u$ of the function $\mathfrak{u} : t \mapsto \mathcal{T}(t)\mathfrak{x}$ automatically satisfies condition (iii) in Definition 3.1 for every initial value $\mathfrak{x} \in D(\mathcal{A})$. We then consider $\mathcal{A}$ on $\mathfrak{X}$ with its "maximal" domain, i.e.,

$$\text{(3.2)} \qquad \mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}, \qquad D(\mathcal{A}) := \big(D(A) \cap D(B)\big) \times D(B).$$

It is now easy to verify that in this setting solutions of (ACP$_2$) can be obtained from solutions of (ACP).

**3.3 Lemma.** *Condition* (S) *holds for $\mathcal{A}$ defined by* (3.2) *on the space $\mathfrak{X} := X_1^B \times X$.*

Due to this lemma and Theorem II.6.7, we obtain a (classical) solution of (ACP$_2$), provided that $\big(\mathcal{A}, D(\mathcal{A})\big)$ generates a strongly continuous semigroup on $\mathfrak{X} = X_1^B \times X$. Since in this case the assumptions of Proposition 3.2 are satisfied, this solution will be also unique.

In order to study the generator property of $\mathcal{A}$, we first assume that $A$ is $B$-bounded, i.e., $D(B) \subseteq D(A)$ and $A \in \mathcal{L}(X_1^B, X)$ (cf. Definition III.2.1). This means that the damping operator $B$ is "more unbounded" than $A$ and is usually referred to as the *overdamped case*.

**3.4 Corollary.** *If $B$ generates a strongly continuous semigroup on $X$ and $A$ is $B$-bounded, then the second-order Cauchy problem* (ACP$_2$) *has a unique classical solution for all initial values $x, y \in D(B)$.*

Proof. We need to show only that the assumptions imply that $\mathcal{A}$ with domain $D(\mathcal{A}) = D(B) \times D(B)$ is a generator on $\mathfrak{X} = X_1^B \times X$. To this end we decompose $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_1 + \mathcal{B}_2$, where

$$\mathcal{A}_0 := \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \qquad D(\mathcal{A}_0) := D(\mathcal{A}),$$

and

$$\mathcal{B}_1 := \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{B}_2 := \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}.$$

Obviously, $\mathcal{A}_0$ is a generator on $\mathfrak{X}$, and the corresponding Sobolev space of order 1 is $\mathfrak{X}_1^{\mathcal{A}_0} = X_1^B \times X_1^B$. Hence $\mathcal{B}_1 \in \mathcal{L}(\mathfrak{X}_1^{\mathcal{A}_0})$, and from Corollary III.1.5 applied to the generator $\mathcal{A}_0$ and the perturbation $\mathcal{B}_1$ we obtain that $\mathcal{A}_1 := \mathcal{A}_0 + \mathcal{B}_1$ with domain $D(\mathcal{A}_1) := D(\mathcal{A})$ is a generator on $\mathfrak{X}$. Finally, $\mathcal{B}_2 \in \mathcal{L}(\mathfrak{X})$ and the Bounded Perturbation Theorem III.1.3 implies that $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}_2$ is a generator on $\mathfrak{X}$. ☐

**3.5 Example.** The previous result applies in particular if $A = CB$ for some bounded operator $C \in \mathcal{L}(X)$. More specific examples of this type are given by the Euler–Bernoulli beam with Kelvin–Voigt damping (cf. [Rus91, (1.04)], [Rus86, (3.8)]) or the Kirchhoff plate (see [Lag89, Chap. 6] or [LLP94]).

If we assume some additional regularity, we can even consider operators $A$ that are not $B$-bounded. In the following the product of operators is always defined as in Proposition B.2.

**3.6 Corollary.** *Assume that there exists $\lambda \in \rho(B)$ such that*

(i) *$AR(\lambda, B)$ is $B$-bounded with $B$-bound zero, and*

(ii) *$B$ and $AR(\lambda, B)$ both generate analytic semigroups.*

*Then the second-order Cauchy problem* (ACP$_2$) *has a unique classical solution for all initial values $x \in D(A) \cap D(B)$ and $y \in D(B)$.*

Proof. Again it suffices to prove that the assumptions imply that $\mathcal{A}$ generates a strongly continuous semigroup on $\mathfrak{X} = X_1^B \times X$, which, in fact, turns out to be analytic.

First, we observe that due to the Bounded Perturbation Theorem III.1.3, we may assume that $B$ is invertible. Otherwise, we can replace $B$ by $B-\lambda$ for some $\lambda \in \rho(B)$. For $\varepsilon > 0$ define the bounded operator $\mathcal{V}_\varepsilon$ from $X \times X$ to $\mathcal{X}$ by

$$\mathcal{V}_\varepsilon := \begin{pmatrix} -\varepsilon B^{-1} & B^{-1} \\ 0 & I \end{pmatrix}.$$

It is invertible with bounded inverse from $\mathcal{X}$ to $X \times X$ given by

$$\mathcal{V}_\varepsilon^{-1} = \begin{pmatrix} -{}^{B}\!/\!_{\varepsilon} & {}^{I}\!/\!_{\varepsilon} \\ 0 & I \end{pmatrix}.$$

It then follows from Paragraph II.2.1 that $\mathcal{A}$ is a generator on $\mathcal{X}$ if and only if $\mathcal{A}_\varepsilon := \mathcal{V}_\varepsilon^{-1}\mathcal{A}\mathcal{V}_\varepsilon$ is a generator on $X \times X$. Let $A_0 := -AB^{-1}$. A simple matrix calculation shows that $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \mathcal{B}_\varepsilon$ for

$$\mathcal{A}_0 := \begin{pmatrix} A_0 & 0 \\ 0 & B - A_0 \end{pmatrix} \qquad \text{with domain} \quad D(\mathcal{A}_0) := D(A_0) \times D(B), \text{ and}$$

$$\mathcal{B}_\varepsilon := \begin{pmatrix} 0 & -{}^{A_0}\!/\!_{\varepsilon} \\ \varepsilon A_0 & 0 \end{pmatrix} \qquad \text{with domain} \quad D(\mathcal{A}_0) := D(A_0) \times D(A_0).$$

By assumption, the operator $A_0$ generates an analytic semigroup on $X$. Moreover, by Theorem III.2.10, we know that $\big(B - A_0, D(A)\big)$; hence $\mathcal{A}_0$ generates an analytic semigroup. The assertion then follows from another application of Theorem III.2.10 to $\mathcal{A}_0$ and the perturbation $\mathcal{B}_\varepsilon$ if we can show that the $\mathcal{A}_0$-bound of $\mathcal{B}_\varepsilon$ converges to zero as $\varepsilon \downarrow 0$. To do so it suffices to prove that $A_0$ is $(B - A_0)$-bounded with $(B - A_0)$-bound equal to zero. Observe that for every $\delta \in (0,1)$ there exists $b_\delta > 0$ such that

$$\|A_0 z\| \le \delta\, \|Bz\| + b_\delta \|z\| \le \delta\big(\|(B - A_0)z\| + \|A_0 z\|\big) + b_\delta \|z\|;$$

hence

$$\|A_0 z\| \le \tfrac{\delta}{1-\delta}\, \|(B - A_0)z\| + \tfrac{b_\delta}{1-\delta}\, \|z\| \quad \text{for all } z \in D(B).$$

Since ${}^{\delta}\!/\!_{1-\delta}$ converges to 0 for $\delta \downarrow 0$, this proves the assertion. □

By doing some extra bookkeeping of the constants in the above proof one can also consider operators $A$ for which $AR(\lambda, B)$ is only $B$-bounded with sufficiently small $B$-bound; cf. Exercise 3.9.(2). In particular, one can deal with the following situation.

**3.7 Example.** Assume that $A = -\alpha B^2$ for $B$ the generator of an analytic semigroup and some $\alpha > 0$. Then, for sufficiently small $\alpha$, the operator matrix $\mathcal{A}$ defined by (3.2) generates an analytic semigroup on $\mathcal{X} = X_1^B \times X$; hence the associated Cauchy problem (ACP$_2$) permits a unique classical solution for all $x \in D(B^2)$, $y \in D(B)$. A specific example of this type is given by the Euler–Bernoulli beam with structural damping where

$$B = \tfrac{d^2}{dx^2}$$

with suitable boundary conditions and defined on the space $\mathrm{C}[0,1]$. For a detailed analysis of this model see [DBKS93].

**3.8 Remark.** The fact that the semigroup generated by the reduction matrix $\mathcal{A}$ of a second-order Cauchy problem is analytic is closely related to so-called *structural* (or *frequency proportional) damping*. The basic property of this kind of damping mechanism is that the damping rates of the eigenmodes of vibration are proportional to their frequencies, a feature that is consistent with extensive empirical studies. In terms of the operator $\mathcal{A}$ this means that its eigenvalues are contained in two half-lines bordering a sector in the left half-plane. For a detailed account of elastic systems exhibiting structural damping we refer to [CR82], [CT89], [DBKS93], [Hua90], [LLP94], and the references therein.

**3.9 Exercises.** (1) Prove Lemma 3.3.

(2) If in Corollary 3.6 we assume in condition (i) the operator $AR(\lambda, B)$ to be merely $B$-bounded, then the matrix

$$\mathcal{A}_\alpha := \begin{pmatrix} 0 & I \\ \alpha A & B \end{pmatrix}, \qquad D(\mathcal{A}) := \big(D(A) \cap D(B)\big) \times D(B),$$

generates an analytic semigroup on $\mathfrak{X} = X_1^B \times X$ for sufficiently small $\alpha > 0$. In particular, for those $\alpha$ the second-order Cauchy problem

$$\begin{cases} \ddot{u}(t) = B\dot{u}(t) + \alpha A u(t) & \text{for } t \geq 0, \\ u(0) = x, \ \dot{u}(0) = y \end{cases}$$

has a unique classical solution for all $x \in D(A) \cap D(B)$ and $y \in D(B)$.

(3) Assume that $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}$ defined by (3.2) generates a strongly continuous semigroup on $\mathfrak{X} := X_1^B \times X$. Then $\mathcal{A}_D := \begin{pmatrix} 0 & I \\ A+D & B \end{pmatrix}$ with domain $D(\mathcal{A}_D) := D(\mathcal{A})$ generates a strongly continuous semigroup for every $D \in \mathcal{L}(X_1^B, X)$. Discuss the consequences of this result for the perturbed second-order Cauchy problem

$$\begin{cases} \ddot{u}(t) = B\dot{u}(t) + (A + D)u(t) & \text{for } t \geq 0, \\ u(0) = x, \ \dot{u}(0) = y. \end{cases}$$

## b. The State Space $\mathfrak{X} = X \times X$

At first glance $X \times X$ seems to be the most natural candidate for the state space $\mathfrak{X}$. However, it turns out that for this choice of $\mathfrak{X}$ the generator property of $\mathcal{A}$ will produce a solution only for an "extended" second-order Cauchy problem $(\overline{\mathrm{ACP}_2})$.

In order to define the operator $\mathcal{A}$ on $\mathfrak{X} = X \times X$ we first recall from Exercise II.5.9.(5) which operators $A$ on $X$ can be extended to a bounded operator $\overline{A}$ from $X$ to the extrapolated Sobolev space $X_{-1}^B$ with respect to $B$ (cf. Section II.5.a).

**3.10 Lemma.** *If $A$ and $B$ are densely defined operators with $\rho(B) \neq \emptyset$, then the following assertions are equivalent.*

(a) $D(B') \subseteq D(A')$.

(b) $\overline{R(\lambda, B)A} \in \mathcal{L}(X)$ *for one (hence all)* $\lambda \in \rho(B)$.

(c) $A : D(A) \subseteq X \to X_{-1}^B$ *is bounded.*

*If one of these assertions is satisfied, we denote by $\overline{A}$ the unique bounded extension of $A$ to an operator from $X$ to $X_{-1}^B$.*

We now assume that one of the conditions (a)–(c) is satisfied and define

$$\tag{3.3} \begin{aligned} \mathcal{A} &:= \begin{pmatrix} 0 & I \\ \overline{A} & B_{-1} \end{pmatrix}, \\ D(\mathcal{A}) &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X \times X : \overline{A}x + B_{-1}y \in X \right\}. \end{aligned}$$

Then we have the following result for the operator $\mathcal{A}$.

**3.11 Proposition.** *Let $A$ and $B$ be densely defined with $\rho(B) \neq \emptyset$ and $D(B') \subset D(A')$. Then the reduction matrix $\mathcal{A}$ defined by* (3.3) *is a generator on $\mathfrak{X} = X \times X$ if and only if $B$ is a generator on $X$.*

Proof. By the Bounded Perturbation Theorem III.1.3 it suffices to show that

$$\widetilde{\mathcal{A}} := \begin{pmatrix} 0 & 0 \\ \overline{A} & B_{-1} \end{pmatrix}, \qquad D(\widetilde{\mathcal{A}}) := D(\mathcal{A}),$$

is a generator if and only if $B$ is a generator. As before, we assume $B$ to be invertible and then factorize $\widetilde{\mathcal{A}}$ as

$$\widetilde{\mathcal{A}} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & 0 \\ \overline{B^{-1}A} & I \end{pmatrix} =: \widetilde{\mathcal{A}}_0 \mathcal{S},$$

where $D(\widetilde{\mathcal{A}}_0) := X \times D(B)$. Since $\mathcal{S}$ is invertible, we conclude by Paragraph II.2.1 that $\widetilde{\mathcal{A}}$ is a generator if and only if

$$\mathcal{S}(\widetilde{\mathcal{A}}_0 \mathcal{S})\mathcal{S}^{-1} = \mathcal{S}\widetilde{\mathcal{A}}_0 = \widetilde{\mathcal{A}}_0$$

is. Obviously, the latter is the case if and only if $B$ is a generator, and the proposition is proved. □

Due to our choice of the state space $\mathfrak{X} = X \times X$ and the domain $D(\mathcal{A})$ of $\mathcal{A}$, a solution of (ACP) corresponding to $\mathcal{A}$ will not yield a solution of ($ACP_2$) in general. In fact, while it is clear that $\mathcal{D} := D(A) \times D(B)$ is contained in $D(\mathcal{A})$, there is no need for the solution $\mathfrak{u} : \mathbb{R}_+ \ni t \mapsto \mathcal{T}(t)\binom{x}{y} \in \mathfrak{X}$ of (ACP) to stay in $\mathcal{D}$ if it starts with an initial value $\binom{x}{y} \in \mathcal{D}$. For the first coordinate $u$ of $\mathfrak{u}$, which is our candidate for the solution of ($ACP_2$), this means that in general we *do not* necessarily have that $u(t) \in D(A)$ and $\dot{u}(t) \in D(B)$, conditions that are imposed in Definition 3.1.

These difficulties can be circumvented if instead of ($\mathrm{ACP}_2$) we consider the *extended abstract second-order Cauchy problem*

$$(\overline{\mathrm{ACP}_2}) \qquad \begin{cases} \ddot{u}(t) = B_{-1}\dot{u}(t) + \overline{A}u(t) & \text{for } t \geq 0, \\ u(0) = x, \ \dot{u}(0) = y. \end{cases}$$

For this problem we have the following result, whose proof is left as an exercise.

**3.12 Corollary.** *If $B$ generates a strongly continuous semigroup on $X$ and $A'$ is $B'$-bounded, then the extended second-order Cauchy problem ($\overline{\mathrm{ACP}_2}$) has a unique classical solution $u \in \mathrm{C}^2(\mathbb{R}_+, X)$ for all initial values $x, y \in X$ satisfying $\overline{A}x + B_{-1}y \in X$.*

Concrete examples of this type can easily be given using Exercise 3.13.(2).

**3.13 Exercises.** (1) Show that the reduction matrix $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}$ with domain $D(\mathcal{A}) := D(A) \times D(B)$ is a generator on $\mathfrak{X} := X \times X$ if and only if $A \in \mathcal{L}(X)$ and $B$ is a generator on $X$.
(2) Show that the conditions in Lemma 3.10 are satisfied if $B$ is densely defined with $\rho(B) \neq \emptyset$ and $A = BA_1 + A_2$ for bounded operators $A_1, A_2 \in \mathcal{L}(X)$.
(3) Prove Corollary 3.12.

## c. The State Space $\mathfrak{X} = X_1^C \times X$

The results obtained so far cover a wide variety of overdamped second-order Cauchy problems. However, to an undamped equation $\ddot{u}(t) = Au(t)$ these results can be applied only in the "trivial" situation of a bounded operator $A$. Moreover, for the previous choices of $\mathfrak{X}$, the norm in $\mathfrak{X}$ is, in general, not related to the energy of the system described by ($\mathrm{ACP}_2$).

Thus, we now study second-order problems where the elastic operator $A$ is, in some sense, the principal coefficient of the equation ($\mathrm{ACP}_2$). To this end, we assume that $X$ is a Hilbert space and that $A$ can be written as

$$A = -C^*C$$

for a densely defined, invertible operator $\big(C, D(C)\big)$ on $X$. This implies that $A$ is self-adjoint and negative definite (see [Wei80, Thm. 5.39]). In addition, we will assume that $B$ is dissipative on $X$. Under these assumptions the appropriate state space is the Hilbert space

$$\mathfrak{X} := X_1^C \times X$$

equipped with the scalar product

$$\left( \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ v \end{pmatrix} \right) := (Cx \,|\, Cu) + (y \,|\, v) \qquad \text{for } x, u \in D(C),\ y, v \in X.$$

In order to find conditions implying that the operator $\mathcal{A} = \left(\begin{smallmatrix} 0 & I \\ A & B \end{smallmatrix}\right)$ is a generator, we consider its inverse, which is (formally) given by

$$\mathcal{A}^{-1} := \begin{pmatrix} -\overline{A^{-1}B} & A^{-1} \\ I & 0 \end{pmatrix}. \tag{3.4}$$

This should be a *bounded* operator on $\mathfrak{X}$, which is the case if and only if we assume that

$$\begin{cases} D(A^{-1}B) = D(B) \cap D(C) \text{ is dense in } X_1^C, \text{ and} \\ \overline{A^{-1}B} \in \mathcal{L}(X_1^C) \end{cases} \tag{3.5}$$

or, equivalently,

$$\begin{cases} D\big((C^*)^{-1}BC^{-1}\big) = C\big(D(B) \cap D(C)\big) \text{ is dense in } X, \text{ and} \\ Q := \overline{(C^*)^{-1}BC^{-1}} \in \mathcal{L}(X). \end{cases} \tag{3.6}$$

Under these assumptions, the operator

$$\mathcal{A}_0 := \begin{pmatrix} 0 & I \\ -C^*C & B \end{pmatrix}, \qquad D(\mathcal{A}_0) := D(C^*C) \times \big(D(B) \cap D(C)\big),$$

is densely defined on $\mathfrak{X}$. Moreover, a simple calculation shows that $\mathcal{A}_0$ is dissipative; hence by Proposition II.3.14.(iv) it is closable, and its closure

$$\mathcal{A} := \overline{\mathcal{A}_0} \tag{3.7}$$

is dissipative as well.

Before discussing the relationship between the solutions of (ACP$_2$) and (ACP), we study the generator property of $\mathcal{A}$.

**3.14 Proposition.** *Let $A = -C^*C$ for a densely defined, invertible operator $\big(C, D(C)\big)$ on the Hilbert space $X$. Moreover, assume that $B$ is dissipative and that* (3.5) *(or* (3.6)*) is satisfied. Then $\mathcal{A}$ defined by* (3.7) *generates a contraction semigroup on $\mathfrak{X} := X_1^C \times X$.*

Proof. We introduce the operator

$$\begin{aligned} \widetilde{\mathcal{A}} &:= \begin{pmatrix} C^{-1} & 0 \\ 0 & C^* \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & Q \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \\ D(\widetilde{\mathcal{A}}) &:= \left\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in D(C) \times D(C) : Cx - QCy \in D(C^*) \right\}. \end{aligned} \tag{3.8}$$

Then $\mathcal{A}_0 \subseteq \widetilde{\mathcal{A}}$ and

$$\begin{aligned} \widetilde{\mathcal{A}} D(\mathcal{A}_0) &= \left\{ \left(\begin{smallmatrix} y \\ -C^*Cx + By \end{smallmatrix}\right) : \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in D(C^*C) \times \big(D(B) \cap D(C)\big) \right\} \\ &\supset \big(D(B) \cap D(C)\big) \times X \end{aligned}$$

is dense in $\mathfrak{X}$. Moreover, $\widetilde{\mathcal{A}}$ is invertible with inverse given by the right-hand side of (3.4). Hence, by Exercise II.1.15.(2), $D(\mathcal{A}_0)$ is a core for $\widetilde{\mathcal{A}}$, and therefore $\widetilde{\mathcal{A}} = \overline{\mathcal{A}_0} = \mathcal{A}$. Since the assumptions on $A$ and $B$ imply that $\mathcal{A}$ is dissipative and densely defined, the assertion follows from the Lumer–Phillips Theorem II.3.15. □

As in the previous subsection, we now have the problem that a solution of (ACP) corresponding to $\mathcal{A}$ will, in general, not give rise to a solution of ($\mathrm{ACP}_2$). In fact, if $\mathcal{A}$ is a generator on $\mathfrak{X} = X_1^C \times X$, we are only able to find a solution $u \in \mathrm{C}^2(\mathbb{R}_+, X)$ of the second-order problem

$$\begin{cases} \ddot{u}(t) = C^*\big(QC\,\dot{u}(t) - C\,u(t)\big) & \text{for } t \geq 0, \\ u(0) = x,\ \ \dot{u}(0) = y, \end{cases}$$

with initial values $x, y \in D(C)$ satisfying $Cx - QCy \in D(C^*)$. However, if we impose an additional assumption on the domain of $B$, Condition (S) will be satisfied.

**3.15 Lemma.** *If $B$ is dissipative and $D(C) \subseteq D(B)$, then* (3.5) *(or, equivalently,* (3.6)*) holds, $D(\mathcal{A}) = D(A) \times D(C)$, and Condition* (S) *is satisfied.*

The proof is left as Exercise 3.21.(2). Together with Proposition 3.2 we now immediately obtain the following result.

**3.16 Corollary.** *Let $A = -C^*C$ for a densely defined, invertible operator $\big(C, D(C)\big)$ on a Hilbert space $X$. Moreover, assume that $B$ is dissipative and $D(C) \subseteq D(B)$. Then the second-order Cauchy problem* ($\mathrm{ACP}_2$) *has a unique classical solution for all initial values $x \in D(A)$ and $y \in D(C)$.*

**3.17 Example.** On the interval $[0, 1]$ we consider the second-order Cauchy problem

$$\text{(3.9)}\qquad \begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} = b\,\dfrac{\partial^3 u(t,x)}{\partial t\,\partial x^2} - \dfrac{\partial}{\partial x}\Big(a(x)\,\dfrac{\partial u(t,x)}{\partial x}\Big) & \\ \qquad\qquad - c\,\dfrac{\partial^4 u(t,x)}{\partial x^4}, & t \geq 0,\ x \in [0,1], \\ u(t,x) = 0 = \dfrac{\partial^2 u(t,x)}{\partial x^2}, & t \geq 0,\ x = 0, 1, \\ u(0,x) = u_0(x), \quad \dfrac{\partial u(0,x)}{\partial t} = u_1(x), & x \in [0,1], \end{cases}$$

for $a \in \mathrm{C}^1[0,1]$, $b \in \mathbb{C}$ with $\operatorname{Re} b \geq 0$ and $c > 0$.

In order to apply our previous results, we rewrite (3.9) as an abstract second-order Cauchy problem ($\mathrm{ACP}_2$) on $X := \mathrm{L}^2[0,1]$. To that purpose we introduce the operators

$$\begin{aligned} A &:= -c\,\Delta^2, & D(A) &:= \big\{f \in \mathrm{H}_0^4[0,1] : f''(0) = 0 = f''(1)\big\}, \\ B &:= b\,\Delta, & D(B) &:= \mathrm{H}_0^2[0,1], \\ C &:= \sqrt{c}\,\Delta, & D(C) &:= \mathrm{H}_0^2[0,1], \\ D &:= -D_m M_a D_0, & D(D) &:= \mathrm{H}_0^2[0,1]. \end{aligned}$$

We refer to Appendix A for the definition of the spaces $\mathrm{H}_0^k[0,1]$. Moreover, $D_m$ and $D_0$ denote the first derivative with maximal domain and Dirichlet boundary conditions, respectively, $\Delta := D_m D_0$ is the Laplacian with Dirichlet boundary conditions, and $M_a$ stands for the multiplication operator induced by the function $a$. Then $A = -C^*C$, where $C = C^* = \sqrt{c}\, D_m D_0$ is positive definite (cf. Exercise II.4.12.(12)), $B$ is dissipative with $D(C) = D(B)$, and $D \in \mathcal{L}(X_1^C, X)$. Hence, by Corollary 3.16 and Exercise 3.21.(3) the partial differential equation (3.9) has a unique classical solution for all initial values $u_0 \in D(A)$, $u_1 \in D(C)$. The uniform exponential stability of this solution is treated in Exercise 3.21.(6).

Next, we are interested in the asymptotic behavior of the solution of $(\mathrm{ACP}_2)$. To this end, we suppose that for $\big(B, D(B)\big)$ there exist constants $\gamma \geq 0$ and $\delta > 0$ such that

$$\big|\mathrm{Im}(By \,|\, y)\big| \leq \gamma\, \mathrm{Re}(-By \,|\, y) \qquad \text{and} \qquad \delta \|y\|^2 \leq \mathrm{Re}(-By \,|\, y) \tag{3.10}$$

for all $y \in D(B)$. We then obtain the following result.

**3.18 Theorem.** *In addition to the assumptions made in Proposition* 3.14, *let* $\big(B, D(B)\big)$ *satisfy the estimates* (3.10). *Then the following holds.*

(i) *If* $\gamma > 0$, *then*

$$\omega_0(\mathcal{A}) \leq w,$$

*where* $w \in (-\delta/2, 0)$ *is the unique solution of the equation*

$$w^2 + \frac{w^2\gamma^2\delta^2}{(\delta + 2w)^2} = \left\|\mathcal{A}^{-1}\right\|^{-2},$$

*cf. Figure* 8.

(ii) *If* $\gamma = 0$, *then*

$$\omega_0(\mathcal{A}) \leq w := \max\Big\{-\frac{\delta}{2},\, -\left\|\mathcal{A}^{-1}\right\|^{-1}\Big\}.$$

*In particular, the solution of* $(\mathrm{ACP}_2)$ *tends exponentially to zero as* $t \to \infty$.

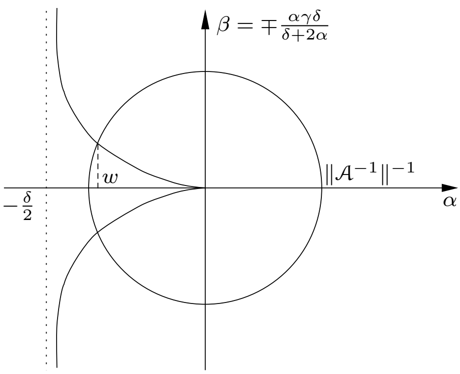



**Figure 8**

The proof of this result is based on the theorem of Gearhart–Greiner–Prüss V.1.11 and the following technical lemma.

**3.19 Lemma.** *Let* $\varepsilon > 0$ *and* $\alpha \in (-\delta/2 + \varepsilon, 0]$. *If*

$$\inf_{\mathfrak{x}\in D(\mathcal{A}),\, \|\mathfrak{x}\|=1} \|(\alpha + \mathrm{i}\beta - \mathcal{A})\mathfrak{x}\| < \varepsilon,$$

*then*

$$|\beta| < \frac{(\varepsilon - \alpha)\gamma + 3\varepsilon}{\delta - 2(\varepsilon - \alpha)} \cdot \delta.$$

Proof. Let $\mathfrak{x} := \binom{x}{y} \in D(\mathcal{A})$ satisfy $\|\mathfrak{x}\| = 1$ and $\|(\alpha + \mathrm{i}\beta - \mathcal{A})\mathfrak{x}\| < \varepsilon$. Then, since $D(\mathcal{A}_0) = D(C^*C) \times \big(D(B) \cap D(C)\big)$ is a core for $\mathcal{A}$, we may assume that $\mathfrak{x} \in D(\mathcal{A}_0)$ and therefore obtain

$$\|(\alpha + \mathrm{i}\beta - \mathcal{A})\mathfrak{x}\| = \left\| \begin{pmatrix} \alpha x + \mathrm{i}\beta x - y \\ C^*Cx + \alpha y + \mathrm{i}\beta y - By \end{pmatrix} \right\| < \varepsilon. \tag{3.11}$$

Thus $\big|\big((\alpha + \mathrm{i}\beta - \mathcal{A})\mathfrak{x} \,|\, \mathfrak{x}\big)\big| < \varepsilon$, i.e.,

$$\big|\alpha + \mathrm{i}\beta + 2\mathrm{i}\,\mathrm{Im}(Cx \,|\, Cy) - \mathrm{Re}(By \,|\, y) - \mathrm{i}\,\mathrm{Im}(By \,|\, y)\big| < \varepsilon, \tag{3.12}$$

and therefore

$$\big|\mathrm{Re}(By \,|\, y) - \alpha\big| < \varepsilon.$$

This, together with (3.10), implies

$$\delta\|y\|^2 \le -\mathrm{Re}(By \,|\, y) < \varepsilon - \alpha; \tag{3.13}$$

hence

$$\|y\|^2 < \tfrac{\varepsilon-\alpha}{\delta} \quad \text{and} \quad \|Cx\|^2 = \|x\|_C^2 = 1 - \|y\|^2 > 1 - \tfrac{\varepsilon-\alpha}{\delta}.$$

This shows that

$$1 - 2\,\|x\|_C^2 < 2\,\tfrac{\varepsilon-\alpha}{\delta} - 1. \tag{3.14}$$

On the other hand, we obtain from (3.11) the estimate

$$\|y - (\alpha + \mathrm{i}\beta)x\|_C = \|Cy - C(\alpha + \mathrm{i}\beta)x\| < \varepsilon,$$

which gives

$$\big|\mathrm{Im}(Cx \,|\, Cy) - \mathrm{Im}(Cx \,|\, \mathrm{i}\beta Cx)\big| < \varepsilon.$$

Next, we combine this with (3.12) and conclude, by taking imaginary parts, that

$$|\beta|\cdot\big|1 - 2\,\|x\|_C^2\big| - \big|\mathrm{Im}(By \,|\, y)\big| \le \big|\beta\big(1 - 2\,\|x\|_C^2\big) - \mathrm{Im}(By \,|\, y)\big| < 3\varepsilon. \tag{3.15}$$

However, by (3.14) we have for $\alpha \in (-\delta/2 + \varepsilon, 0]$

$$1 - 2\,\|x\|_C^2 < 2\,\tfrac{\varepsilon-\alpha}{\delta} - 1 < 0,$$

hence

$$|\beta| \cdot \big(1 - 2\,\tfrac{\varepsilon-\alpha}{\delta}\big) < |\beta| \cdot \big|1 - 2\,\|x\|_C^2\big|.$$

Together with (3.10), (3.13), and (3.15) this implies

$$|\beta| \cdot \big(1 - 2\,\tfrac{\varepsilon-\alpha}{\delta}\big) < 3\varepsilon - \gamma\,\mathrm{Re}(By \,|\, y) < 3\varepsilon + \gamma(\varepsilon - \alpha),$$

and the assertion follows. □

PROOF OF THEOREM 3.18. (i) By Exercise V.1.13.(1) it suffices to show that

$$(3.16)\qquad \begin{cases} \alpha + \mathrm{i}\mathbb{R} \subset \rho(\mathcal{A}) & \text{and} \\ \sup_{\beta\in\mathbb{R}} \|R(\alpha+\mathrm{i}\beta, \mathcal{A})\| < \infty & \text{for all } \alpha > w. \end{cases}$$

Since by Proposition 3.14 the matrix $\mathcal{A}$ generates a contraction semigroup, we know that $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \subset \rho(\mathcal{A})$. On the other hand, by Proposition IV.1.10 the boundary $\partial\sigma(\mathcal{A})$ is always contained in the approximate point spectrum $A\sigma(\mathcal{A})$. However, from Lemma IV.1.9 and Lemma 3.19 it follows that for $\lambda := \alpha + \mathrm{i}\beta$ satisfying $\alpha \in (-\delta/2, 0]$ and

$$|\beta| \geq -\frac{\alpha\gamma\delta}{\delta + 2\alpha}$$

we have $\inf_{\mathfrak{z}\in D(\mathcal{A}),\, \|\mathfrak{z}\|=1} \|(\lambda - \mathcal{A})\mathfrak{z}\| > 0$, and therefore $\lambda \notin \partial\sigma(\mathcal{A}) \subset \sigma(\mathcal{A})$. Moreover, by Proposition IV.1.3.(i) the set $\{\mu \in \mathbb{C} : |\mu| < 1/\|\mathcal{A}^{-1}\|\}$ is contained in $\rho(\mathcal{A})$, and a simple geometric argument, cf. Figure 8, implies the first part of (3.16). The second part then follows from the inequality

$$\|\mathfrak{x}\| = \big\|(\lambda - \mathcal{A})[R(\lambda,\mathcal{A})\mathfrak{x}]\big\| \geq \inf_{\mathfrak{z}\in D(\mathcal{A}),\, \|\mathfrak{z}\|=1} \|(\lambda - \mathcal{A})\mathfrak{z}\| \cdot \|R(\lambda,\mathcal{A})\mathfrak{x}\|$$

for all $\mathfrak{x} \in \mathfrak{X}$.

Assertion (ii) follows from (i) by taking the limit $\gamma \downarrow 0$. □

We close this section with a typical application.

**3.20 Example.** (*Damped vibrating string*). On the interval $[0,1]$ we consider the second-order Cauchy problem

$$(3.17)\qquad \begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} = q(x)\dfrac{\partial u(t,x)}{\partial t} + \dfrac{\partial^2 u(t,x)}{\partial x^2}, & t \geq 0,\ x \in [0,1], \\ u(t,0) = 0 = u(t,1), & t \geq 0, \\ u(0,x) = u_0(x),\ \dfrac{\partial u(0,x)}{\partial t} = u_1(x), & x \in [0,1], \end{cases}$$

describing the motion of a damped vibrating string fixed at its endpoints $x = 0$ and $x = 1$.

As in Example 3.17, we rewrite (3.17) as an abstract second-order Cauchy problem $(\mathrm{ACP}_2)$ on $X := \mathrm{L}^2[0,1]$ for the operators

$$(3.18)\qquad \begin{aligned} A &:= \Delta, & D(A) &:= \mathrm{H}^2_0[0,1], \\ B &:= M_q, & D(M_q) &:= \{f \in X : qf \in X\}. \end{aligned}$$

Moreover, we assume that the measurable function $q : [0,1] \to \mathbb{C}$ satisfies

$$(3.19)\qquad |\operatorname{Im} q(x)| \leq -\gamma \operatorname{Re} q(x) \quad \text{and} \quad \operatorname{Re} q(x) \leq -\delta \quad \text{a.e.}$$

for some constants $\gamma \geq 0$, $\delta > 0$. Then $A$ is self-adjoint and negative definite, $B$ is dissipative, and the conditions in (3.10) are satisfied. Our next goal is to find conditions on $q$ implying (3.6) for $B = M_q$ and the (unique) positive definite square root

$$C := (-A)^{1/2} \tag{3.20}$$

of $-A$ (cf. Section II.5.c). Since by Exercise II.5.36.(4) we have $D(C) = \mathrm{H}_0^1[0,1]$, we first study the relation between the domain of a multiplication operator and $\mathrm{H}_0^1[0,1]$.

**Lemma.** *For a measurable function $p : [0,1] \to \mathbb{C}$ the following hold.*

(i) *$D(M_p) \cap \mathrm{H}_0^1[0,1]$ is dense in $\mathrm{H}_0^1[0,1]$ if and only if $p \in \mathrm{L}^2[\varepsilon, 1-\varepsilon]$ for every $0 < \varepsilon < 1/2$.*

(ii) *$\mathrm{H}_0^1[0,1] \subset D(M_p)$ if the map $x \mapsto x(1-x)\, p^2(x)$ belongs to $\mathrm{L}^1[0,1]$.*

Proof. (i) We show only that $D(M_p) \cap \mathrm{H}_0^1[0,1]$ is dense in $\mathrm{H}_0^1[0,1]$ if $p \in \mathrm{L}^2[\varepsilon, 1-\varepsilon]$ for every $0 < \varepsilon < 1/2$. The converse implication is left as Exercise 3.21.(7).

For arbitrary fixed $f \in \mathrm{H}_0^1[0,1]$ and $0 < \varepsilon < 1/2$ we define the function

$$g(x) := \begin{cases} 0 & \text{if } x \in [0, \varepsilon/2] \cup [1-\varepsilon/2, 1], \\ (2x/\varepsilon - 1) f(\varepsilon) & \text{if } x \in [\varepsilon/2, \varepsilon], \\ f(x) & \text{if } x \in [\varepsilon, 1-\varepsilon], \\ (2(1-x)/\varepsilon - 1) f(1-\varepsilon) & \text{if } x \in [1-\varepsilon, 1-\varepsilon/2]. \end{cases}$$

Then $g \in \mathrm{H}_0^1[0,1]$, and the assumption $p \in \mathrm{L}^2[\varepsilon/2, 1-\varepsilon/2]$ combined with the boundedness of $g$ implies $g \in D(M_p)$.

In order to estimate $\|f-g\|_{\mathrm{H}_0^1[0,1]}$, we first observe that Hölder's inequality implies

$$\begin{aligned} |f(\varepsilon)| &\leq \|f'\|_{\mathrm{L}^1[0,\varepsilon]} \leq \sqrt{\varepsilon}\, \|f'\|_{\mathrm{L}^2[0,\varepsilon]}, \\ |f(1-\varepsilon)| &\leq \|f'\|_{\mathrm{L}^1[1-\varepsilon,1]} \leq \sqrt{\varepsilon}\, \|f'\|_{\mathrm{L}^2[1-\varepsilon,1]} \end{aligned} \tag{3.21}$$

for all $\varepsilon \in (0, 1/2)$. Using these estimates we obtain

$$\begin{aligned} \|f-g\|_{\mathrm{H}_0^1[0,1]} &= \|f'-g'\|_{\mathrm{L}^2[0,1]} = \|f'-g'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} \\ &\leq \|f'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} + \|g'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} \\ &= \|f'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} + \Big[\tfrac{2}{\varepsilon} f(\varepsilon)^2 + \tfrac{2}{\varepsilon} f(1-\varepsilon)^2\Big]^{1/2} \\ &\leq \|f'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} + \Big[2\|f'\|^2_{\mathrm{L}^2[0,\varepsilon]} + 2\|f'\|^2_{\mathrm{L}^2[1-\varepsilon,1]}\Big]^{1/2} \\ &\leq (1+\sqrt{2})\|f'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])}. \end{aligned}$$

Since $f' \in \mathrm{L}^2[0,1]$, we conclude that

$$\|f'\|_{\mathrm{L}^2([0,\varepsilon]\cup[1-\varepsilon,1])} \to 0, \qquad \text{as } \varepsilon \downarrow 0,$$

and hence $D(M_p) \cap \mathrm{H}_0^1[0,1]$ is dense in $\mathrm{H}_0^1[0,1]$.

(ii) For $f \in \mathrm{H}_0^1[0,1]$ we obtain from (3.21)

$$
\begin{aligned}
\|M_p f\|^2_{\mathrm{L}^2[0,1]} &= \int_0^{1/2} \big|f^2(x)p^2(x)\big|\,dx + \int_{1/2}^1 \big|f^2(x)p^2(x)\big|\,dx \\
&\le \Big(\int_0^{1/2} \big|xp^2(x)\big|\,dx + \int_{1/2}^1 \big|(1-x)p^2(x)\big|\,dx\Big)\cdot \|f'\|^2_{\mathrm{L}^2[0,1]} \\
&\le \Big(\int_0^{1/2} \big|xp^2(x)\cdot 2(1-x)\big|\,dx + \int_{1/2}^1 \big|(1-x)p^2(x)\cdot 2x\big|\,dx\Big)\cdot \|f'\|^2_{\mathrm{L}^2[0,1]} \\
&= 2\int_0^1 \big|x(1-x)\,p^2(x)\big|\,dx \cdot \|f'\|^2_{\mathrm{L}^2[0,1]} < \infty,
\end{aligned}
$$

and hence $f \in D(M_p)$ as claimed. □

We are now able to prove the following result on the well-posedness and the stability of the damped vibrating string from (3.17).

**Proposition.** *If* $q : [0,1] \to \mathbb{C}$ *satisfies* (3.19), *then the following holds.*

(i) *If* $q \in \mathrm{L}^2[\varepsilon, 1-\varepsilon]$ *for every* $0 < \varepsilon < 1/2$ *and* $x \mapsto x(1-x)q(x)$ *belongs to* $\mathrm{L}^1[0,1]$, *then the closure* $\mathcal{A}$ *of the operator matrix*

$$
\mathcal{A}_0 := \begin{pmatrix} 0 & I \\ \Delta & M_q \end{pmatrix}, \qquad D(\mathcal{A}_0) := \mathrm{H}_0^2[0,1] \times \big(D(M_q) \cap \mathrm{H}_0^1[0,1]\big),
$$

*generates an exponentially stable contraction semigroup on* $\mathcal{X} := \mathrm{H}_0^1[0,1] \times \mathrm{L}^2[0,1]$.

(ii) *If the function* $x \mapsto x(1-x)\,q^2(x)$ *belongs to* $\mathrm{L}^1[0,1]$, *then* $D(\mathcal{A}) = \mathrm{H}_0^2[0,1] \times \mathrm{H}_0^1[0,1]$ *and Condition* (S) *is satisfied. Moreover, for every* $u_0 \in \mathrm{H}_0^2[0,1]$, $u_1 \in \mathrm{H}_0^1[0,1]$, *the problem* (3.17) *has a unique classical solution* $u$ *that tends exponentially to* $0$ *as* $t \to \infty$.

PROOF. (i) To verify condition (3.6), we first observe that by statement (i) in the previous lemma with $p := q$ the intersection $D(M_q) \cap \mathrm{H}_0^1[0,1]$ is dense in $\mathrm{H}_0^1[0,1]$. Hence, the operator $(C^*)^{-1}BC^{-1} = (-\Delta)^{-1/2} M_q (-\Delta)^{-1/2}$ is densely defined on $X := \mathrm{L}^2[0,1]$. In order to show that it has a bounded closure $Q \in \mathcal{L}(X)$, we write

$$
(-\Delta)^{-1/2} M_q (-\Delta)^{-1/2} = (-\Delta)^{-1/2} M_{\sqrt{r}} \cdot M_a \cdot M_{\sqrt{r}} (-\Delta)^{-1/2},
$$

where $q(x) = a(x)r(x)$ for a measurable function $a$ with $|a(x)| = 1$ and $r(x) := |q(x)|$. Since $x \mapsto x(1-x)r(x)$ belongs to $\mathrm{L}^1[0,1]$, it follows by statement (ii) of the previous lemma with $p := \sqrt{r}$ that $\mathrm{H}_0^1[0,1] \subset D(M_{\sqrt{r}})$. Proposition B.2.(i) and the closed graph theorem then imply that $M_{\sqrt{r}}(-\Delta)^{-1/2} \in \mathcal{L}(X)$. On the other hand, $(-\Delta)^{-1/2} M_{\sqrt{r}}$ is densely defined and $\big((-\Delta)^{-1/2} M_{\sqrt{r}}\big)^* = M_{\sqrt{r}}(-\Delta)^{-1/2} \in \mathcal{L}(X)$. From this we conclude that $(-\Delta)^{-1/2} M_{\sqrt{r}}$ is closable and its closure is bounded. Since $\|M_a\| = 1$, this proves the second part of (3.6). The assertion now follows from Proposition 3.14 and Theorem 3.18.

(ii) Assume that the function $x \mapsto x(1-x)\, q^2(x)$ belongs to $\mathrm{L}^1[0,1]$. Then statement (ii) of the previous lemma applied to $p := q$ implies that $D(C) = \mathrm{H}^1_0[0,1] \subset D(M_q) = D(B)$. The assertion now follows from Corollary 3.16 and Theorem 3.18. □

**3.21 Exercises.** (1) Give an explicit formula for the semigroup $(\mathcal{T}(t))_{t\ge 0}$ generated by $\mathcal{A}$ in the undamped case, i.e., for $B = 0$. Moreover, show that $(\mathcal{T}(t))_{t\ge 0}$ can be extended to a group. (Hint: Show that $\mathcal{A}$ and $-\mathcal{A}$ are similar.)

(2) Prove Lemma 3.15.

(3) Take $D \in \mathcal{L}(X_1^C, X)$ and reconsider Exercise 3.9.(3) under the hypotheses of this subsection. In particular, show that in the case $D(C) \subseteq D(B)$ Condition (S) is also satisfied for the perturbed generator $\mathcal{A}_D$.

(4) Show that under the assumptions of Theorem 3.18 one has the estimates $\omega_0(\mathcal{A}) \le \max\{\mathrm{s}(\mathcal{A}), -{}^{\delta}/_{2}\} < 0$ for the growth bound of $\mathcal{A}$.

(5) (*Uniformly damped wave equation*). Let $A = -C^*C$ be self-adjoint negative definite and let $B := -2b$ for some $b > 0$. Show that for the generator

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times D(C)$$

on $\mathcal{X} := X_1^C \times X$ one has

$$\omega_0(\mathcal{A}) = \mathrm{s}(\mathcal{A}) = -b + \mathrm{Re}\,\sqrt{b^2 + \mathrm{s}(A)}.$$

(Hint: Verify first that $\sigma(\mathcal{A}) = \left\{-b \pm \sqrt{b^2+\mu} \;:\; \mu \in \sigma(A)\right\}$ and then apply Exercise (4).)

(6) Show that the solution of the second-order Cauchy problem from Example 3.17 is exponentially stable if $\mathrm{Re}\, b > 0$. (Hint: Modifying Example 3.17, define the operator $A := -c\Delta^2 - D_m M_a D_0$. Then $A$ is self-adjoint and negative definite, and for $C := (-A)^{1/2}$ the conditions of Theorem 3.18 are satisfied. Observe also that $D(C) = \mathrm{H}^2_0[0,1]$.)

(7) Prove the "only if" part in statement (i) of the lemma in Example 3.20.

## *Notes and Further Reading to Section* 3

As general references on second-order (and, more generally, higher-order) Cauchy problems we refer to the monographs by Fattorini [Fat85] and Xiao–Liang [XL98]. Among the methods used in the literature we mention:

- The theory of *operator cosine functions* and *$M,N$-families*, which can be viewed as the semigroup analogue for second-order Cauchy problems, cf. Sova [Sov66], Kisyński [Kis72], Lutz [Lut82], Mel'nikova–Filinkov [MF88], Goldstein [Gol85, Sec. 2.8], Zheng [Zhe94].
- *Hille–Yosida–type theorems*, which impose growth conditions on the inverse of the operator pencil $\lambda^2 - \lambda B - A$; see Xiao–Liang [XL90] and Zheng [Zhe92].
- The theory of parabolic second-order equations, which uses a holomorphic functional calculus to obtain representations for the solution; see Favini–Obrecht [FO91].
- Factorized equations of the form $({}^{d}/_{dt} - A)({}^{d}/_{dt} - B)u(t) = 0$ for which an extensive theory of "equipartition of energy" was developed by Goldstein–Sandefur [GS82].
- Perturbation methods for $m$-dissipative operators; see [CP89] and [Eng94].

For still other approaches we refer to [deL94], [FY99], and [Neu89].

The first-order reduction used in this section is standard. The existence of an appropriate state space for the operator matrix $\mathcal{A} = \left(\begin{smallmatrix} 0 & I \\ A & B \end{smallmatrix}\right)$ was discussed in detail by Fattorini [Fat81], [Fat85, Chap. VIII], and Obrecht [Obr91]. The results presented in Section 3.a are quite similar to those in [Kre71, Sec. III.3], where a different reduction matrix on the state space $X \times X$ is considered. The situation in Section 3.b is, in some sense, dual to the one studied in [Neu86], where operators $A$ and $B$ with $D(B) \subset D(A)$ are treated. In Section 3.c we follow [Hua97] and [BE99]. Example 3.17 is taken from [XL90, (3.7), p. 193]. For related results on the well-posedness and the stability of second-order Cauchy problems see also [Eng97] and [Wyl92].

# 4. Semigroups for Ordinary Differential Operators

**(by M. Campiti, G. Metafune, D. Pallara, and S. Romanelli)**

In this section we present some results on the existence and regularity of semigroups generated by second-order ordinary differential operators on spaces of continuous functions. The general operator is given by

$$Au = mu'' + qu' + ru \tag{4.1}$$

on some interval $J$ and for continuous functions $m, q, r$. We say that $A$ is *nondegenerate* if $m, q, r$ are bounded, $\inf_{x\in J} m(x) > 0$, and the limits of $m(x)$ exist when $x$ approaches the real endpoints of $J$. In this case, several generation results already appeared in the previous chapters, and hence we concentrate on the analyticity of the generated semigroup, which we prove under general boundary conditions. To make the exposition clearer, we distinguish the case of an unbounded interval from that of an bounded interval $J$. In both cases, we first study the generator $Au = u''$ with elementary methods, and then use a perturbation argument to deduce the general case. The theory can be developed along the same lines in the $\mathrm{L}^p$-setting, as outlined in Exercises 4.4.(2)–(3) and Exercises 4.7.(1)–(2).

Next, we consider *degenerate* operators $A$ keeping $m > 0$ in $J$ and $r$ bounded, but allowing degeneracy of $m$ and $q$ at the endpoints. In this case, the boundary conditions that lead to the generation in $\mathrm{C}(\overline{J})$ depend on the behavior of the coefficients at the endpoints. This is the content of Feller's theory, partly presented in Section 4.c, which relies on more delicate arguments. Clearly, the regularity of the semigroups generated by these operators is even more difficult. We discuss in the last section a few analyticity results in spaces of continuous functions, omitting most of the proofs and giving only some indications on the main ideas and techniques involved.

To shorten the notation, we set $\overline{\mathbb{R}} := [-\infty, \infty]$ and $\overline{\mathbb{R}}_+ := [0, \infty]$. Accordingly, we write $u \in \mathrm{C}^k(\overline{\mathbb{R}})$ if $u \in \mathrm{C}^k(\mathbb{R})$ and the derivatives of $u$ up to the order $k$ have finite limits at $\pm\infty$, and similarly for $\mathrm{C}^k(\overline{\mathbb{R}}_+)$.

## *a. Nondegenerate Operators on* $\mathbb{R}$ *and* $\mathbb{R}_+$

We start with the case of the second-order derivative operator. For clarity's sake, we denote this operator by $A_0$, $A_1$, regarded on $\mathbb{R}$, $\mathbb{R}_+$ respectively, with suitable domains. In each case, we prove that they generate analytic semigroups $\big(T(t)\big)_{t\geq 0}$ of angle $\pi/2$, the (one-dimensional) diffusion semigroups already treated in Paragraph II.2.12, Paragraph II.3.30, and Example II.4.8.

We first look at the case of the whole real line and consider $A_0$ with the maximal domain, i.e., $D(A_0) := \mathrm{C}^2(\overline{\mathbb{R}})$. Then the operator $\big(A_0, D(A_0)\big)$ is densely defined in $\mathrm{C}(\overline{\mathbb{R}})$, and from Lemma III.2.4 and Example III.2.2 it easily follows that it is also closed.

There are many ways to verify that $\big(A_0, D(A_0)\big)$ is the generator of a bounded analytic semigroup in $\mathrm{C}(\overline{\mathbb{R}})$, as one likely expects in such an archetypical case. The simplest one, cf. Example II.4.10, is to apply Corollary II.4.9 viewing $A_0$ as the square of the first derivative operator, which generates the bounded translation group. Another way, which seems to be worth mentioning, is to write down explicitly the solution of the Cauchy problem

$$\begin{cases} u_t(t,x) = u_{xx}(t,x) & \text{for } t>0,\ x\in\mathbb{R},\\ u(0,x) = f(x) & \text{for } x\in\mathbb{R}, \end{cases}$$

and to deduce the estimate (4.9) in Theorem II.4.6. Using the Fourier transform, as is done in much greater generality in Section 5, we obtain (see Section II.2, (2.10))

$$u(t,x) = T(t)f(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{|x-y|^2}{4t}} f(y)\,dy,$$

and it is readily seen that Theorem II.4.6.(c) applies. Although these arguments are sufficient in the present situation, we give still another proof of the analyticity, based on the estimate of the resolvent operator, according to Theorem II.4.6.(d). In fact, this method can be applied in the general situation that we discuss later on.

**4.1 Theorem.** *The spectrum of the operator* $\big(A_0, D(A_0)\big)$ *is contained in the interval* $(-\infty, 0]$*, and if* $\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\vartheta}$ *with* $|\vartheta| < \pi$*, then*

$$\|R(\lambda, A_0)\| \leq \frac{1}{|\lambda|\cos(\vartheta/2)}.$$

*Hence,* $\big(A_0, D(A_0)\big)$ *generates a bounded analytic semigroup of angle* $\pi/2$.

Proof. Let $\lambda \notin (-\infty, 0]$ and write $\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\vartheta}$ for $|\vartheta| < \pi$. Since no nonzero solution of the equation $\lambda u - u'' = 0$ is in $\mathrm{C}(\overline{\mathbb{R}})$, the operator $\lambda - A_0$ is injective.

To show the surjectivity of $\lambda - A_0$ we write $\lambda = \mu^2$ with $\operatorname{Re}\mu > 0$. Assuming for a moment $f \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{C}(\overline{\mathbb{R}})$, we can apply the Fourier transform and obtain $\mathcal{F}u(y) = \frac{\mathcal{F}f(y)}{\mu^2+y^2}$ for the solution $u$ of the equation $\lambda u - u'' = f$. By taking the inverse Fourier transform, we obtain

$$u(x) = \frac{1}{2\mu} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu|x-s|} f(s)\, ds = (f * h_\mu)(x) \qquad \text{for } x \in \mathbb{R}, \tag{4.2}$$

where $h_\mu(x) := \frac{1}{2\mu}\mathrm{e}^{-\mu|x|}$. Let us check that the above formula gives a solution $u = R(\lambda, A_0)f \in D(A_0)$ for any $f \in \mathrm{C}(\overline{\mathbb{R}})$. In fact, the integral in (4.2) converges, since $\operatorname{Re}\mu > 0$. Splitting it into the integral from $-\infty$ to $x$ and from $x$ to $\infty$ and differentiating, we obtain

$$u'(x) = -\frac{1}{2} \int_{-\infty}^{x} \mathrm{e}^{\mu(s-x)} f(s)\, ds + \frac{1}{2} \int_{x}^{\infty} \mathrm{e}^{\mu(x-s)} f(s)\, ds. \tag{4.3}$$

Differentiating again, we obtain $\lambda u - u'' = f$. Writing

$$u(x) = \frac{1}{2\mu} \int_{-\infty}^{\infty} \mathrm{e}^{-\mu|s|} f(x-s)\, ds$$

and letting $x \to \pm\infty$, we deduce the existence of the limits $\lim_{x\to\pm\infty} u(x)$. Then the equation $\lambda u - u'' = f$ implies that $u \in D(A_0)$. Finally, a direct computation yields

$$\|u\| = \|R(\lambda, A_0)f\| \le \|h_\mu\|_{\mathrm{L}^1} \cdot \|f\| = \frac{\|f\|}{|\mu| \operatorname{Re}\mu} = \frac{\|f\|}{|\lambda| \cos(\vartheta/2)}.$$

□

We now consider the operator $A_1 u = u''$ in $\mathrm{C}(\overline{\mathbb{R}}_+)$ under general boundary conditions. To that purpose fix two real numbers $\alpha$ and $\beta$ with $\alpha^2+\beta^2 \ne 0$ and define the domain of $A_1$ by

$$D(A_1) := \bigl\{u \in \mathrm{C}^2(\overline{\mathbb{R}}_+) : \alpha u(0) + \beta u'(0) = 0\bigr\} \subset \mathrm{C}(\overline{\mathbb{R}}_+). \tag{4.4}$$

Then the operator $\bigl(A_1, D(A_1)\bigr)$ is closed by Lemma III.2.4 and Example III.2.2, and its domain is dense in $\mathrm{C}(\overline{\mathbb{R}}_+)$ in the case $\beta \ne 0$.

If $\beta = 0$, then $D(A_1)$ is not dense in $\mathrm{C}(\overline{\mathbb{R}}_+)$. However, if we take the part of $\bigl(A_1, D(A_1)\bigr)$ in the subspace $X := \{u \in \mathrm{C}(\overline{\mathbb{R}}_+) : u(0) = 0\}$, then $A_1$ becomes densely defined, and the following results are valid in $X$.

**4.2 Theorem.** *The operator $\bigl(A_1, D(A_1)\bigr)$ generates an analytic semigroup of angle $\pi/2$.*

Proof. We proceed as in Theorem 4.1 and take $\lambda = |\lambda| e^{i\vartheta} \notin (-\infty, 0]$ with $|\vartheta| < \pi$. We write $\lambda = \mu^2$ with $\operatorname{Re} \mu > 0$. Then the function $x \mapsto u_\mu(x) := e^{-\mu x}$ is (up to a constant) the only nonzero solution of the equation $\lambda u - u'' = 0$ in $C(\overline{\mathbb{R}}_+)$ and satisfies the boundary condition at $x = 0$ if and only if $\alpha\beta > 0$ and $\lambda = (\alpha/\beta)^2$. Therefore, we also assume $\lambda \neq (\alpha/\beta)^2$. For $f \in C(\overline{\mathbb{R}}_+)$, we put

$$u(x) := (f * h_\mu)(x) = \frac{1}{2\mu} \int_0^\infty e^{-\mu|x-s|} f(s)\, ds \qquad \text{for } x \geq 0.$$

By extending $f \equiv 0$ for $x < 0$, we infer from the proof of Theorem 4.1 that $u \in C^2(\overline{\mathbb{R}}_+)$ and that $\|u\| \leq \frac{1}{|\lambda| \cos(\vartheta/2)} \|f\|$. Moreover, we have that

$$\alpha u(0) + \beta u'(0) = (\alpha + \beta\mu)\gamma,$$

where

$$\gamma := \frac{1}{2\mu} \int_0^\infty e^{-\mu s} f(s)\, ds.$$

We define $w := R(\lambda, A_1)f = u + c u_\mu$ with $c := \frac{\alpha + \beta\mu}{\beta\mu - \alpha}\gamma$. Then $w \in D(A_1)$ and $(\lambda - A_1)w = f$. It remains to estimate $\|c u_\mu\|$. To this aim we observe that $\|u_\mu\| = 1$ and that $|\gamma| \leq \|f\|/(2|\mu| \operatorname{Re} \mu)$. From these inequalities we deduce that there exists $k > 0$ such that

$$\|R(\lambda, A_1)\| \leq \frac{k}{|\lambda| \cos(\vartheta/2)}$$

for $\lambda \notin (-\infty, 0]$ if $\alpha\beta \leq 0$, and for $\lambda \notin (-\infty, 0]$ with $|\lambda|$ large if $\alpha\beta > 0$. The assertion then follows from Proposition II.4.3. □

We remark that the semigroup generated by $(A_1, D(A_1))$ is *bounded* analytic if and only if $\alpha\beta \leq 0$.

We can extend the above results to the general nondegenerate operator by using a perturbation argument.

Let $J = \mathbb{R}$ or $J = \mathbb{R}_+$, $m \in C^1(J)$ with $\inf_{x \in J} m(x) > 0$ and $m, m' \in C_b(J)$, $q \in C_b(J)$, $r \in C(\overline{J})$, and consider the operator $A$ given by

$$Au := mu'' + qu' + ru.$$

If $J = \mathbb{R}$, we define $D(A) := D(A_0) = C^2(\overline{\mathbb{R}})$, while for $J = \mathbb{R}_+$ we take $D(A) := D(A_1)$ (see (4.4)) or the part of $A$ in the space $X := \{u \in C(\overline{\mathbb{R}}_+) : u(0) = 0\}$ in the case $\beta = 0$.

**4.3 Theorem.** *The operator $(A, D(A))$ generates an analytic semigroup of angle $\pi/2$ in $C(\overline{J})$.*

Proof. We first consider the case $m \equiv 1$ and define $Bu := qu' + ru$ with domain $D(B) := C^1(\overline{J})$. Then, by Example III.2.2, the operator $B$ is $A_i$-bounded for $i = 0, 1$ with $A_i$-bound 0. Hence, the assertion follows from Exercise III.2.18.(3), Theorem 4.1, and Theorem 4.2.

The general case can be deduced from the previous one by a similarity transformation arising from a change of variables. In fact, let

$$\varphi(x) := \int_0^x \frac{1}{\sqrt{m(t)}}\, dt \qquad \text{for } x \in \overline{J}. \tag{4.5}$$

Then $\varphi : \overline{J} \to \overline{J}$ is bijective, and hence $Q_\varphi \in \mathcal{L}\big(\mathrm{C}(\overline{J})\big)$ defined by

$$Q_\varphi u := u \circ \varphi, \quad u \in \mathrm{C}(\overline{J}), \tag{4.6}$$

is invertible with $Q_\varphi^{-1} = Q_{\varphi^{-1}}$. Next, we apply Exercise 4.4.(1) and obtain

$$\widetilde{A}v := Q_\varphi^{-1} A Q_\varphi v = v'' + \frac{\widetilde{q} - \widetilde{m'}/2}{\sqrt{\widetilde{m}}}\, v' + \widetilde{r}v, \tag{4.7}$$

where $\widetilde{h} := Q_\varphi^{-1}h = h \circ \varphi^{-1}$ for a function $h \in \mathrm{C}(\overline{J})$. If $J = \mathbb{R}_+$, then $v \in D(\widetilde{A}) = Q_{\varphi^{-1}}D(A)$ if and only if $v \in \mathrm{C}^2(\overline{\mathbb{R}}_+)$ and the boundary condition $\alpha v(0) + \big(\beta/\sqrt{m(0)}\big)v'(0) = 0$ holds, while in the case $J = \mathbb{R}$ we obtain $D(A) = \mathrm{C}^2(\overline{\mathbb{R}}) = D(A_0)$. Since the similarity transformation preserves the relevant properties, we can apply the previous argument to $\widetilde{A}$ and conclude that $\big(A, D(A)\big)$ generates an analytic semigroup of angle $\pi/2$. □

**4.4 Exercises.** (1) For two intervals $J, K \subseteq \overline{\mathbb{R}}$ and a twice continuously differentiable, bijective function $\varphi : J \to K$ define the operator $Q_\varphi : \mathrm{C}(K) \to \mathrm{C}(J)$ by $Q_\varphi v := v \circ \varphi$. Then $Q_\varphi$ is invertible, and for the operator $(A, D(A))$ on $\mathrm{C}(\overline{J})$ given by $Au := mu'' + qu' + ru$ with $m, q, r$ as above the following holds.

(i) The operator $\widetilde{A} := Q_\varphi^{-1} A Q_\varphi$ is given by

$$\widetilde{A}v = \widetilde{m} \cdot \widetilde{(\varphi')}^2 \cdot v'' + \big(\widetilde{m}\widetilde{\varphi''} + \widetilde{q}\widetilde{\varphi'}\big) \cdot v' + \widetilde{r} \cdot v \quad \text{for } v \in D(\widetilde{A}) = Q_\varphi^{-1}D(A),$$

where we write $\widetilde{h} := h \circ \varphi^{-1} \in \mathrm{C}(K)$ for $h \in \mathrm{C}(J)$.

(ii) There exists a function $\varphi$ that transforms the operator $A$ into $\widetilde{A}$ of the form $\widetilde{A}v = m_1 v'' + rv$.

(2*) Let $1 \le p < \infty$ and define $A_0$ on $X := \mathrm{L}^p(\mathbb{R})$ by $A_0 u := u''$ for $u \in D_p(A_0) := \mathrm{W}^{2,p}(\mathbb{R})$.

(a) Prove that the spectrum of $(A_0, D_p(A_0))$ is contained in $(-\infty, 0]$.

(b) Using a suitable cutoff function, construct for every $\lambda < 0$ an approximate eigenvector, thus proving that $\sigma(A_0) = (-\infty, 0]$.

(c) Use Young's inequality in (4.2) to estimate $\|R(\lambda, A_0)\|_p$ for $\lambda \notin (-\infty, 0]$ and deduce that $(A_0, D_p(A_0))$ generates a bounded analytic semigroup of angle $\pi/2$.

(3) For $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ and $1 \le p < \infty$ define the operator $A_1$ on $X := \mathrm{L}^p(\mathbb{R}_+)$ by $A_1 u := u''$ for $u$ in its domain

$$D_p(A_1) := \big\{u \in W^{2,p}(\mathbb{R}_+) : \alpha u(0) + \beta u'(0) = 0\big\}.$$

Following the path of Exercise (2), prove that $\sigma(A_1) = (-\infty, 0]$ if $\alpha\beta \le 0$, $\sigma(A_1) = (-\infty, 0] \cup \{(\alpha/\beta)^2\}$ if $\alpha\beta > 0$, and that $(A_1, D_p(A_1))$ generates an analytic semigroup. When is this semigroup *bounded* analytic?

## b. Nondegenerate Operators on Bounded Intervals

In this subsection we consider the operator $A$ given by $Au := mu''+qu'+ru$ on a *bounded* interval $J$ assuming $m, q, r \in \mathrm{C}(\overline{J})$ and $\inf_{x\in J} m(x) > 0$. Without loss of generality, we always take $J = (0, 1)$.

Given $\alpha_0, \beta_0, \alpha_1, \beta_1 \in \mathbb{R}$ with $\alpha_0^2 + \beta_0^2 > 0$, $\alpha_1^2 + \beta_1^2 > 0$, we put

$$D(A) := \bigl\{u \in \mathrm{C}^2(\overline{J}) : \alpha_0 u(0) + \beta_0 u'(0) = 0,\ \alpha_1 u(1) + \beta_1 u'(1) = 0\bigr\}$$

and obtain a closed operator $\bigl(A, D(A)\bigr)$ on $\mathrm{C}(\overline{J})$.

We first consider the operator $A_2$ given by

$$A_2 u := u'' \qquad \text{for } u \in D(A_2) := D(A). \tag{4.8}$$

In order to determine its spectrum we take $0 \neq \lambda = \mu^2$ with $\operatorname{Re}\mu \geq 0$. Then the functions $x \mapsto u_1(x) := \mathrm{e}^{-\mu x}$ and $x \mapsto u_2(x) := \mathrm{e}^{\mu x}$ are two linearly independent solutions of the equation $\lambda u - u'' = 0$, whence every solution $u$ of this equation can be written as $u = c_1 u_1 + c_2 u_2$. A direct computation shows that $u \in D(A_2)$ if and only if the coefficients $c_1$, $c_2$ satisfy a linear homogeneous system whose determinant $\xi(\mu)$ is given by

$$\xi(\mu) = \mathrm{e}^{\mu}(\alpha_0 - \mu\beta_0)(\alpha_1 + \mu\beta_1) - \mathrm{e}^{-\mu}(\alpha_0 + \mu\beta_0)(\alpha_1 - \mu\beta_1). \tag{4.9}$$

Therefore, $\lambda$ is an eigenvalue if and only if it satisfies the *characteristic equation* $\xi(\mu) = 0$. Since $\xi(\cdot)$ is an entire function, we deduce that the point spectrum $P\sigma(A_2)$ of $\bigl(A_2, D(A_2)\bigr)$ is (at most) countable. Now a simple integration by parts shows that for every $u, v \in D(A_2)$ we have

$$\int_0^1 \bigl(A_2 u(x)\bigr) \cdot \overline{v(x)}\, dx = \int_0^1 u(x) \cdot \overline{\bigl(A_2 v(x)\bigr)}\, dx;$$

hence $P\sigma(A_2)$ is real. Moreover, the above formula for $\xi(\mu)$ easily implies the existence of a constant $l > 0$ such that every solution of the equation $\xi(\mu) = 0$ satisfies $\operatorname{Re}\mu \leq l$, and therefore we obtain $P\sigma(A_2) \subseteq (-\infty, l^2]$.

Finally, we show that the spectrum coincides with the point spectrum. In fact, let $\lambda \in \mathbb{C}$ not be an eigenvalue. Then for $f \in \mathrm{C}(\overline{J})$ we take a solution $w \in \mathrm{C}^2(\overline{J})$ of the equation $\lambda w - w'' = f$. Since $\xi(\mu) \neq 0$, we can find $k_1$, $k_2$ such that $w + k_1 u_1 + k_2 u_2 \in D(A_2)$, whence $\lambda - A_2$ is surjective. Since $A_2$ is closed, this implies that $\lambda$ belongs to the resolvent set $\rho(A_2)$.

Next, we prove that $A_2$ generates an analytic semigroup on $\mathrm{C}(\overline{J})$. However, if $\beta_0\beta_1 = 0$ the domain $D(A_2)$ is not dense in $C(\overline{J})$, and as before we have to consider the part of $A_2$ in $X := \overline{D(A_2)}$.

**4.5 Theorem.** *The operator $\bigl(A_2, D(A_2)\bigr)$ generates an analytic semigroup of angle $\pi/2$.*

Proof. We write $\lambda \in \rho(A_2)$ as $\lambda = \mu^2 = |\lambda| \mathrm{e}^{\mathrm{i}\vartheta}$ with $\operatorname{Re}\mu > 0$. Then, for $f \in \mathrm{C}(\overline{J})$ define

$$u(x) := \frac{1}{2\mu} \int_0^1 \mathrm{e}^{-\mu|x-s|} f(s)\, ds \qquad \text{for } x \in [0,1].$$

We already know from the proof of Theorem 4.1 that $u$ is a $\mathrm{C}^2$-solution of the equation $\lambda u - u'' = f$ satisfying $\|u\| \le \frac{\|f\|}{|\lambda| \cos(\vartheta/2)}$. Define next

$$\gamma_0 := \frac{1}{2\mu} \int_0^1 \mathrm{e}^{-\mu s} f(s)\, ds, \qquad \gamma_1 := \frac{1}{2\mu} \int_0^1 \mathrm{e}^{-\mu(1-s)} f(s)\, ds$$

and observe that $u'(0) = \mu\gamma_0$ and $u'(1) = -\mu\gamma_1$.

We now set $v := c_1 u_1 + c_2 u_2 + u$, where $u_1(x) := \mathrm{e}^{-\mu x}$ and $u_2(x) := \mathrm{e}^{\mu x}$. Since $\lambda$ is not an eigenvalue, we can find $c_1, c_2 \in \mathbb{C}$ such that $v \in D(A_2)$. A straightforward computation yields

(4.10)

$$c_1 = \frac{1}{\xi(\mu)} \big[(\alpha_0 + \mu\beta_0)(\alpha_1 - \mu\beta_1)\gamma_1 - \mathrm{e}^{\mu}(\alpha_0 + \mu\beta_0)(\alpha_1 + \mu\beta_1)\gamma_0\big],$$

$$c_2 = \frac{1}{\xi(\mu)} \big[(-\alpha_0 + \mu\beta_0)(\alpha_1 - \mu\beta_1)\gamma_1 + \mathrm{e}^{-\mu}(\alpha_0 + \mu\beta_0)(\alpha_1 - \mu\beta_1)\gamma_0\big],$$

where $\xi(\mu)$ is defined in (4.9). We have

$$\|u_1\| = 1, \qquad \|u_2\| = \mathrm{e}^{\operatorname{Re}\mu},$$

and

$$|\gamma_0| \le \frac{\|f\|}{2|\mu|(\operatorname{Re}\mu)}, \qquad |\gamma_1| \le \frac{\|f\|}{2|\mu|(\operatorname{Re}\mu)}.$$

Moreover, if $|\vartheta| \le |\vartheta_0| < \pi$, then $\operatorname{Re}\mu \ge |\mu| \cos(\vartheta_0/2)$, and we obtain that the coefficients of $\gamma_0, \gamma_1$ in the formulas (4.10) for $c_1, c_2$ are bounded, as is easily seen by taking the limit as $|\mu| \to \infty$. As a consequence, it follows that

$$\|c_1 u_1\| \le \frac{k}{|\lambda|} \|f\| \qquad \text{and} \qquad \|c_2 u_2\| \le \frac{k}{|\lambda|} \|f\|$$

for a suitable $k > 0$ and sufficiently large $|\lambda|$. This shows that $\|v\| \le k\|f\|/|\lambda|$, and the result follows from Proposition II.4.3. □

We now obtain the same result for the general operator $A$ by using perturbation and similarity arguments. As before, in the case $\beta_0\beta_1 = 0$ we have to consider the part of $A$ in $X := \overline{D(A)}$ instead of the operator $A$.

**4.6 Theorem.** *The operator $\big(A, D(A)\big)$ generates an analytic semigroup of angle $\pi/2$.*

Proof. Set $Bu := qu' + ru$ for $u \in D(B) := \mathrm{C}^1(\overline{J})$. As in the proof of Theorem 4.3, we conclude that the operator $A := A_2 + B$ with domain $D(A)$ generates an analytic semigroup of angle $\pi/2$.

For the general case we use the similarity transformation $Q_\varphi^{-1} A Q_\varphi =: \widetilde{A}$ given by (4.5) and (4.6). Then $\widetilde{A}v$ is given by (4.7) for $v$ in the domain

$$D(\widetilde{A}) = \big\{ v \in \mathrm{C}^2[0,b] : \alpha_0 v(0) + \widetilde{\beta}_0 v'(0) = 0,\ \alpha_1 v(b) + \widetilde{\beta}_1 v'(b) = 0 \big\},$$

where $b = \varphi(1)$, $\widetilde{\beta}_0 = \beta_0/\sqrt{m(0)}$, and $\widetilde{\beta}_1 = \beta_1/\sqrt{m(1)}$. Since by the first part of the proof $\widetilde{A}$ generates an analytic semigroup of angle $\pi/2$, by similarity the same conclusion holds for $A$. □

**4.7 Exercises.** (1) For $1 \le p < \infty$ and $\alpha_0, \beta_0, \alpha_1, \beta_1 \in \mathbb{R}$ with $\alpha_0^2 + \beta_0^2 > 0$, $\alpha_1^2 + \beta_1^2 > 0$ take the domain

$$D_p(A_2) := \big\{ u \in W^{2,p}(0,1) : \alpha_0 u(0) + \beta_0 u'(0) = 0,\ \alpha_1 u(1) + \beta_1 u'(1) = 0 \big\}$$

for the operator $A_2$ on $X := \mathrm{L}^p[0,1]$ given by $A_2 u := u''$. Check that the computations leading to Theorem 4.5 can be performed in $X$, and adapt the proof of Theorem 4.5 to the $\mathrm{L}^p$-setting.

(2) State the analogues of Theorem 4.3 and Theorem 4.6 in $\mathrm{L}^p$. Prove them using the similarity transformation with $Q_\varphi$ for $\varphi$ in (4.5) and the usual perturbation argument.

(3) Let $(A_2, D(A_2))$ be the operator defined in (4.8). Show that $A_2$ is dissipative if $\alpha_0\beta_0 \le 0$, $\alpha_1\beta_1 \ge 0$ and that if one of these inequalities fails, the operator $A_2 - \lambda$ is not dissipative for arbitrary $\lambda \in \mathbb{R}$.

## *c. Degenerate Operators*

In this subsection we study the boundary conditions for which a *degenerate* second-order differential operator $A$ as in (4.1) generates a strongly continuous semigroup in the space of continuous functions. Since the generation is not affected by bounded perturbations, we restrict ourselves to operators having the form

$$Au := mu'' + qu' \tag{4.11}$$

in some interval $J = (r_1, r_2)$, $-\infty \le r_1 < r_2 \le \infty$. We assume $m, q : J \to \mathbb{R}$ to be continuous functions with $m$ strictly positive on $J$, but do not impose any condition at the endpoints. We will choose boundary conditions in order to define the appropriate domain $D(A) \subset \mathrm{C}(\overline{J}) \cap \mathrm{C}^2(J)$.

We start by studying the behavior of the solutions of the differential equation

$$\lambda u - (mu'' + qu) = f, \tag{4.12}$$

with $f \in \mathrm{C}(\overline{J})$ for $\lambda > 0$. The general solution of (4.12) can be written in the form $u = F + c_1 v_1 + c_2 v_2$, where $F$ is a particular solution and $v_1$, $v_2$ are two linearly independent solutions of the homogeneous equation

$$\lambda u - (mu'' + qu) = 0. \tag{4.13}$$

We fix a point $x_0 \in J$ and introduce the *Wronskian*

$$W(x) = \exp\left(-\int_{x_0}^{x} \frac{q(s)}{m(s)}\, ds\right), \qquad x \in J. \tag{4.14}$$

To simplify the notation, henceforth we drop the dependence on $\lambda$. Once two linearly independent solutions $u_1, u_2$ of (4.13) have been found, we obtain $F$ by the variation of parameters formula in the form $F = \gamma_1 u_1 + \gamma_2 u_2$. Since $u_1' u_2 - u_1 u_2' = w_0 W$ with $w_0 \neq 0$, an easy computation gives (formally)

$$\gamma_1(x) = \int_x^{r_2} \frac{u_2(s) f(s)}{w_0 m(s) W(s)}\, ds, \qquad \gamma_2(x) = \int_{r_1}^{x} \frac{u_1(s) f(s)}{w_0 m(s) W(s)}\, ds, \tag{4.15}$$

but the convergence of the integrals must be justified. We define the *Green's function*

$$G(x,s) = \begin{cases} \dfrac{u_1(x) u_2(s)}{w_0 W(s) m(s)} & \text{for } x, s \in J \text{ with } x \leq s, \\[2ex] \dfrac{u_1(s) u_2(x)}{w_0 W(s) m(s)} & \text{for } x, s \in J \text{ with } x \geq s. \end{cases} \tag{4.16}$$

By this (up to now formal) argument we can represent $F$ as the image of $f$ under the integral operator $T_\lambda$ defined by the kernel $G$, i.e.,

$$F(x) = T_\lambda f(x) := \int_{r_1}^{r_2} G(x,s) f(s)\, ds. \tag{4.17}$$

To close the gaps in the preceding formal argument, we construct two positive monotone solutions $u_1, u_2$ of (4.13), so that $w_0 > 0$ and the Green's function $G$ becomes positive.

The construction of the functions $u_1, u_2$ is based on the elementary observation that a real solution of (4.13) can have neither a positive maximum nor a negative minimum in $J$ (in fact, $u(x)$ and $u''(x)$ have the same sign if $u'(x) = 0$), and therefore it cannot vanish at two distinct points of $J$.

If a nonzero solution $u$ of (4.13) vanishes at some point, then it is strictly monotone on all of $J$. In fact, if $u(x_1) = 0$, then $u'(x_1) \neq 0$ by the uniqueness of the solution of (4.13) with initial conditions $u(x_1) = 0$, $u'(x_1) = 0$. Assuming $u'(x_1) > 0$, suppose that $u'$ vanishes in some point, and let $x_2$ be the point nearest to $x_1$ where $u'(x_2) = 0$. Then, if $x_2 > x_1$, $u$ is strictly increasing in $(x_1, x_2)$, and $x_2$ cannot be a relative minimum of $u$. On the other hand, we have $u(x_2) > 0$, whence $u''(x_2) > 0$ which implies that $x_2$ is a relative minimum of $u$. An analogous argument can be used if $x_2 < x_1$, and therefore $u$ is strictly monotone as claimed.

Moreover, $u'$ can vanish at most once, and then any solution is definitely monotone near the endpoints, where, as a consequence, its limits exist, possibly $\pm\infty$.

Notice that (4.13) can be written in the form

$$\left(\frac{u'}{W}\right)' = \lambda\frac{u}{mW}. \tag{4.18}$$

Hence, if $u$ is a positive solution, then ${}^{u'}/_{W}$ is increasing and admits limits at the endpoints (possibly $\pm\infty$ again). In particular, we have for every positive solution $u$ the equivalence

$$u \text{ decreasing} \iff \lim_{x\to r_2}\frac{u'(x)}{W(x)} \le 0, \tag{4.19}$$

since the limit is the supremum of ${}^{u'}/_{W}$ and $W$ is positive.

**4.8 Lemma.** *There exist a positive increasing solution $u_1$ and a positive decreasing solution $u_2$ of* (4.13)*. The limits $l_{ij} := \lim_{x\to r_i} {}^{u_j'(x)}/_{W(x)}$ exist for $i,j = 1,2$, and if for $i \neq j$ we have $\lim_{x\to r_i} u_j(x) = \infty$, then $l_{jj} = 0$.*

Proof. For every $\gamma \in \mathbb{R}$ denote by $u_\gamma$ the solution of (4.13) satisfying the initial conditions $u_\gamma(x_0) = 1$ and $u_\gamma'(x_0) = \gamma$. For the set

$$\Upsilon := \bigl\{\gamma : u_\gamma \text{ vanishes at some point of } (x_0, r_2)\bigr\},$$

we show that $\Upsilon \neq \emptyset$. Let $v_1, v_2$ be two linearly independent solutions of (4.13). For every $x_1 \in (x_0, r_2)$ it is possible to determine $c_1, c_2$ such that for $v := c_1v_1 + c_2v_2$ we have $v(x_0) = 1$ and $v(x_1) = 0$ (hence $v'(x_0) \in \Upsilon$). This follows from Cramer's rule because the only solution with $v(x_0) = v(x_1) = 0$ is $v \equiv 0$.

If $\gamma \in \Upsilon$, then $u_\gamma$ is decreasing, since it vanishes at some point $x_1 > x_0$. Moreover, if $\gamma_1 < \gamma_2$, then $u_{\gamma_1} > u_{\gamma_2}$ in $(r_1, x_0)$ and $u_{\gamma_1} < u_{\gamma_2}$ in $(x_0, r_2)$. Indeed, $u_{\gamma_1}$ and $u_{\gamma_2}$ cannot attain the same value at any point $x_1 \neq x_0$, since otherwise their difference would vanish twice. Then $\gamma_2 \in \Upsilon$ implies $\gamma_1 \in \Upsilon$ as well, and $\Upsilon$ is an interval.

The solution of (4.13) with $u(x_0) = 1$ and $u'(x_0) = 0$ has an absolute minimum at $x_0$ because $u''(x_0) > 0$ and $u'$ can vanish at most once. As a consequence, we obtain $0 \notin \Upsilon$, hence $c := \sup\Upsilon \le 0$. We now take $u_2 := u_c$. The continuity with respect to the initial value $u'(x_0)$ shows that $u_c$ is the limit of solutions $u_\gamma$, $\gamma \in \Upsilon$, and therefore it is decreasing. Finally, if there were $x_1 \in (x_0, r_2)$ with $u_c(x_1) = 0$, we could find for each $x_2 \in (x_1, r_2)$ a solution $u$ with $u(x_0) = 1$ and $u(x_2) = 0$. Then $u'(x_0) \in \Upsilon$ and $u'(x_0) > c$, which is a contradiction.

Arguing in an analogous way in $(r_1, x_0)$ we can construct $u_1$.

The existence of the limits $l_{ij}$ has already been observed above. Assuming, e.g., $u_1(x) \to \infty$ as $x \to r_2$, we show that $\lim_{x\to r_2} {}^{u_2'(x)}/_{W(x)} = 0$. In fact, $u_1'u_2 - u_1u_2' = w_0W$, whence $0 \le -{}^{u_1u_2'}/_{W} \le w_0$ and the assertion follows. □

We now know that $T_\lambda$ is a positive operator. In the next step we show that it is bounded.

**4.9 Proposition.** *For every $\lambda > 0$ the operator $T_\lambda : \mathrm{C}(\overline{J}) \to \mathrm{C}(\overline{J})$ is bounded and satisfies $\|T_\lambda\| \le 1/\lambda$.*

Proof. We first prove that $T_\lambda$ is a bounded operator from $\mathrm{C}(\overline{J})$ to $\mathrm{L}^\infty(J)$. Since $T_\lambda$ is positive, it suffices to show that the function $T_\lambda \mathbb{1}$ is bounded. We have

$$\begin{aligned}
T_\lambda \mathbb{1}(x) &= \int_{r_1}^{r_2} G(x,s)\,ds \\
&= \frac{u_2(x)}{w_0\lambda}\int_{r_1}^{x}\left(\frac{u_1'(s)}{W(s)}\right)' ds + \frac{u_1(x)}{w_0\lambda}\int_{x}^{r_2}\left(\frac{u_2'(s)}{W(s)}\right)' ds \\
&= \frac{u_2(x)}{w_0\lambda}\left(\frac{u_1'(x)}{W(x)} - \lim_{s\to r_1}\frac{u_1'(s)}{W(s)}\right) + \frac{u_1(x)}{w_0\lambda}\left(\lim_{s\to r_2}\frac{u_2'(s)}{W(s)} - \frac{u_2'(x)}{W(x)}\right) \\
&= \frac{u_1(x)}{w_0\lambda}\lim_{s\to r_2}\frac{u_2'(s)}{W(s)} - \frac{u_2(x)}{w_0\lambda}\lim_{s\to r_1}\frac{u_1'(s)}{W(s)} + \frac{1}{\lambda} \le \frac{1}{\lambda}.
\end{aligned}$$

The above limits are always finite by the elementary properties of $u_1, u_2$. Moreover, by Lemma 4.8, $\lim_{s\to r_2} u_2'(s)/W(s) = 0$ if $u_1$ is unbounded at $s = r_2$. A similar argument can be used at $r_1$. This shows that $T_\lambda\mathbb{1} \in \mathrm{C}(\overline{J})$ and $\|T_\lambda\| \le 1/\lambda$.

Now let $f \in \mathrm{C}(\overline{J})$ be such that $f \equiv 0$ in a neighborhood of $r_2$. Then $u = T_\lambda f$ is a bounded solution of (4.13) near $r_2$, whence it is eventually monotone and has a finite limit. The existence of $\lim_{x\to r_2} T_\lambda f(x) \in \mathbb{R}$ for functions $f$ vanishing at $r_2$ easily follows by a density argument and the boundedness of $T_\lambda$. Writing $f = (f - l) + l$, with $l \equiv f(r_2)$, we obtain the general result by linearity. □

As a consequence of the above proposition, the integrals in (4.15) are convergent, and for every $f \in \mathrm{C}(\overline{J})$ the formula $F = T_\lambda f$ gives a solution of the equation (4.12) that belongs to $\mathrm{C}(\overline{J}) \cap \mathrm{C}^2(J)$. Moreover, if $f$ has compact support in $J$, then the function $F$ coincides with a linear combination of $u_1$ and $u_2$ near $r_1$ and $r_2$, respectively. Hence, its boundary behavior can be studied by looking at the solutions of the homogeneous equation. In order to give a complete description of the possible cases, we introduce the functions

$$(4.20)\qquad \begin{aligned}
Q(x) &:= \frac{1}{m(x)W(x)}\int_{x_0}^{x} W(s)\,ds, \\
R(x) &:= W(x)\int_{x_0}^{x}\frac{1}{m(s)W(s)}\,ds \qquad \text{for } x \in J.
\end{aligned}$$

In the sequel we deal only with the endpoint $r_2$.

**4.10 Remark.** If $R \in \mathrm{L}^1(x_0, r_2)$, then $W \in \mathrm{L}^1(x_0, r_2)$. Analogously, $Q \in \mathrm{L}^1(x_0, r_2)$ implies $(mW)^{-1} \in \mathrm{L}^1(x_0, r_2)$. Moreover, if $W \in \mathrm{L}^1(x_0, r_2)$ and $(mW)^{-1} \in \mathrm{L}^1(x_0, r_2)$, then $R, Q \in \mathrm{L}^1(x_0, r_2)$.

**4.11 Lemma.** *All solutions of* (4.13) *are bounded near* $r_2$ *if and only if* $R \in \mathrm{L}^1(x_0, r_2)$.

PROOF. Since $u_2$ is certainly bounded near $r_2$, it is enough to prove that $u_1$ is bounded if and only if $R \in \mathrm{L}^1(x_0, r_2)$. Using (4.18), we write

$$u_1'(x) = W(x)\Big[u_1'(x_0) + \lambda \int_{x_0}^{x} \frac{u_1(s)}{m(s)W(s)}\,ds\Big] \tag{4.21}$$

and remark that $u_1$ is bounded if and only if $u_1' \in \mathrm{L}^1(x_0, r_2)$. Note that all terms on the right-hand side of (4.21) are positive, $u_1(x_0) = 1$, and that $u_1$ is increasing. Therefore, we have

$$\begin{aligned}\lambda R(x) \le \lambda W(x) \int_{x_0}^{x} \frac{u_1(s)}{m(s)W(s)}\,ds &\le u_1'(x)\\ &\le u_1'(x_0)W(x) + \lambda u_1(x)R(x) \qquad \text{for } x \ge x_0.\end{aligned}$$

This shows that $u_1' \in \mathrm{L}^1(x_0, r_2)$ implies $R \in \mathrm{L}^1(x_0, r_2)$. Conversely, $R \in \mathrm{L}^1(x_0, r_2)$ implies $W \in \mathrm{L}^1(x_0, r_2)$. We conclude that $u_1$ is bounded near $r_2$ by comparing it with the solution of the Cauchy problem

$$v'(s) = u_1'(x_0)W(s) + \lambda R(s)v(s), \quad v(x_0) = 1,$$

given by

$$v(x) = \mathrm{e}^{\lambda \int_{x_0}^{x} R(s)\,ds}\Big[1 + u_1'(x_0)\int_{x_0}^{x} W(s)\mathrm{e}^{-\lambda \int_{x_0}^{s} R(\tau)\,d\tau}\,ds\Big].$$

□

**4.12 Lemma.** *A decreasing solution $u$ of* (4.13) *with* $l := \lim_{x\to r_2} u(x) > 0$ *exists if and only if* $Q \in \mathrm{L}^1(x_0, r_2)$. *If* $Q \in \mathrm{L}^1(x_0, r_2)$ *and* $R \notin \mathrm{L}^1(x_0, r_2)$, *then* $\lim_{x\to r_2} {}^{u'(x)}/_{W(x)} = 0$ *for every positive, decreasing solution $u$ of* (4.13).

PROOF. Assume $l > 0$. Since $u' \le 0$, we deduce from (4.21) that

$$l \int_{x_0}^{x} \frac{1}{m(s)W(s)}\,ds \le \int_{x_0}^{x} \frac{u(s)}{m(s)W(s)}\,ds \le \frac{|u'(x_0)|}{\lambda};$$

hence $(mW)^{-1} \in \mathrm{L}^1(x_0, r_2)$. Integrating (4.18) in $(x, r_2)$ we obtain

$$-u'(x) = W(x)\Big[k + \lambda \int_{x}^{r_2} \frac{u(s)}{m(s)W(s)}\,ds\Big], \tag{4.22}$$

where the limit $k := -\lim_{x\to r_2} {}^{u'(x)}/_{W(x)} \ge 0$ is finite because $u(mW)^{-1}$ is integrable. Since the function $u$ is decreasing and $k \ge 0$, we deduce that $\lambda W(x)\int_x^{r_2} \frac{1}{m(s)W(s)}\,ds \le -{}^{u'(x)}/_{l}$, and this implies $Q \in \mathrm{L}^1(x_0, r_2)$.

Conversely, take a positive, decreasing solution $u$ of (4.13). If $Q$ (and then, by Remark 4.10, $(mW)^{-1}$) is integrable, then (4.22) holds. If $k = 0$, then

$$-u'(x) \le \lambda u(x) W(x) \int_x^{r_2} \frac{1}{m(s)W(s)}\, ds;$$

hence ${}^{u'}\!/_{u}$ is integrable and $-\log u$ is bounded, so that $l > 0$. If $k > 0$, the boundedness of $u$ and (4.22) imply that $W \in \mathrm{L}^1(x_0, r_2)$, and the existence of a solution $u$ with the stated properties is proved in the next lemma.

Finally, let $Q \in \mathrm{L}^1(x_0, r_2)$, $R \notin \mathrm{L}^1(x_0, r_2)$ (hence, by Remark 4.10, $W \notin \mathrm{L}^1(x_0, r_2)$) and let $u$ be a positive decreasing solution of (4.13). Then equation (4.22) holds, and $k = -\lim_{x\to r_2} {}^{u'(x)}\!/_{W(x)} = 0$, since otherwise we would obtain $W \in \mathrm{L}^1(x_0, r_2)$. □

**4.13 Lemma.** *If* $Q, R \in \mathrm{L}^1(x_0, r_2)$, *then for every solution* $u$ *of* (4.13) *the limits* $\lim_{x\to r_2} u(x)$ *and* $\lim_{x\to r_2} {}^{u'(x)}\!/_{W(x)}$ *are finite. Moreover, there exist two decreasing solutions* $\bar u, \bar v$ *of* (4.13) *such that*

$$\lim_{x\to r_2} \bar u(x) = 0, \qquad \lim_{x\to r_2} \frac{\bar u'(x)}{W(x)} = -1,$$

$$\lim_{x\to r_2} \bar v(x) = 1, \qquad \lim_{x\to r_2} \frac{\bar v'(x)}{W(x)} = 0.$$

Proof. By Lemma 4.11, all solutions $u$ of equation (4.13) are bounded; hence the limit $\lim_{x\to r_2} u(x)$ is finite. Moreover, identity (4.21) implies that $\lim_{x\to r_2} {}^{u'(x)}\!/_{W(x)}$ is finite.

Let $u_1, u_2$ be the solutions constructed in Lemma 4.8 and define $u := u_2 - cu_1$, where $c \ge 0$ is chosen so that $u$ vanishes at $r_2$. Since $u_1 > u_2$ in $(x_0, r_2)$, we have $c < 1$ and $u(x_0) > 0$. Thus, the nonexistence of positive maxima and negative minima for $u$ (noted above) implies that $u$ is decreasing on the whole of $J$. As in Lemma 4.12, the limit $-k := \lim_{x\to r_2} {}^{u'(x)}\!/_{W(x)}$ exists and cannot vanish because otherwise we would have $u(x) \to l > 0$ as $x \to r_2$. Then we can take $\bar u := {}^{u}\!/_{k}$.

Now let $w$ be the solution of (4.13) with $w(x_0) = 0$ and $w'(x_0) = 1$. Since it is increasing and positive in $[x_0, r_2)$, we obtain by (4.19) that $\lim_{x\to r_2} {}^{w'(x)}\!/_{W(x)} > 0$. We choose $\tau > 0$ such that the function $v := \bar u + \tau w$ satisfies $\lim_{x\to r_2} {}^{v'(x)}\!/_{W(x)} = 0$. Then $v$ is positive in $[x_0, r_2)$ and decreasing by (4.19), hence positive and decreasing in $J$, so that we can take $\bar v := {}^{v(x)}\!/_{v(r_2)}$. □

The above discussion shows that the behavior of the solutions of (4.13) near the endpoints depends on the integrability of the functions $Q$ and $R$. Accordingly, we classify the boundary points into four types and say that

the *boundary point* $r_2$ is

$$
\text{(4.23)}\qquad
\begin{array}{lll}
\textit{regular} & \text{if } Q \in \mathrm{L}^1(x_0, r_2), & R \in \mathrm{L}^1(x_0, r_2),\\
\textit{exit} & \text{if } Q \notin \mathrm{L}^1(x_0, r_2), & R \in \mathrm{L}^1(x_0, r_2),\\
\textit{entrance} & \text{if } Q \in \mathrm{L}^1(x_0, r_2), & R \notin \mathrm{L}^1(x_0, r_2),\\
\textit{natural} & \text{if } Q \notin \mathrm{L}^1(x_0, r_2), & R \notin \mathrm{L}^1(x_0, r_2).
\end{array}
$$

Analogous definitions can be given for $r_1$ by considering the interval $(r_1, x_0)$ instead of $(x_0, r_2)$. Observe that $\pm\infty$ are natural boundary points for the operators of Section 4.a (this is obvious for $A_0$ and $A_1$; for general $A$ see Exercise 4.19.(7)). On the other hand, 0 and 1 are regular boundary points for the operators of Section 4.b. Other examples are listed in Exercise 4.19.(3). We summarize the results proved so far as follows.

**4.14 Theorem.** (i) *The boundary point $r_2$ is regular if and only if there exist two positive, decreasing solutions $v_1$ and $v_2$ of* (4.13) *satisfying*

$$\lim_{x\to r_2} v_1(x) = 0, \quad \lim_{x\to r_2} \frac{v_1'(x)}{W(x)} = -1, \quad \lim_{x\to r_2} v_2(x) = 1, \quad \lim_{x\to r_2} \frac{v_2'(x)}{W(x)} = 0.$$

*In this case, every solution of* (4.13) *is bounded near $r_2$.*

(ii) *The boundary point $r_2$ is exit if and only if every solution of* (4.13) *is bounded at $r_2$ and every positive decreasing solution $v_1$ satisfies*

$$\lim_{x\to r_2} v_1(x) = 0, \quad \lim_{x\to r_2} \frac{v_1'(x)}{W(x)} \le 0.$$

(iii) *The boundary point $r_2$ is entrance if and only if there exists a positive, decreasing solution $v_1$ of* (4.13) *satisfying*

$$\lim_{x\to r_2} v_1(x) = 1, \quad \lim_{x\to r_2} \frac{v_1'(x)}{W(x)} = 0$$

*and every solution of* (4.13) *independent of $v_1$ is unbounded at $r_2$. In this case, no nonzero solution tends to* 0 *as $x \to r_2$.*

(iv) *The boundary point $r_2$ is natural if and only if there exists a positive, decreasing solution $v_1$ of* (4.13) *satisfying*

$$\lim_{x\to r_2} v_1(x) = 0, \quad \lim_{x\to r_2} \frac{v_1'(x)}{W(x)} = 0$$

*and every solution of* (4.13) *independent of $v_1$ is unbounded at $r_2$.*

Analogous statements hold for the boundary point $r_1$. One has to replace decreasing by increasing everywhere, the condition $\lim_{x\to r_2} {v_1'(x)}/{W(x)} = -1$ by $\lim_{x\to r_1} {v_1'(x)}/{W(x)} = 1$ in (i), and finally $\lim_{x\to r_2} {v_1'(x)}/{W(x)} \le 0$ by $\lim_{x\to r_1} {v_1'(x)}/{W(x)} \ge 0$ in (ii).

We now apply the preceding results to the generation problem for the operator $A$ given by $Au := mu'' + qu'$ on $\mathrm{C}(\overline{J})$. We start by choosing for $A$ the maximal domain

$$D_M(A) := \{u \in \mathrm{C}(\overline{J}) \cap \mathrm{C}^2(J) : \ Au \in \mathrm{C}(\overline{J})\}.$$

We observe that $T_\lambda f \in D_M(A)$ for every $f \in \mathrm{C}(\overline{J})$, since $(\lambda - A)T_\lambda f = f$ and $T_\lambda f \in \mathrm{C}(\overline{J})$, by Proposition 4.9.

**4.15 Theorem.** *The operator $(A, D_M(A))$ generates a strongly continuous semigroup on $\mathrm{C}(\overline{J})$ if and only if each of the points $r_1$ and $r_2$ is either of entrance or of natural type. In this case, the semigroup is positive and contractive.*

PROOF. We leave as an exercise to show that $A$ is closed and densely defined. By Lemma 4.11 the operator $\lambda - A$ is injective for $\lambda > 0$ on $D_M(A)$ if and only if $R \notin \mathrm{L}^1(r_1, x_0)$ and $R \notin \mathrm{L}^1(x_0, r_2)$, that is, if both endpoints are of entrance or of natural type. In this case, the resolvent is given by $T_\lambda$; hence $(A, D_M(A))$ generates a contraction semigroup by Proposition 4.9 and Generation Theorem II.3.5. Finally, the generated semigroup is positive by Theorem 1.8. □

If $r_1$ and $r_2$ are regular boundary points, we define the domain of $A$ by using general boundary conditions. Given $\alpha_i, \beta_i \in \mathbb{R}$ such that $\alpha_i^2 + \beta_i^2 > 0$, $i = 1, 2$, we consider the so-called *elastic barrier conditions*, i.e., we put

$$D(A) := \Big\{u \in D_M(A) : \ \alpha_i u(r_i) + \beta_i \lim_{x \to r_i} \frac{u'(x)}{W(x)} = 0, \ i = 1, 2\Big\}. \tag{4.24}$$

Observe that if $W$ is bounded away from 0 (which is certainly true for nondegenerate operators), the above conditions reduce to those considered in Section 4.b. Moreover, if $\beta_1 = \beta_2 = 0$, we are imposing *Dirichlet* (or "*absorbing barrier*") conditions, while if $\alpha_1 = \alpha_2 = 0$, then we are imposing generalized *Neumann* (or "*reflecting barrier*") conditions.

The operator $(A, D(A))$ is not always dissipative (see Exercise 4.7.(3)). Even though the general case can be treated completely, we confine ourselves to the contractive case and impose the sign conditions $\alpha_1\beta_1 \le 0$ and $\alpha_2\beta_2 \ge 0$, guaranteeing dissipativity. For simplicity, we also suppose $\beta_1\beta_2 > 0$, so that $D(A)$ is dense in $C(\overline{J})$. The remaining case can be recovered as in Section 4.b. We state the following theorem, whose proof is outlined in Exercise 4.19.(1).

**4.16 Theorem.** *If $r_1$ and $r_2$ are regular boundary points and for the domain given by (4.24) with $\alpha_1\beta_1 \le 0$, $\alpha_2\beta_2 \ge 0$, and $\beta_1\beta_2 > 0$, the operator $(A, D(A))$ generates a strongly continuous contraction semigroup on $\mathrm{C}(\overline{J})$.*

Finally, we turn to *Ventcel* (or "*adhesive*") boundary conditions, i.e., the domain of $A$ is

$$D_V(A) := \big\{u \in \mathrm{C}(\overline{J}) \cap \mathrm{C}^2(J) : \lim_{x\to r_1,r_2} Au(x) = 0\big\}. \tag{4.25}$$

**4.17 Lemma.** *The operator $\big(A, D_V(A)\big)$ is closed, densely defined, and dissipative. Moreover, $(\lambda - A)u \geq 0$ implies $u \geq 0$.*

Proof. The verification that $A$ is closed and densely defined is standard and left as an exercise.

Now, for $u \in D_V(A)$, $u$ real, and $\lambda > 0$, we prove that

$$\inf_{x\in J}(\lambda - A)u(x) \leq \lambda \inf_{x\in J} u(x), \qquad \lambda \sup_{x\in J} u(x) \leq \sup_{x\in J}(\lambda - A)u(x). \tag{4.26}$$

If the maximum (respectively, the minimum) of $u$ is attained at an internal point $x_1 \in J$, we have $Au(x_1) \leq 0$ (respectively, $Au(x_1) \geq 0$), and hence (4.26) is valid. If the supremum or infimum of $u$ is attained at $r_1$ or $r_2$, the inequalities (4.26) hold again because $Au$ vanishes at $r_1, r_2$. □

By the preceding lemma and Theorem II.3.15, the operator $\big(A, D_V(A)\big)$ generates a strongly continuous contraction semigroup on $\mathrm{C}(\overline{J})$ if and only if the operator $I - A$ is surjective.

**4.18 Theorem.** *The operator $\big(A, D_V(A)\big)$ generates a strongly continuous semigroup on $\mathrm{C}(\overline{J})$ if and only if both the endpoints $r_1$ and $r_2$ are not of entrance type. In this case, the semigroup is positive and contractive.*

Proof. It suffices to use the above results for $\lambda = 1$ only. In particular, we write $T := T_1$ for the operator defined in (4.17).

Let $f \in \mathrm{C}(\overline{J})$ and $u$ be a solution of the equation $u - Au = f$. Then $u = Tf + c_1v_1 + c_2v_2$, where $v_1$ and $v_2$ are two linearly independent solutions of the homogeneous equation and $c_1, c_2 \in \mathbb{C}$. Writing $Au = g + c_1v_1 + c_2v_2$ with $g := Tf - f$, we see that the problem is to find constants $c_1, c_2$ such that $Au$ vanishes at the boundary.

Suppose first that $r_1, r_2$ are regular or exit boundaries. In this case all the solutions of (4.13) are bounded, and we can take $v_1$ positive increasing and vanishing at $r_1$, and $v_2$ positive decreasing and vanishing at $r_2$. This implies the existence of the constants as above.

Before discussing the case where a natural boundary point $r_i$ occurs, we check that in this case the equality $Tf(r_i) = f(r_i)$ holds for every $f \in \mathrm{C}(\overline{J})$. In fact, assume that $r_2$ is a natural boundary point. If $f$ vanishes in a neighborhood of $r_2$, then $Tf$ is a bounded solution of (4.13) near $r_2$; hence $Tf$ vanishes at $r_2$ by Theorem 4.14.(iv). The same holds if $f(r_2) = 0$ by the same density argument used at the end of the proof of Proposition 4.9. It remains to check the statement for the constant function $f = \mathbb{1}$. In this case, since both $\mathbb{1}$ and $T\mathbb{1}$ are solutions of $u - Au = \mathbb{1}$, we can write $T\mathbb{1} = \mathbb{1} + c_1v_1 + c_2v_2$, with $v_1$ as in Theorem 4.14.(iv) and $v_2$ unbounded. Then $c_2 = 0$, and the assertion follows, since $v_1$ vanishes at $r_2$.

If $r_1$ is regular or exit and $r_2$ is natural, then $g(r_2) = 0$ by the above argument. By Theorem 4.14, we can take a positive, decreasing solution $v_2$ of (4.13) vanishing at $r_2$ and with a finite positive limit at $r_1$. The function $u := Tf + c_2 v_2$ then belongs to $D_V(A)$ for an appropriate constant $c_2$.

If both the endpoints are natural, then, using again that $Tf(r_i) = f(r_i)$, we have $g(r_i) = 0$ for $i = 1, 2$ and can take $u := Tf$.

This shows that $I - A$ is surjective if both $r_1, r_2$ are not of entrance type.

We now assume that $I - A$ is surjective and show that $r_2$ is not an entrance boundary point. To this aim, by Theorem 4.14.(iii), it suffices to show that there exists a nonzero solution $u$ of $u - Au = 0$ vanishing at $r_2$. Choose $f \geq 0$, $f \not\equiv 0$, vanishing in $[x_1, r_2)$ and let $u \in D_V(A)$ be the solution of $u - Au = f$. Then $u \geq 0$ by Lemma 4.17, and $u$ is a solution of $u - Au = 0$ in $(x_1, r_2)$. If we show that $u \not\equiv 0$ in $(x_1, r_2)$, we can take as $v$ the continuation of $u$ to the whole of $J$ and we are done. In fact, we prove that if $K = \{x \in J : u(x) = 0\} \neq \emptyset$, then $u \equiv 0$. Hence $u$ never vanishes in $J$. Assume that $u(x_2) = 0$ and observe that $u'(x_2) = 0$, since $u$ does not change sign. From the equation we have $Au \leq u$, that is,

$$\left(\frac{u'}{W}\right)' \leq \frac{u}{mW}.$$

Then

$$u'(x) \leq W(x) \int_{x_2}^{x} \frac{u(s)}{m(s)W(s)}\, ds$$

for $x \in (x_2, r_2)$. Choose $\delta < 1$ such that $W(x) \int_{x_2}^{x} \frac{1}{m(s)W(s)}\, ds < 1$ for $x \in (x_2, x_2+\delta)$ and take $M = \sup\{u(x) : x \in (x_2, x_2+\delta)\}$. Then $u'(x) \leq M$; hence $u(x) \leq M\delta$ for $x \in (x_2, x_2 + \delta)$, and $M \leq M\delta$ implies $M = 0$. A similar argument shows that $u$ vanishes in a left neighborhood of $x_2$. Therefore $K$ is open, and since it is trivially closed, $K = J$.

Finally, the positivity of the generated semigroup follows from Lemma 4.17 and Theorem 1.8. □

Looking at the evolution problem $u_t = Au$, $u(0) = u_0$, Ventcel conditions mean that the solution $u(t, x)$ is constant at $x = r_1, r_2$. In particular, if $u_0(r_1) = u_0(r_2) = 0$, then $u(t, x)$ satisfies the classical homogeneous Dirichlet boundary conditions $u(t, r_1) = u(t, r_2) = 0$.

**4.19 Exercises.** (1) Let $r_1, r_2$ be regular boundary points, and $\alpha_i, \beta_i \in \mathbb{R}$ such that $(-1)^i \alpha_i \beta_i \geq 0$, $\alpha_i^2 + \beta_i^2 > 0$ for $i = 1, 2$ and $\beta_1 \beta_2 \neq 0$. Moreover, let $(A, D(A))$ be defined as in (4.24).

(i) Show that $(A, D(A))$ is closed, densely defined, and dissipative.

(ii) Using Theorem 4.14, choose two monotone solutions $w_1, w_2$ of $\lambda u - Au = 0$ such that

$$w_1(r_1) = w_2(r_2) = 0, \qquad \lim_{x \to r_1} \frac{w_1'(x)}{W(x)} = 1, \qquad \lim_{x \to r_2} \frac{w_2'(x)}{W(x)} = -1.$$

Show that for every $f \in C(\overline{J})$ there exist constants $c_1, c_2$ such that $u = T_\lambda f + c_1 w_1 + c_2 w_2 \in D(A)$. Finally, deduce Theorem 4.16.

(iii) Arguing as in Lemma 4.17, show that the resolvent of $(A, D(A))$ is positive, so that the generated semigroup is positive.

(2) Show, using Lemma III.2.4 and Example III.2.2 in any compact interval $K \subset J$, that the operators $(A, D_V(A))$ and $(A, D_M(A))$ are closed and densely defined.

(3) Classify, according to Theorem 4.14, the endpoints of $J$ in the following situations: $Au(x) := u''(x)$, $J := (0,\infty)$; $Au(x) := x^\alpha u''(x)$, $J := (0,\infty)$; $Au(x) := (x^\alpha u'(x))'$, $J := (0,\infty)$; $Au(x) := u''(x) + xu'(x)$, $J := \mathbb{R}$; $Au(x) := x(1-x)u''(x) + b(x)u'(x)$, $J := (0,1)$; $Au(x) := u''(x) + \frac{b(x)}{x(1-x)}u'(x)$, $J := (0,1)$, where $\alpha \in \mathbb{R}$ and $b$ is assumed to be Hölder continuous at the endpoints.

(4) Show that $D_M(A) = D_V(A)$ if the endpoints are natural boundaries.

(5) Let $A$ be a nondegenerate operator on $\mathrm{C}(\overline{\mathbb{R}})$ as in Section 4.a. Show that $\mathrm{C}^2(\overline{\mathbb{R}}) = D_M(A)$. (Hint: Use Example III.2.2 together with elementary arguments to control the limits at $\pm\infty$.)

(6) Show that the classification (4.23) of the endpoints is invariant under similarity transformations arising from a change of variables as in Exercise 4.4.(1). (Hint: Use Lemmas 4.11 and 4.12.)

(7*) Consider the operator $A$ with $Au := mu'' + qu'$ on $J := [0,\infty)$ studied in Section 4.a. Prove that $\infty$ is a natural boundary point. (Hint: Assume $R \in \mathrm{L}^1(0,\infty)$ and observe that $W \in \mathrm{L}^1(0,\infty)$ and $1/(mW) \notin \mathrm{L}^1(0,\infty)$, since $\inf_{x\geq 0} m(x) > 0$. Then take a sequence $x_n \to \infty$ such that $R(x_n) \to 0$ and apply the generalized mean value theorem to the ratio $\left(\int_0^{x_n} 1/(m(x)W(x))\,dx\right)/\left(1/W(x_n)\right)$ to deduce a contradiction. In a similar way, prove that $Q \notin \mathrm{L}^1(0,\infty)$.)

## *d. Analyticity of Degenerate Semigroups*

On $\mathrm{C}[0,1]$ we consider a highly degenerate operator $B$ given by

$$Bu := mu'' + qu',$$

where $m(0) = m(1) = 0$, $\sqrt{m} \in \mathrm{C}^1[0,1]$, $q \in \mathrm{C}[0,1]$, and $q/\sqrt{m}$ is bounded in $(0,1)$. Observe that $m(x) = O(x^2)$ as $x \to 0$ and $m(x) = O((1-x)^2)$ as $x \to 1$. Therefore, the map $\varphi : [0,1] \to \overline{\mathbb{R}}$ with

$$\varphi(x) := \int_{1/2}^{x} \frac{1}{\sqrt{m(t)}}\,dt, \quad x \in [0,1],$$

is bijective, and hence the corresponding operator $Q_\varphi$ given by (4.6) is invertible as a map from $\mathrm{C}(\overline{\mathbb{R}})$ to $\mathrm{C}[0,1]$. Moreover, as in (4.7) (or see Exercise 4.4.(1)), $Q_\varphi$ transforms the operator $B$ into $A := Q_\varphi^{-1} B Q_\varphi$ given by

$$Av = v'' + \frac{\widetilde{q} - \widetilde{m'}/2}{\sqrt{\widetilde{m}}}\,v',$$

where we write $\widetilde{h} := Q_\varphi^{-1} h \in \mathrm{C}(\overline{\mathbb{R}})$ for $h \in \mathrm{C}[0,1]$. The hypotheses on the coefficients $m$ and $q$ are exactly those for which the operator $A$ belongs to the class considered in Section 4.a. By Theorem 4.3 we obtain

that $(A, D(A))$, with $D(A) := \mathrm{C}^2(\overline{\mathbb{R}})$, generates an analytic semigroup on $\mathrm{C}(\overline{\mathbb{R}})$. Moreover, $D(A)$ is the maximal domain of $A$, that is, $D(A) = \{u \in \mathrm{C}(\overline{\mathbb{R}}) \cap \mathrm{C}^2(\mathbb{R}) : Au \in \mathrm{C}(\overline{\mathbb{R}})\}$ (see Exercise 4.19.(5)). Therefore, we take the domain of $B$ as

$$D_M(B) := \{u \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : Bu \in \mathrm{C}[0,1]\}$$

and obtain $D_M(B) = Q_\varphi D(A) = D(Q_\varphi A Q_\varphi^{-1})$. By similarity, this implies the following result.

**4.20 Theorem.** *The operator $(B, D_M(B))$ generates an analytic semigroup of angle $\pi/2$ in* $\mathrm{C}[0,1]$.

Observe that both 0 and 1 are natural boundary points for $B$, so that $D_M(B)$ coincides with the Ventcel domain $D_V(B)$ (see Exercise 4.25).

We now consider the case of first-order zeros at $x = 0, 1$, that is, the differential operator

$$Lu(x) := m(x)[x(1-x)u''(x) + b(x)u'(x)] \tag{4.27}$$

with Ventcel boundary conditions, i.e., we take the domain

$$D_V(L) := \{u \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : \lim_{x \to 0,1} Lu(x) = 0\}. \tag{4.28}$$

In addition, we suppose $m, b \in \mathrm{C}[0,1]$, $m$ strictly positive and $b$ Hölder continuous at the endpoints. By Theorem 4.18 we know that $(L, D_V(L))$ generates a $C_0$-semigroup if and only if both 0 and 1 are not entrance boundary points, that is, if and only if $b(0) < 1$ and $b(1) > -1$ (see Exercise 4.19.(3)).

The analyticity of the semigroup is stated in the following theorem.

**4.21 Theorem.** *Under the above assumptions the operator $(L, D_V(L))$ generates a bounded analytic semigroup of angle $\pi/2$.*

We point out only the main techniques used in the proof of the above theorem. For details see [Met98] and [CM98].

The aim is to solve in $D_V(L)$ the stationary equation $\lambda u - Lu = f$ for each $f \in \mathrm{C}[0,1]$ and to obtain the right decay (in $\lambda$) of the norm of the resolvent. It is clear that the difficulties come from the degeneration at the endpoints. If we can solve the equation (with the given boundary conditions) in neighborhoods of 0 and 1, a partition of unity argument yields a global solution. In this approach, the easiest way to localize near $x = 0$ is to consider an auxiliary problem in $\mathbb{R}_+$ with degeneration at $x = 0$. Therefore, we define

$$L_1 u(x) := m_1(x)[xu''(x) + b_1(x)u'(x)]$$

for $u \in D_V(L_1) := \{u \in \mathrm{C}[0,\infty] \cap \mathrm{C}^2(0,\infty] : \lim_{x\to 0} L_1 u(x) = 0\}$. We assume $m_1$ to be bounded and uniformly continuous with $\inf_{x>0} m_1(x) > 0$, and $b_1$ to be continuous, bounded, and Hölder continuous at $x = 0$ with $b_1(0) < 1$.

The similarity transformation with $Q_\varphi$ for $\varphi(x) := \sqrt{x}$ transforms the operator $L_1$ into the operator $L_2$ given by

$$L_2u := m_2\Big[u'' + \frac{b_2}{s}u'\Big],$$

where $m_2(s) := {}^1\!/\!_4 m_1(s^2)$ and $b_2(s) := 2b_1(s^2) - 1$, with domain

$$D_V(L_2) := \Big\{u \in \mathrm{C}[0,\infty] \cap \mathrm{C}^2(0,\infty] : \lim_{s\to 0} L_2u(s) = 0\Big\}.$$

The coefficients of the operator $L_2$ satisfy the same properties as the coefficients of $L_1$, and moreover, $b_2(0) < 1$.

Writing $b_2(s) = b_2(0) + c(s)$, it is possible to deduce the general case from the case where $b_2$ is constant, treating the additional term as a perturbation. However, the needed perturbation argument is not standard because of the degeneracy of the operator. Here, we discuss only the case of "constant" coefficients, i.e., $m_2 \equiv 1$ and $b_2 \equiv b < 1$, and

$$L_0u(s) := u''(s) + \frac{b}{s}u'(s)$$

with domain $D_V(L_0) := \{u \in \mathrm{C}[0,\infty] \cap \mathrm{C}^2(0,\infty] : \lim_{s\to 0} L_0u(s) = 0\}$. To prove the analyticity of the semigroup generated by $L_0$, we argue as in Section 4.a, localizing first its spectrum and then proving estimate (4.9) in Theorem II.4.6.

The location of the spectrum of $L_0$ heavily depends on the properties of Bessel functions and is stated in the following lemma.

**4.22 Lemma.** *The spectrum of $\big(L_0, D_V(L_0)\big)$ is contained in $(-\infty, 0]$.*

We now prove that the norm of the resolvent $R(\lambda, L_0)$ decays like $|\lambda|^{-1}$. In our situation such an estimate, which in general is quite difficult to obtain, can be easily obtained by a simple rescaling argument. The possibility of using this technique is the main reason for which we consider these operators in $[0,\infty)$.

**4.23 Theorem.** *The operator $\big(L_0, D_V(L_0)\big)$ generates a bounded analytic semigroup of angle ${}^\pi\!/\!_2$.*

PROOF. Consider the group of isometries $(I_t)_{t>0}$ on $\mathrm{C}[0,\infty]$ defined by $I_tf(x) := f(t^{1/2}x)$. It is easily verified that $L_0I_t = tI_tL_0$, and, as a consequence, $I_t\big(D_V(L_0)\big) = D_V(L_0)$. If $\lambda = t\omega$ with $|\omega| = 1$ and $\omega \neq -1$, then

$$(\lambda - L_0)^{-1} = t^{-1}I_t(\omega - L_0)^{-1}I_{t^{-1}}$$

and

$$\big\|(\lambda - L_0)^{-1}\big\| \leq |\lambda|^{-1}C(\omega),$$

where $C(\omega) := \|(\omega - L_0)^{-1}\|$ depends continuously on $\omega$. This estimate, together with the preceding lemma, shows that $\big(L_0, D(L_0)\big)$ generates a bounded analytic semigroup of angle ${}^\pi\!/\!_2$. $\square$

We conclude this section with an application to a more general degenerate operator in the space C[0, 1] given by

$$A_{\alpha\beta}u(x) := m(x)[x^\alpha(1-x)^\beta u''(x)],$$

where $\alpha$ and $\beta$ are positive real numbers, $m$ is continuous and strictly positive on $[0,1]$. We assume Ventcel boundary conditions, and we define

$$D_V(A_{\alpha\beta}) := \left\{u \in \mathrm{C}[0,1] \cap \mathrm{C}^2(0,1) : \lim_{x\to 0,1} A_{\alpha\beta}u(x) = 0\right\}.$$

**4.24 Proposition.** *The operator* $\big(A_{\alpha\beta}, D_V(A_{\alpha\beta})\big)$ *generates an analytic semigroup of angle* $\pi/2$ *for every* $\alpha, \beta > 0$.

Sketch of Proof. For $\alpha, \beta \geq 2$ the result follows from Theorem 4.20. Suppose $0 < \alpha, \beta < 2$ and put

$$\varphi(x) := \int_0^x t^{-\alpha/2}(1-t)^{-\beta/2}\, dt \qquad \text{for } x \in [0,1].$$

Then the operator $\big(A_{\alpha\beta}, D_V(A_{\alpha\beta})\big)$ on C[0, 1] transforms via the similarity transformation with $Q_\varphi$ into a degenerate operator $A_1 := Q_\varphi^{-1} A_{\alpha\beta} Q_\varphi$ on $\mathrm{C}[0,l]$ for $l := \varphi(1)$. More precisely, $A_1$ is of the form

$$A_1 v(s) = m(s)\left[v''(s) + \frac{b_1(s)}{s(l-s)} v'(s)\right], \qquad s \in [0,l],$$

with Ventcel boundary conditions. Near the endpoints $x = 0$ and $l$ the operator $A_1$ behaves like the operator $L_2$ near $x = 0$; hence it can be proved, using the strategy outlined above, that it generates an analytic semigroup of angle $\pi/2$. The general case can be deduced by a partition of unity argument. □

**4.25 Exercise.** Use Exercise 4.19.(7), together with an appropriate similarity transformation, to show that 0 and 1 are natural boundaries for the operator $B$ and deduce that $D_M(B) = D_V(B)$. Prove that $\lim_{x\to 0,1} \sqrt{m(x)}u'(x) = 0$ for every $u \in D_M(B)$.

## *Notes and Further Reading to Section* 4

**Sections** 4.a **and** 4.b**.** Here we have presented in the framework of semigroup theory some classical results on evolution problems in one space dimension with general boundary conditions. All the results hold with a uniformly continuous coefficient $m$. Since this requires more technical proofs based on a partition of unity argument, we assumed $m \in \mathrm{C}^1$ to simplify the exposition.

**Section** 4.c. This part is meant as an introduction to Feller's theory of semigroups of positive contractions in spaces of continuous functions. Most of the material is already contained in [Fel52]. We give his classification of the endpoints in a different but, by Remark 4.10, equivalent way. Theorem 4.15 and Theorem 4.16 are in [Fel52]. Theorem 4.18, which we prove more in the spirit of Feller, is in [CT86]. The whole theory originated from applications to one-dimensional diffusion processes in probability theory. In this setting, the evolution equation $u_t = Au$ is Kolmogorov's backward equation. We refer to [Fel54], [Man68], and [Lam77] for a detailed analysis of this matter as well as for an explanation of the terminology and a discussion of the probabilistic meaning of the type of boundaries and boundary conditions.

For references on Feller semigroups in the case of higher dimensions we refer to the Notes of Section II.2.b and, for the discussion of boundary conditions, to [Tai92], [Tai95], and [Tai97].

**Section** 4.d. This part is devoted to some regularity results for Feller's semigroups. A version of Theorem 4.20 is in [FR98]; the case $q \equiv 0$ is in [AR94]. The first analyticity result for first-order degeneration in spaces of continuous functions is due to Angenent (see [Ang88]) in the case of Neumann boundary conditions. Complete proofs of Theorems 4.21 and 4.23 are given in [Met98] and [CM98].

# 5. Semigroups for Partial Differential Operators

**(by Abdelaziz Rhandi)**

Starting with Hadamard's work (see [Had23] or the quotations in Section II.6), initial value problems for partial differential equations were one of the motivating forces for the development of semigroup theory as well as the main source for applications of the results. Today, most books on partial differential equations (e.g., [RR93], [Eva98]) contain a section on semigroups. Conversely, the semigroup books by Goldstein [Gol85] and Pazy [Paz83] devote a large part to applications to partial differential equations, and in the monographs by Amann [Ama95] and Lunardi [Lun95] semigroups are a main tool for a systematic study of parabolic equations.

Therefore, this section is meant only as a brief introduction to this field. We restrict our attention to partial differential operators on all of $\mathbb{R}^N$, thus avoiding the delicate discussion of boundary conditions (already encountered in the one-dimensional situation in Section 4). Our aim is to solve initial value problems for parabolic partial differential equations with constant coefficients

$$\text{(cPDE)} \qquad \frac{\partial u(x,t)}{\partial t} = \sum_{|\alpha| \le m} a_\alpha \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}$$

or with variable coefficients

$$\text{(vPDE)} \qquad \frac{\partial u(x,t)}{\partial t} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x,t)}{\partial x_j}\Big)$$

for $x \in \mathbb{R}^N$ and $t \geq 0$. To that purpose, we use operators and semigroups on the Hilbert space $\mathrm{L}^2(\mathbb{R}^N)$. For deeper and more general results we refer to the literature quoted above and in the notes.

The section is organized as follows. In Section 5.a we fix some notation and recall various properties of the Fourier transform. Section 5.b contains a discussion of partial differential operators with constant coefficients using the Fourier transform approach. Finally, the last subsection is concerned with second-order elliptic differential operators with symmetric variable coefficients using variational methods.

## *a. Notation and Preliminary Results*

In this subsection we briefly discuss the Fourier transform on $\mathrm{L}^2(\mathbb{R}^N)$ and on the corresponding Sobolev spaces. We first fix some notation.

For a multi-index $\alpha := (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, we define

$$|\alpha| := \sum_{k=1}^{N} \alpha_k, \qquad x^\alpha := x_1^{\alpha_1} \cdot \ldots \cdot x_N^{\alpha_N},$$
$$|x|^2 := \sum_{k=1}^{N} x_k^2, \qquad xy := \sum_{k=1}^{N} x_k y_k \text{ for } x, y \in \mathbb{R}^N,$$

and $D^\alpha := D_1^{\alpha_1} \cdots D_N^{\alpha_N}$, where $D_k := \partial/\partial x_k$ for $k \in \{1, \ldots, N\}$. We now introduce the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.

**5.1 Definition.** *A function $f : \mathbb{R}^N \to \mathbb{C}$ is said to be rapidly decreasing if it is infinitely many times differentiable, i.e., $f \in \mathrm{C}^\infty(\mathbb{R}^N)$, and*

$$\lim_{|x|\to\infty} |x|^n D^\alpha f(x) = 0 \quad \textit{for all } n \in \mathbb{N} \textit{ and } \alpha \in \mathbb{N}^N.$$

*The space*

$$\mathcal{S}(\mathbb{R}^N) := \left\{ f \in \mathrm{C}^\infty(\mathbb{R}^N) : f \text{ is rapidly decreasing} \right\}$$

*is called the Schwartz space.*

When endowed with the family of seminorms

$$|f|_{n\alpha} := \sup_{x\in\mathbb{R}^N} |x^n D^\alpha f(x)|,$$

the space $\mathcal{S}(\mathbb{R}^N)$ becomes a Fréchet space containing $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N)$ as a dense subspace (see [RR93, Def. 5.15 and the following remark] or [DL88, App. Distributions, 3.2]).

**5.2 Definition.** *The Fourier transform* $\mathcal{F}f$ *of* $f \in \mathcal{S}(\mathbb{R}^N)$ *is defined by*[3]

$$(\mathcal{F}f)(y) := \int_{\mathbb{R}^N} \mathrm{e}^{-ixy} f(x)\,dx \quad \text{for } y \in \mathbb{R}^N.$$

**5.3 Example.** The function $\mu_t : \mathbb{R}^N \to \mathbb{R}$ given by

$$\mu_t(x) := (4\pi t)^{-N/2}\,\mathrm{e}^{-|x|^2/4t}, \quad x \in \mathbb{R}^N,$$

belongs to $\mathcal{S}(\mathbb{R}^N)$ for each $t > 0$. Moreover,

$$(\mathcal{F}\mu_t)(y) = \mathrm{e}^{-t|y|^2}, \quad y \in \mathbb{R}^N$$

(see, e.g., [RS75, Sec. IX.1, Expl. 1]).

The Fourier transform leaves $\mathcal{S}(\mathbb{R}^N)$ invariant and transforms derivations into multiplications. This is the content of the following lemma. For the proof we refer to [DL88, App. Distributions, 3.2, Prop. 2].

**5.4 Lemma.** *For all* $f \in \mathcal{S}(\mathbb{R}^N)$ *and* $\alpha \in \mathbb{N}^N$ *the following properties hold.*

(i) $\mathcal{F}f \in \mathrm{C}^\infty(\mathbb{R}^N)$ *and* $D^\alpha(\mathcal{F}f) = (-\mathrm{i})^{|\alpha|}\mathcal{F}(x^\alpha f)$.

(ii) $\mathcal{F}(D^\alpha f) = \mathrm{i}^{|\alpha|} y^\alpha \mathcal{F}f$.

(iii) $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^N)$.

Since $|(\mathcal{F}f)(y)| \le \|f\|_1$ for all $y \in \mathbb{R}^N$, it follows that the Fourier transform can be continuously extended to $\mathrm{L}^1(\mathbb{R}^N)$. Since $\mathcal{S}(\mathbb{R}^N)$ is dense in $\mathrm{L}^1(\mathbb{R}^N)$ and contained in $\mathrm{C}_0(\mathbb{R}^N)$, we obtain from Lemma 5.4.(iii) the following important property, whose vector-valued analogue is stated in Theorem C.8.

**5.5 Riemann–Lebesgue Lemma.** *The extended Fourier transform satisfies* $\mathcal{F}\,\mathrm{L}^1(\mathbb{R}^N) \subset \mathrm{C}_0(\mathbb{R}^N)$, *i.e.,* $\lim_{|y|\to\infty}(\mathcal{F}f)(y) = 0$ *for all* $f \in \mathrm{L}^1(\mathbb{R}^N)$.

Next, we give an explicit representation of the inverse Fourier transform. For the proof we refer again to [DL88, App. Distributions, 3.2, Rem. 3].

**5.6 Theorem.** *For all* $f \in \mathcal{S}(\mathbb{R}^N)$ *we have*

$$\big(\mathcal{F}(\mathcal{F}f)\big)(x) = (2\pi)^N f(-x), \quad x \in \mathbb{R}^N.$$

*In particular, the Fourier transform* $\mathcal{F} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ *is bijective, and its inverse* $\mathcal{F}^{-1}$ *is given by*

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}xy} f(y)\,dy, \quad x \in \mathbb{R}^N.$$

*Moreover, we have that*

$$(2\pi)^{-N}\,(\mathcal{F}f \,|\, \mathcal{F}g)_{\mathrm{L}^2} = (f \,|\, g)_{\mathrm{L}^2} := \int_{\mathbb{R}^N} f(x)\,\overline{g(x)}\,dx$$

*for all* $f, g \in \mathcal{S}(\mathbb{R}^N)$.

[3] In the context of differential equations it is more common to define the Fourier transform with the factor $(2\pi)^{-N/2}$ in front of the integral. However, for reasons explained in Appendix C.b we prefer the above definition.

Since $\mathcal{S}(\mathbb{R}^N)$ is also dense in $\mathrm{L}^2(\mathbb{R}^N)$, it follows from Theorem 5.6 that $\mathcal{F}$ can be extended continuously to $\mathrm{L}^2(\mathbb{R}^N)$. This extension will still be denoted by $\mathcal{F}$ and satisfies *Plancherel's equation*

$$\text{(PE)} \qquad (2\pi)^{-N}\,(\mathcal{F}f\,|\,\mathcal{F}g)_{\mathrm{L}^2} = (f\,|\,g)_{\mathrm{L}^2} \quad \text{for all } f,g \in \mathrm{L}^2(\mathbb{R}^N),$$

i.e., $(2\pi)^{-N/2}\,\mathcal{F}$ is a unitary operator.

Next, we define the (classical) Sobolev spaces needed in the sequel.

**5.7 Definition.** *The space* $\mathrm{H}^n(\mathbb{R}^N) :=$

$$\left\{ f \in \mathrm{L}^2(\mathbb{R}^N) : \begin{array}{l} \text{for every } |\alpha| \le n \text{ there exists } g_\alpha \in \mathrm{L}^2(\mathbb{R}^N) \text{ such that} \\ (f\,|\,D^\alpha\varphi)_{\mathrm{L}^2} = (-1)^{|\alpha|}\,(g_\alpha\,|\,\varphi)_{\mathrm{L}^2} \text{ for all } \varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N) \end{array} \right\}$$

*is called the (classical) Sobolev space of order* $n$. *For* $f \in \mathrm{H}^n(\mathbb{R}^N)$ *and* $|\alpha| \le n$ *we call* $D^\alpha f := g_\alpha$ *the weak derivative of order* $\alpha$ *of* $f$.

Equipped with the inner product

$$\text{(5.1)} \qquad (f\,|\,g)_{\mathrm{H}^n} := \sum_{|\alpha|\le n} (D^\alpha f\,|\,D^\alpha g)_{\mathrm{L}^2}$$

and the associated norm $\|\cdot\|_{\mathrm{H}^n}$ the space $\mathrm{H}^n(\mathbb{R}^N)$ is a Hilbert space.

Note that for an $n$-times continuously differentiable function $f \in \mathrm{H}^n(\mathbb{R}^N)$ its weak derivative $D^\alpha f$ and its classical derivative coincide for all $|\alpha| \le n$.

Next, we relate $\mathrm{H}^n(\mathbb{R}^N)$ to

$$X_{n,q} := \big(D(M_q^n), \|\cdot\|_n\big),$$

the $n$th abstract Sobolev space associated to multiplication operator $M_q$ on $X := \mathrm{L}^2(\mathbb{R}^N)$, where

$$\text{(5.2)} \qquad q(y) := (1+|y|^2)^{1/2}, \quad y \in \mathbb{R}^N$$

(see Definition II.5.1). In Example II.5.7 we have shown that

$$X_{n,q} = \big\{ f \in \mathrm{L}^2(\mathbb{R}^N) : q^n f \in \mathrm{L}^2(\mathbb{R}^N) \big\},$$

where the corresponding norm $\|\cdot\|_n$ is given by

$$\|f\|_{n,q} := \left( \int_{\mathbb{R}^N} (1+|y|^2)^n |f(y)|^2 \, dy \right)^{1/2} \quad \text{for } f \in X_{n,q}.$$

By the above results the Fourier transform is an isomorphism on $\mathrm{L}^2(\mathbb{R}^N)$ and transforms the weak derivative $D^\alpha$ into multiplication by $\mathrm{i}^{|\alpha|} y^\alpha$. These two facts imply the following result (see Exercise 5.11).

**5.8 Theorem.** *With the above definitions we have*

$$\mathcal{F}\,\mathrm{H}^n(\mathbb{R}^N) = X_{n,q}.$$

*Moreover, the norm* $\|\cdot\|_{\mathrm{H}^n}$ *is equivalent to the norm defined by*

$$|f|_n := \|\mathcal{F}f\|_{n,q} \quad \text{for } f \in \mathrm{H}^n(\mathbb{R}^N);$$

*hence the Fourier transform is an isomorphism from* $\mathrm{H}^n(\mathbb{R}^N)$ *onto* $X_{n,q}$.

It is easy to verify that $\mathcal{S}(\mathbb{R}^N)$ is dense in $X_{n,q}$; hence by the previous theorem it is dense in $\mathrm{H}^n(\mathbb{R}^N)$. Since we already observed that $\mathrm{C}_c^\infty(\mathbb{R}^N)$ is dense in $\mathcal{S}(\mathbb{R}^N)$, we obtain the following result.

**5.9 Corollary.** *The spaces* $\mathrm{C}_c^\infty(\mathbb{R}^N)$ *and* $\mathcal{S}(\mathbb{R}^N)$ *are dense in* $\mathrm{H}^n(\mathbb{R}^N)$ *for each* $n \in \mathbb{N}$.

Moreover, the Sobolev space $\mathrm{H}^n(\mathbb{R}^N)$ for $n$ sufficiently large consists of continuous functions vanishing at infinity. This is the content of the following embedding theorem. For its proof, which relies on Theorem 5.8, we refer to [RR93, Cor. 6.92].

**5.10 Sobolev's Embedding Theorem.** *Let* $n \in \mathbb{N}$ *and* $k \in \mathbb{N}_0$ *with* $n > k + {}^{N}\!/_{2}$. *If* $f \in \mathrm{H}^n(\mathbb{R}^N)$, *then* $D^\alpha f \in \mathrm{C}_0(\mathbb{R}^N)$ *for all* $\alpha \in \mathbb{N}^N$ *with* $|\alpha| \le k$, *and the embedding*

$$\mathrm{H}^n(\mathbb{R}^N) \hookrightarrow \mathrm{C}_0^k(\mathbb{R}^N)$$

*is continuous.*

**5.11 Exercise.** Give the details of the proof of Theorem 5.8. (Hint: For the inclusion $\mathcal{F}\mathrm{H}^n(\mathbb{R}^N) \subseteq X_{n,q}$ observe that $1 + |y|^2 \le c(1 + y_1^{2n} + \cdots + y_N^{2n})^{1/n}$ for all $y \in \mathbb{R}^N$ and a suitable constant $c > 0$. To show the equivalence of the norms $\|\cdot\|_{\mathrm{H}^n}$ and $|\cdot|_n$ use Plancherel's equation and the open mapping theorem.)

## *b. Elliptic Differential Operators with Constant Coefficients*

In this subsection we apply semigroup theory and the Fourier transform to the *formal* differential operator

$$A(D) := \sum_{|\alpha| \le m} a_\alpha D^\alpha,$$

where $a_\alpha \in \mathbb{C}$. In order to define an operator on $\mathrm{L}^2(\mathbb{R}^N)$ corresponding to $A(D)$ we use Lemma 5.4 and the multiplication operator $M_a$ on $\mathrm{L}^2(\mathbb{R}^N)$ defined by the polynomial

$$a(\xi) := \sum_{|\alpha| \le m} a_\alpha \cdot (\mathrm{i}\xi)^\alpha, \quad \xi \in \mathbb{R}^N.$$

Then the operator $A$ is defined by

$$(5.3)\qquad \begin{aligned} A &:= \mathcal{F}^{-1} M_a \mathcal{F}, \\ D(A) &:= \mathcal{F}^{-1} D(M_a) = \big\{ f \in \mathrm{L}^2(\mathbb{R}^N) : a(\cdot)\, \mathcal{F} f \in \mathrm{L}^2(\mathbb{R}^N) \big\}. \end{aligned}$$

As a consequence of Propositions I.4.11, I.4.12 and Paragraph II.2.1 we immediately obtain the following result.

**5.12 Theorem.** *Let $\big(A, D(A)\big)$ be the operator on $\mathrm{L}^2(\mathbb{R}^N)$ defined in* (5.3). *Then the following assertions are equivalent.*

(a) *$A$ generates a strongly continuous semigroup.*

(b) *$M_a$ generates a strongly continuous semigroup.*

(c) $\sup_{\xi \in \mathbb{R}^N} \operatorname{Re} a(\xi) < \infty$.

In order to obtain special properties such as analyticity of the semigroup generated by $A$, it therefore suffices to look at the function $a(\cdot)$ and the properties of the corresponding multiplication semigroup. This is the idea behind the following definition.

**5.13 Definition.** (i) *The operator $A$ defined in* (5.3) *is called elliptic if the principal part $a_m(\xi) := \sum_{|\alpha|=m} a_\alpha \cdot (\mathrm{i}\xi)^\alpha$ of $a(\cdot)$ satisfies*

$$a_m(\xi) \neq 0 \quad \textit{for all } 0 \neq \xi \in \mathbb{R}^N.$$

(ii) *If $a(\xi) = a_m(\xi)$, then $A$ is called homogeneous.*

We start by proving some estimates for the polynomial defining an elliptic operator.

**5.14 Lemma.** *Suppose $A$ to be an elliptic operator defined by* (5.3). *For the corresponding polynomial $a$ the following properties hold.*

(i) *There are positive constants $c_1, c_2$, and $R$ such that*

$$c_1 |\xi|^m \leq |a(\xi)| \leq c_2 |\xi|^m \quad \textit{for all } |\xi| \geq R.$$

(ii) *Let $A$ be homogeneous with real coefficients $a_\alpha$. If $N \geq 2$, then $m$ is even. If $m$ is even, then $a(\xi) < 0$ or $a(\xi) > 0$ for all $0 \neq \xi \in \mathbb{R}^N$.*

Proof. (i) For $|\xi| \geq 1$ we have

$$|a(\xi)| \leq \sum_{|\alpha| \leq m} |a_\alpha| \cdot |\xi|^{|\alpha|} \leq \Big( \sum_{|\alpha| \leq m} |a_\alpha| \Big) \cdot |\xi|^m =: c_2 |\xi|^m.$$

On the other hand, since $A$ is elliptic, we have $c := \inf\{|a_m(\xi)| : |\xi| = 1\} > 0$. Hence,

$$|a_m(\xi)| = |\xi|^m |a_m\big(\tfrac{\xi}{|\xi|}\big)| \geq c|\xi|^m \quad \text{for all } 0 \neq \xi \in \mathbb{R}^N.$$

Combining both estimates we obtain

$$\begin{aligned} |a(\xi)| &\geq |a_m(\xi)| - |a_{m-1}(\xi)| - \cdots - |a_0(\xi)| \\ &\geq c|\xi|^m - c_2\big(|\xi|^{m-1} + \cdots + 1\big) \geq \tfrac{c}{2}\, |\xi|^m \quad \text{for all } |\xi| \geq R \end{aligned}$$

and a sufficiently large constant $R > 0$. This proves (i) for $c_1 := {}^{c}\!/_{2}$.

(ii) Let $N \geq 2$. By the ellipticity of $A$ we can take $0 \neq \xi_0 \in \mathbb{R}^N$ such that $a(\mathrm{i}\xi_0) \neq 0$. Choose $\xi_1 \in \mathbb{R}^N$ such that the vectors $\xi_1$ and $\xi_0$ are independent. If we put $\widetilde{a}(\xi) := a(\mathrm{i}\xi)$ and $P(s) := \widetilde{a}(\xi_1 + s\xi_0)$ for $s \in \mathbb{R}$, then $P$ is a real $m$th-degree polynomial in $s$ with leading coefficient $\widetilde{a}(\xi_0) \neq 0$. Such a polynomial of odd order $m$ has at least one real root $s_0$. Hence, the ellipticity of $A$ and the linear independence of $\xi_0$ and $\xi_1$ imply $\xi_0 = \xi_1 = 0$. Therefore, $m$ must be even. Since $a$ is real and $\mathbb{R}^N \setminus \{0\}$ is connected for $N \geq 2$, it follows that $a(\mathbb{R}^N \setminus \{0\})$ is an interval of $\mathbb{R}$. Consequently, due to the ellipticity of $A$, we have $a(\xi) < 0$ or $a(\xi) > 0$ for all $0 \neq \xi \in \mathbb{R}^N$. Since the case $N = 1$ is trivial, the proof is complete. □

With this lemma we characterize elliptic differential operators generating analytic semigroups.

**5.15 Theorem.** *Let $A$ be the operator defined in* (5.3). *Assume that $A$ is elliptic and that condition* (c) *in Theorem* 5.12 *is satisfied. Then the following assertions are true.*

(i) *$D(A) = \mathrm{H}^m(\mathbb{R}^N)$ and $A = A(D)$, i.e., $Af = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f$ for all $f \in D(A)$, where $D^\alpha$ denotes the weak derivative introduced in Definition* 5.7.

(ii) *If there is $\delta \in (0, \pi/2]$ such that $a(\mathbb{R}^N) \subset \mathbb{C} \setminus \Sigma_{\delta + \pi/2}$, then $A$ generates a bounded analytic semigroup $(T(t))_{t\geq 0}$ of angle $\delta$ on $\mathrm{L}^2(\mathbb{R}^N)$.*

(iii) *If $A$ is homogeneous with real coefficients $a_\alpha$, and $m$ is even in the case $N = 1$, then $A$ generates a bounded analytic semigroup $(T(t))_{t\geq 0}$ of angle $\pi/2$ on $\mathrm{L}^2(\mathbb{R}^N)$.*

Proof. (i) By Theorem 5.8 and Lemma 5.4.(ii) it suffices to show that the multiplication operators corresponding to $a(\cdot)$ and $q^m(\cdot)$ defined in (5.2) have the same domain. This, however, follows easily from the inequalities in Lemma 5.14.(i).

Since $\sigma(A) = \sigma(M_a)$, assertion (ii) follows from Proposition I.4.10.(iv) and statement (i) in the theorem in Paragraph II.4.32.

Finally, we prove assertion (iii). If we suppose that $a(\xi) > 0$ for all $0 \neq \xi \in \mathbb{R}^N$, then Lemma 5.14.(i) shows that condition (c) in Theorem 5.12 is violated. Hence, by Lemma 5.14.(ii), we obtain

$$a(\xi) < 0 \quad \text{for all } 0 \neq \xi \in \mathbb{R}^N.$$

Thus, the spectrum $\sigma(M_a)$ is contained in $\mathbb{R}_-$, and assertion (iii) follows from Proposition I.4.10.(iv) and statement (i) in the theorem in Paragraph II.4.32. □

**5.16 Remark.** Suppose that $A$ satisfies the condition (ii) or (iii) in Theorem 5.15. Then $\big(A(D), \mathrm{H}^m(\mathbb{R}^N)\big)$ generates a bounded analytic semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^2(\mathbb{R}^N)$. So, by Theorem II.4.6.(c) we know that

$$T(t)u_0 \in D(A^\infty)$$

for each $t > 0$ and $u_0 \in \mathrm{L}^2(\mathbb{R}^N)$. From this we infer by Theorem 5.10 and Exercise 5.17.(1) below that

$$T(t)u_0 \in \mathrm{C}_0^\infty(\mathbb{R}^N) \quad \text{for all } t > 0.$$

**5.17 Exercises.** (1) Suppose that $A$ is elliptic and that condition (c) in Theorem 5.12 is satisfied. Prove that for every $n \in \mathbb{N}$ the abstract Sobolev space $X_n := D(A^n)$ coincides with $\mathrm{H}^{mn}(\mathbb{R}^N)$ and that the norm $\|\cdot\|_n$ is equivalent to $\|\cdot\|_{\mathrm{H}^{mn}(\mathbb{R}^N)}$. (Hint: Use Theorem 5.8 and the inequalities in Lemma 5.14.(i).)

(2) Discuss the consequences of Theorem 5.12, Theorem 5.15, and Remark 5.16 for the solutions of (cPDE).

## *c. Elliptic Differential Operators with Variable Coefficients*

In this subsection we consider *uniformly elliptic* differential expressions

$$A(D) := \sum_{i,j=1}^{N} D_i a_{ij}(\cdot) D_j$$

with variable, real-valued coefficients $a_{ij}(\cdot)$ satisfying

$$a_{ij}(\cdot) = a_{ji}(\cdot) \in \mathrm{L}^\infty(\mathbb{R}^N) \quad \text{for } i, j = 1, \dots, N, \tag{5.4}$$

$$\sum_{i,j=1}^{N} a_{ij}(x) y_i y_j \geq c|y|^2 \quad \text{for all } x, y \in \mathbb{R}^N \text{ and some } c > 0. \tag{5.5}$$

Since the matrix function $\big(a_{ij}(\cdot)\big)_{N\times N}$ is symmetric and real, one can see that (5.4) is equivalent to

$$\operatorname{Re} \sum_{i,j=1}^{N} a_{ij}(x) z_i \overline{z_j} \geq c|z|^2 \qquad \text{for all } x \in \mathbb{R}^N,\ z \in \mathbb{C}^N.$$

In order to define an operator on $\mathrm{L}^2(\mathbb{R}^N)$ associated to $A(D)$ we introduce the sesquilinear form $a(\cdot,\cdot)$ defined by

(5.6)

$$a(u, v) := \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} a_{ij}(x)\, D_j u(x)\, \overline{D_i v(x)}\, dx \quad \text{for } u, v \in D(a) := \mathrm{H}^1(\mathbb{R}^N),$$

where $D_j$ denotes the weak derivative with respect to the $j$th coordinate.

The main properties of this sesquilinear form are collected in the next theorem.

**5.18 Theorem.** *If* (5.4) *and* (5.5) *are satisfied for* $A(D)$, *then the following properties hold for* $a(\cdot,\cdot)$.

(i) $\overline{D(a)} = \mathrm{L}^2(\mathbb{R}^N)$.

(ii) $a(\cdot,\cdot)$ *is symmetric and positive, i.e.,* $a(u,v) = \overline{a(v,u)}$ *and* $a(u,u) \geq 0$ *for* $u, v \in D(a)$.

(iii) $(u\,|\,v)_a := (u\,|\,v)_{\mathrm{L}^2} + a(u,v)$ *defines an inner product on* $D(a) = \mathrm{H}^1(\mathbb{R}^N)$ *that is equivalent to* $(\cdot\,|\,\cdot)_{\mathrm{H}^1}$ *from* (5.1).

(iv) *There is a unique self-adjoint, dissipative operator* $\bigl(-A, D(A)\bigr)$ *such that* $D(A) \subset D(a)$ *and* $a(u,v) = (Au\,|\,v)_{\mathrm{L}^2}$ *for all* $u \in D(A)$ *and* $v \in D(a)$.

(v) *The operator* $\bigl(-A, D(A)\bigr)$ *generates an analytic contraction semigroup* $\bigl(T(t)\bigr)_{t\geq 0}$ *of angle* $\pi/2$ *on* $\mathrm{L}^2(\mathbb{R}^N)$.

Proof. Assertion (i) follows from the density of $\mathrm{C}_c^\infty(\mathbb{R}^N)$ in $\mathrm{L}^2(\mathbb{R}^N)$, while (ii) is clear from the definition.

From (5.5) we deduce that

$$a(u,u) \geq c \sum_{i=1}^{N} \|D_i u\|_{\mathrm{L}^2}^2 \quad \text{for } u \in D(a).$$

On the other hand, since $a_{ij} \in \mathrm{L}^\infty(\mathbb{R}^N)$, by Hölder's inequality there is a constant $\widetilde{c} > 0$ such that

$$\|u\|_a^2 := \|u\|_{\mathrm{L}^2}^2 + a(u,u) \leq \widetilde{c}\,\|u\|_{\mathrm{H}^1}^2 \quad \text{for } u \in D(a).$$

This proves (iii).

Next, we define the operator $\bigl(A, D(A)\bigr)$ by

$$D(A) := \bigl\{u \in D(a) : \exists\, v \in \mathrm{L}^2(\mathbb{R}^N) \text{ with } a(u,\varphi) = (v\,|\,\varphi)_{\mathrm{L}^2}\ \forall\, \varphi \in D(a)\bigr\},$$
$$Au := v \quad \text{for } u \in D(A).$$

Observe first that

$$(-Au\,|\,u)_{\mathrm{L}^2} = -a(u,u) \leq 0 \quad \text{for } u \in D(A);$$

hence $-A$ is dissipative by Example II.3.26.(iii). In order to verify the range condition of the Lumer–Phillips theorem, we consider the Hilbert space $H_a := \bigl(D(a), (\cdot\,|\,\cdot)_a\bigr)$ and the associated norm $\|\cdot\|_a$. Now, for $v \in \mathrm{L}^2(\mathbb{R}^n)$ define the mapping $L_v$ by

$$H_a \ni \varphi \mapsto L_v\varphi := (v\,|\,\varphi)_{\mathrm{L}^2}\,.$$

Then $L_v \in H_a'$, and the Riesz representation theorem (cf. [Wei80, Thm. 4.8]) yields a unique $u \in D(a)$ such that

$$(u\,|\,\varphi)_a = L_v\varphi \quad \text{for all } \varphi \in D(a).$$

This implies $a(u,\varphi) = (v-u\,|\,\varphi)_{\mathrm{L}^2}$ for all $\varphi \in D(a)$ and consequently $u \in D(A)$ with $(I+A)u = v$. Thus $I+A$ is surjective, and Corollary II.3.20 implies that $A$ generates a contraction semigroup $\big(T(t)\big)_{t\geq 0}$ on $\mathrm{L}^2(\mathbb{R}^N)$. Next, observe that by (ii) the operator $A$ is symmetric, i.e., $A \subseteq A^*$. Moreover, $1 \in \rho(A) \cap \rho(A^*)$, and therefore $A = A^*$ by Exercise IV.1.21.(5). In particular, it follows from Corollary II.4.7 that $\big(T(t)\big)_{t\geq 0}$ is analytic of angle $\pi/2$, and we obtain (iv) and (v) except for the uniqueness of $A$.

Let $\big(B, D(B)\big)$ be another operator satisfying (iv). Then, by the definition of $A$, we have

$$D(B) \subset D(A) \qquad \text{and} \qquad Bu = Au \qquad \text{for } u \in D(B).$$

Since both $(I+B) : D(B) \to \mathrm{L}^2(\mathbb{R}^N)$ and $(I+A) : D(A) \to \mathrm{L}^2(\mathbb{R}^N)$ are bijective, we obtain $A = B$ (use again Exercise IV.1.21.(5)). This proves the uniqueness in assertion (iv), and the proof is complete. □

While it was relatively easy, using the sesquilinear form $a(\cdot,\cdot)$, to define an operator $A$ corresponding to the differential expression $A(D)$, it is by no means clear in which sense this operator is a “differential operator.” By this we mean that its domain $D(A)$ should be a classical Sobolev space and $A$ should be defined by the weak derivatives, i.e., $f \in \mathrm{H}^1(\mathbb{R}^N)$, $a_{ij}(\cdot)D_j f \in \mathrm{H}^1(\mathbb{R}^N)$, and

$$-Af = \sum_{i,j=1}^{N} D_i\big(a_{ij}(\cdot)D_j f\big)$$

for all $f \in D(A)$. Only in this case, the semigroup generated by $-A$ produces solutions of the partial differential equation (vPDE).

To achieve this goal we need additional regularity assumptions on the coefficients $a_{ij}(\cdot)$.

**5.19 Definition.** *For $n \in \mathbb{N}$ consider the vector space* $\mathrm{W}^{n,\infty}(\mathbb{R}^N) :=$

$$\left\{ f \in \mathrm{L}^\infty(\mathbb{R}^N) : \begin{array}{c} \forall\, |\alpha| \leq n \;\exists\, g_\alpha \in \mathrm{L}^\infty(\mathbb{R}^N) \text{ such that } \forall\, \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^N) \\ \displaystyle\int_{\mathbb{R}^N} f(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^N} g_\alpha(x)\varphi(x)\, dx \end{array} \right\}.$$

*Then, for $f \in \mathrm{W}^{n,\infty}(\mathbb{R}^N)$ and $|\alpha| \leq n$, we define the weak derivative of order $\alpha$ of $f$ by $D^\alpha f := g_\alpha$.*

Using functions in $\mathrm{W}^{1,\infty}(\mathbb{R}^N)$ we can prove a product rule for the weak derivatives $D_j$.

**5.20 Lemma.** *For $f \in \mathrm{W}^{1,\infty}(\mathbb{R}^N)$ and $u \in \mathrm{H}^1(\mathbb{R}^N)$ we have $fu \in \mathrm{H}^1(\mathbb{R}^N)$ and $D_j(fu) = (D_j f)u + f D_j u$ for $j = 1, \ldots, N$.*

PROOF. For $u \in \mathrm{C}_c^\infty(\mathbb{R}^N)$ the assertion follows from the definition of the weak derivative and the analogous property of the classical derivative. For arbitrary $u \in \mathrm{H}^1(\mathbb{R}^N)$ choose, using Corollary 5.9, a sequence $(u_n)_{n\in\mathbb{N}} \subset \mathrm{C}_c^\infty(\mathbb{R}^N)$ converging to $u$ in $\mathrm{H}^1(\mathbb{R}^N)$. Then, we have

$$fu_n \in \mathrm{H}^1(\mathbb{R}^N) \quad \text{and} \quad D_j(fu_n) = (D_j f)u_n + fD_j u_n$$

for each $n \in \mathbb{N}$ and $j = 1, \dots, N$. Since $fu_n$ converges to $fu$ in $\mathrm{H}^1(\mathbb{R}^N)$, the assertion follows. □

This lemma allows us to define a differential operator corresponding to the expression $A(D)$ as soon as the coefficients $a_{ij}(\cdot)$ belong to $\mathrm{W}^{1,\infty}(\mathbb{R}^N)$.

**5.21 Definition.** *Let $a_{ij}(\cdot) \in \mathrm{W}^{1,\infty}(\mathbb{R}^N)$ for $i, j = 1, \dots, N$. Then the operator $\big(\widetilde{A}, D(\widetilde{A})\big)$ is defined by*

$$\widetilde{A}f := \sum_{i,j=1}^{N} D_i\big(a_{ij}(\cdot)D_j f\big)$$

*for $f \in D(\widetilde{A}) := \mathrm{H}^2(\mathbb{R}^N)$.*

From this definition it is clear that our operator $\big(-A, D(A)\big)$ defined via the sesquilinear form $a(\cdot,\cdot)$ is an extension of the differential operator $\big(\widetilde{A}, D(\widetilde{A})\big)$. Therefore, in order to prove the following theorem, it remains to show the converse inclusion.

**5.22 Theorem.** *Assuming (5.4), (5.5), and $a_{ij}(\cdot) \in \mathrm{W}^{1,\infty}(\mathbb{R}^N)$ for $i, j = 1, \dots, N$, it follows that the operator $\big(-A, D(A)\big)$ coincides with the differential operator $\big(\widetilde{A}, D(\widetilde{A})\big)$ defined on $D(\widetilde{A}) := \mathrm{H}^2(\mathbb{R}^N)$.*

PROOF. We have only to show that $u \in D(A)$ implies $u \in \mathrm{H}^2(\mathbb{R}^N)$. In order to do so we introduce the strongly continuous (left) translation groups $\big(T_j(t)\big)_{t\in\mathbb{R}}$ in direction $j$ defined by

$$T_j(t)u(x) := u(x + te_j) \quad \text{for } u \in \mathrm{L}^2(\mathbb{R}^N),\ t \in \mathbb{R}, \tag{5.7}$$

and $j = 1, \dots, N$. Its generator $A_j$ coincides with the weak derivative $D_j$ on its maximal domain

$$D(A_j) := \big\{u \in \mathrm{L}^2(\mathbb{R}^N) : D_j u \in \mathrm{L}^2(\mathbb{R}^N)\big\};$$

see Exercise 5.26.(1.i). By Corollary II.5.21, this domain coincides with its Favard space, i.e., $u \in D(A_j)$ if and only if

$$\|\Delta_h^j u\|_{\mathrm{L}^2} := \big\|\tfrac{1}{h}\big(T_j(h)u - u\big)\big\|_{\mathrm{L}^2}$$

remains bounded for $h > 0$. As a consequence (see Exercise 5.26.(1.ii)) we obtain that $u \in \mathrm{H}^1(\mathbb{R}^N)$ if and only if

$$\sup_{0<h\le 1} \|\Delta_h^j u\|_{\mathrm{L}^2} \le \|u\|_{\mathrm{H}^1} \quad \text{for all } j = 1, \dots, N. \tag{5.8}$$

For $u \in \mathrm{H}^1(\mathbb{R}^N)$ this implies that $u \in \mathrm{H}^2(\mathbb{R}^N)$ if and only if

$$\sup_{0<h\le 1} \|\Delta_h^j u\|_{\mathrm{H}^1} < \infty \quad \text{for all } j = 1, \dots, N. \tag{5.9}$$

We now assume $u \in D(A)$ and prove (5.9). Writing $v := \Delta_h^k u$ we then have

$$\begin{aligned}
\|v\|_a^2 &= \big(\Delta_h^k u \,|\, v\big)_a \\
&= -\big(u \,|\, \Delta_{-h}^k v\big)_a + \big(u \,|\, \Delta_{-h}^k v\big)_a + \big(\Delta_h^k u \,|\, v\big)_a \\
&= -\Big[\big(u \,|\, \Delta_{-h}^k v\big)_{\mathrm{L}^2} + \big(Au \,|\, \Delta_{-h}^k v\big)_{\mathrm{L}^2}\Big] + \Big[\big(u \,|\, \Delta_{-h}^k v\big)_a + \big(\Delta_h^k u \,|\, v\big)_a\Big] \\
&=: P_h^k(u) + Q_h^k(u) \quad \text{for all } h > 0,\ k = 1, \dots, N.
\end{aligned}$$

Since $v \in \mathrm{H}^1(\mathbb{R}^N)$, the estimate (5.8), which also holds for $-h$ since $\big(T_j(t)\big)_{t\in\mathbb{R}}$ is a group, implies that

$$\begin{aligned}
|P_h^k(u)| &\le \|u\|_{\mathrm{L}^2}\|v\|_{\mathrm{H}^1} + \|Au\|_{\mathrm{L}^2}\|v\|_{\mathrm{H}^1} \\
&\le \widetilde{c}\,\|v\|_a\big(\|u\|_{\mathrm{L}^2} + \|Au\|_{\mathrm{L}^2}\big) \quad \text{for } h > 0,\ k = 1, \dots, N
\end{aligned} \tag{5.10}$$

and some constant $\widetilde{c} > 0$. On the other hand, the term $Q_h^k(u)$ is the sum of $\big(u \,|\, \Delta_{-h}^k v\big)_{\mathrm{L}^2} + \big(\Delta_h^k u \,|\, v\big)_{\mathrm{L}^2}$ and $N^2$ terms of the form $\big(a_{ij} D_i u \,|\, D_j \Delta_{-h}^k v\big)_{\mathrm{L}^2} + \big(a_{ij} D_i \Delta_h^k u \,|\, D_j v\big)_{\mathrm{L}^2}$. They can be estimated as

$$\begin{aligned}
\Big|\big(a_{ij} D_i u \,|\, D_j \Delta_{-h}^k v\big)_{\mathrm{L}^2} &+ \big(a_{ij} D_i \Delta_h^k u \,|\, D_j v\big)_{\mathrm{L}^2}\Big| \\
&= \Big|\big(a_{ij} \Delta_h^k D_i u - \Delta_h^k (a_{ij} D_i u) \,|\, D_j v\big)_{\mathrm{L}^2}\Big| \\
&= \Big|\frac{1}{h}\big(a_{ij} T_k(h) D_i u - T_k(h)(a_{ij} D_i u) \,|\, D_j v\big)_{\mathrm{L}^2}\Big| \\
&= \Big|\frac{1}{h}\big((a_{ij} - T_k(h) a_{ij}) \cdot (T_k(h) D_i u) \,|\, D_j v\big)_{\mathrm{L}^2}\Big| \\
&\le \Big\|\frac{a_{ij} - T_k(h) a_{ij}}{h}\Big\|_{\mathrm{L}^\infty} \cdot \|u\|_{\mathrm{H}^1} \cdot \|v\|_{\mathrm{H}^1}.
\end{aligned}$$

Applying Exercise 5.26.(1.iii) and Theorem 5.18.(iii), we obtain

$$|Q_h^k(u)| \le d\|u\|_{\mathrm{H}^1}\|v\|_a \quad \text{for } h > 0,\ k = 1, \dots, N \tag{5.11}$$

and some constant $d > 0$. So, we infer from (5.10) that

$$\|\Delta_h^k u\|_a = \|v\|_a \le \widetilde{d}\big(\|u\|_{\mathrm{H}^1} + \|Au\|_{\mathrm{L}^2}\big) \quad \text{for } h > 0,\ k = 1, \dots, N,$$

where $\widetilde{d}$ is a positive constant that is independent of $h$ and $k$. Thus (5.9) holds, and the proof is complete. □

If the coefficients $a_{ij}(\cdot)$ are "sufficiently smooth," we can show that even the higher-order abstract Sobolev spaces $X_n := D(A^n)$ coincide with $\mathrm{H}^{2n}(\mathbb{R}^N)$. To establish this, we first prove the following estimate.

**5.23 Lemma.** *If $a_{ij}(\cdot) \in \mathrm{W}^{2,\infty}(\mathbb{R}^N)$, then there exists a constant $C > 0$ such that the estimate*

$$|\,(D_k u \,|\, v)_a + (u \,|\, D_k v)_a\,| \le C\|u\|_{\mathrm{H}^2}\|v\|_{\mathrm{L}^2}$$

*holds for all $u, v \in \mathrm{H}^2(\mathbb{R}^N)$ and $k = 1, \ldots, N$.*

Proof. For fixed $k$, the expression $(D_k u \,|\, v)_a + (u \,|\, D_k v)_a$ is the sum of $(D_k u \,|\, v)_{\mathrm{L}^2} + (u \,|\, D_k v)_{\mathrm{L}^2}$ and $N^2$ terms of the form

$$L_{ijk}(u,v) := (a_{ij} D_i D_k u \,|\, D_j v)_{\mathrm{L}^2} + (a_{ij} D_i u \,|\, D_j D_k v)_{\mathrm{L}^2}$$

for $i, j = 1, \ldots, N$. So, by Lemma 5.20, we obtain

$$\begin{aligned} |L_{ijk}(u,v)| &= \big|\big(a_{ij} D_k D_i u - D_k(a_{ij} D_i u) \,|\, D_j v\big)_{\mathrm{L}^2}\big| \\ &= \big|-\big((D_k a_{ij})(D_i u) \,|\, D_j v\big)_{\mathrm{L}^2}\big| \\ &= \big|\big((D_j D_k a_{ij})(D_i u) + (D_k a_{ij}) D_j D_i u \,|\, v\big)_{\mathrm{L}^2}\big| \\ &\le C\|u\|_{\mathrm{H}^2}\|v\|_{\mathrm{L}^2} \end{aligned}$$

as claimed □

**5.24 Proposition.** *If (5.4) and (5.5) hold and if $a_{ij}(\cdot) \in \mathrm{W}^{2n-1,\infty}(\mathbb{R}^N)$ for some $n \in \mathbb{N}$, then the operator $\big(-A, D(A)\big)$ coincides with the differential operator $\big(\widetilde{A}, D(\widetilde{A})\big)$, and one has*

$$D(A^n) = \mathrm{H}^{2n}(\mathbb{R}^N). \tag{5.12}$$

Proof. By Theorem 5.22 we only have to verify (5.12). To that purpose we first prove the following claim:

$$\begin{cases} \text{If } a_{ij} \in \mathrm{W}^{n+1,\infty}(\mathbb{R}^N) \text{ for some } n \in \mathbb{N}, \\ \text{then } \big(I + A\big)^{-1}\mathrm{H}^n(\mathbb{R}^N) = \mathrm{H}^{n+2}(\mathbb{R}^N). \end{cases} \tag{5.13}$$

Let $u \in \mathrm{H}^{n+2}(\mathbb{R}^N) \subset \mathrm{H}^2(\mathbb{R}^N)$. Then Lemma 5.20 yields

$$Au = \sum_{i,j=1}^{N} D_i a_{ij} D_j u = \sum_{i,j=1}^{N} [(D_i a_{ij}) D_j u + a_{ij} D_i D_j u].$$

Applying Lemma 5.20 once again yields $Au \in \mathrm{H}^n(\mathbb{R}^N)$ and hence

$$\mathrm{H}^{n+2}(\mathbb{R}^N) \subset (I + A)^{-1}\mathrm{H}^n(\mathbb{R}^N).$$

We now show the converse inclusion for $n = 1$. Let $v \in \mathrm{H}^1(\mathbb{R}^N)$ and set $u := (I + A)^{-1}v$. Then $u \in \mathrm{H}^2(\mathbb{R}^N)$, and for every $\varphi \in \mathrm{H}^2(\mathbb{R}^N)$ and $k = 1, \dots, N$ we have

$$\begin{aligned}
|\,(D_k u \,|\, \varphi)_a\big| &= \big|(-u \,|\, D_k\varphi)_a + (u \,|\, D_k\varphi)_a + (D_k u \,|\, \varphi)_a\big| \\
&\le \big|(-u \,|\, D_k\varphi)_{\mathrm{L}^2} - (Au \,|\, D_k\varphi)_{\mathrm{L}^2}\big| + \big|(u \,|\, D_k\varphi)_a + (D_k u \,|\, \varphi)_a\big| \\
&= \big|(-v \,|\, D_k\varphi)_{\mathrm{L}^2}\big| + \big|(u \,|\, D_k\varphi)_a + (D_k u \,|\, \varphi)_a\big| \\
&= \big|(D_k v \,|\, \varphi)_{\mathrm{L}^2}\big| + \big|(u \,|\, D_k\varphi)_a + (D_k u \,|\, \varphi)_a\big|.
\end{aligned}$$

From Lemma 5.23 we obtain

$$|\,(D_k u \,|\, \varphi)_a\,| \le \|v\|_{\mathrm{H}^1}\|\varphi\|_{\mathrm{L}^2} + C\|u\|_{\mathrm{H}^2}\|\varphi\|_{\mathrm{L}^2}$$

for all $\varphi \in \mathrm{H}^2(\mathbb{R}^N)$. From the density of $\mathrm{H}^2(\mathbb{R}^N)$ in $\mathrm{L}^2(\mathbb{R}^N)$ and the Riesz representation theorem it follows that there is $w \in \mathrm{L}^2(\mathbb{R}^N)$ such that $(D_k u \,|\, \varphi)_a = (w \,|\, \varphi)_{\mathrm{L}^2}$ for all $\varphi \in \mathrm{H}^2(\mathbb{R}^N)$. Since $\mathrm{H}^2(\mathbb{R}^N)$ is dense in $\mathrm{H}^1(\mathbb{R}^N)$, we infer that

$$(D_k u \,|\, \varphi)_a = (w \,|\, \varphi)_{\mathrm{L}^2} \quad \text{for } \varphi \in \mathrm{H}^1(\mathbb{R}^N) = D(a).$$

This implies that $D_k u \in D(A) = \mathrm{H}^2(\mathbb{R}^N)$ for $k = 1, \dots, N$. Hence, $u \in \mathrm{H}^3(\mathbb{R}^N)$, and (5.13) is proved for $n = 1$.

Assume now that (5.13) is satisfied for some $n \in \mathbb{N}$. Let $v \in \mathrm{H}^{n+1}(\mathbb{R}^N)$, set $u := (I + A)^{-1}v$, and suppose that $a_{ij} \in \mathrm{W}^{n+2,\infty}(\mathbb{R}^N)$. Then $u \in \mathrm{H}^{n+2}(\mathbb{R}^N)$, and so

$$\begin{aligned}
D_k u &= (I + A)^{-1}D_k v - (I + A)^{-1}[D_k(I + A) - (I + A)D_k]u \\
&= (I + A)^{-1}D_k v - (I + A)^{-1}[D_k A - AD_k]u \\
&=: f_1 - f_2.
\end{aligned}$$

Since $D_k v \in \mathrm{H}^n(\mathbb{R}^N)$, we have $f_1 \in \mathrm{H}^{n+2}(\mathbb{R}^N)$. For the second term we first note that due to $u \in \mathrm{H}^{n+2}(\mathbb{R}^N) \subset \mathrm{H}^3(\mathbb{R}^N)$, the expression $w := [D_k A - AD_k]u$ is meaningful. By applying Lemma 5.20 and using the assumption $a_{ij} \in \mathrm{W}^{n+2,\infty}(\mathbb{R}^N)$, we infer that $w \in \mathrm{H}^n(\mathbb{R}^N)$. Thus (5.13) yields $f_2 = (I + A)^{-1}w \in \mathrm{H}^{n+2}(\mathbb{R}^N)$. Therefore, $u \in \mathrm{H}^{n+3}(\mathbb{R}^N)$, and (5.13) follows by induction.

We now prove (5.12). By Theorem 5.22, the assertion holds for $n = 1$. Proceeding again by induction and using (5.13), we obtain

$$\begin{aligned}
D(A^{n+1}) &= (I + A)^{-1}D(A^n) \\
&= (I + A)^{-1}\mathrm{H}^{2n}(\mathbb{R}^N) \\
&= \mathrm{H}^{2(n+1)}(\mathbb{R}^N)
\end{aligned}$$

for $a_{ij} \in \mathrm{W}^{2n+1,\infty}(\mathbb{R}^N)$. □

From this proposition and from the analyticity of the semigroup generated by $-A$ we draw the following conclusion yielding classical solutions of the partial differential equation (vPDE).

**5.25 Corollary.** *If (5.4) and (5.5) hold and if* $a_{ij}(\cdot) \in \mathrm{C}_{\mathrm{b}}^{\infty}(\mathbb{R}^N)$ *for all* $i, j = 1, \ldots, N$*, then* $\big(-A, D(A)\big)$ *with domain* $D(A) = \mathrm{H}^2(\mathbb{R}^N)$ *generates an analytic semigroup* $\big(T(t)\big)_{t\geq 0}$ *on* $\mathrm{L}^2(\mathbb{R}^N)$ *such that* $T(t)u_0 \in \mathrm{C}_0^{\infty}(\mathbb{R}^N)$ *for all* $t > 0$ *and* $u_0 \in \mathrm{L}^2(\mathbb{R}^N)$.

Proof. Arguing as in Remark 5.16 and using Proposition 5.24 we obtain

$$T(t)u_0 \in D(A^{\infty}) = \bigcap_{n\in\mathbb{N}} \mathrm{H}^{2n}(\mathbb{R}^N) \subset \mathrm{C}_0^{\infty}(\mathbb{R}^N)$$

for all $t > 0$ and $u_0 \in \mathrm{L}^2(\mathbb{R}^N)$. □

**5.26 Exercises.** (1) Let $(T_j(t))_{t\in\mathbb{R}}$ denote the (left) translation group in direction $j \in \{1, \ldots, N\}$ as defined in (5.7).

(i) Show that the generator $A_j$ of $(T_j(t))_{t\in\mathbb{R}}$ coincides with the weak derivative $D_j$ on its maximal domain $D(A_j) := \big\{u \in \mathrm{L}^2(\mathbb{R}^N) : D_j u \in \mathrm{L}^2(\mathbb{R}^N)\big\}$.

(ii) Show that $\mathrm{H}^1(\mathbb{R}^N) = \cap_{j=1}^{N} D(D_j)$.

(iii) Show that $\|{}^{(T_j(h)f-f)}/_{h}\|_{\mathrm{L}^\infty} \leq \|f\|_{\mathrm{W}^{1,\infty}}$ for all $f \in \mathrm{W}^{1,\infty}(\mathbb{R}^N)$ and $h > 0$. (Hint: Use Proposition II.5.19.)

(2) Let the assumptions of Theorem 5.22 be satisfied and denote by $(T(t))_{t\geq 0}$ the analytic semigroup generated by $-A$ on $\mathrm{L}^2(\mathbb{R}^N)$.

(i) Show that there exists a constant $d > 0$ such that

$$\|D_j T(t)\| \leq d\, t^{^1\!/_2} \quad \text{for all } t > 0 \text{ and } j = 1, \ldots, N.$$

(ii) Define an operator $(B, D(B))$ by

$$Bu := \sum_{i=1}^{N} a_i(\cdot) D_i u + a_0(\cdot)u$$

for $u \in D(B) := \mathrm{H}^1(\mathbb{R}^N)$ and with $a_i(\cdot) \in \mathrm{L}^\infty(\mathbb{R}^N)$ for $i = 0, \ldots, N$. Show that the operator $(-A + B, D(A))$ generates an analytic semigroup on $\mathrm{L}^2(\mathbb{R}^N)$. (Hint: Use Theorem III.2.10.)

(iii) Discuss the consequences of these results for the solutions of the initial value problem for the second-order partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x,t)}{\partial x_j}\Big) + \sum_{i=1}^{N} a_i(x)\frac{\partial u(x,t)}{\partial x_i} + a_0(x)u(x,t)$$

for $x \in \mathbb{R}^N$ and $t \geq 0$.

## *Notes and Further Reading to Section* 5

The literature on semigroups and partial differential equations is enormous. Besides the references given in the introduction to this section, we mention the books by Dautray–Lions [DL88], Fattorini [Fat83], Friedman [Fri69], Jacob [Jac99], Taira [Tai88], [Tai95], and Tanabe [Tan79], [Tan97].

The generation of analytic semigroups by elliptic differential operators was proved for $X = \mathrm{L}^p(\Omega,\mu)$, $\Omega \subset \mathbb{R}^N$, and $1 < p < \infty$ by [ADN59], [Agm62], and for $X = \mathrm{C}(\overline{\Omega})$ by [Ste74] (see also [AT87a] and [Tan97]). For more detailed information we refer to the books [Lun95] and [Ama95]; see also [CV87], [CV88].

Form methods (as used in Theorem 5.18 to define an operator associated to a formal differential expression) can be found, e.g., in [Kat80, Chap. VI], [RS72, Chap. VIII] and [RS75, Chap. X]. See also [LV91].

# 6. Semigroups for Delay Differential Equations

As explained in the Epilogue and verified in the previous sections, deterministic systems should be described by an abstract Cauchy problem of the form

$$\text{(ACP)}\qquad \begin{cases} \dot{u}(t) = Au(t) & \text{for } t \geq 0,\\ u(0) = x, \end{cases}$$

where the infinitesimal change of $u$ at time $t$ depends, via the operator $A$, only on the state $u(t)$ at time $t$.

In contrast to this situation, there are many examples where the change of $u$ at time $t$ also depends on the history of the system. A very simple example is the scalar *population equation*

$$\dot{u}(t) = -du(t) + bu(t-r) \tag{6.1}$$

with constants $d, b \geq 0$ and $r > 0$. Here, $u(t)$ denotes the number of individuals of a population at time $t$, while $d$ is the death rate, $b$ is the birth rate, and $r$ is the delay due to pregnancy (cf. Example 6.18 below).

If we consider this equation in the state space $Y := \mathbb{C}$, then it is not deterministic and therefore not accessible to the semigroup theory developed above. In order to resolve this problem and in agreement with the biological interpretation, we have to enlarge the state space, such that a state also contains the relevant information on the history of the system. For equation (6.1), e.g., we can take $X := \mathrm{C}[-r,0]$ and consider the time evolution of the history segments

$$u_t : [-r,0] \to Y, \quad u_t(s) := u(t+s); \tag{6.2}$$

cf. Figure 9.

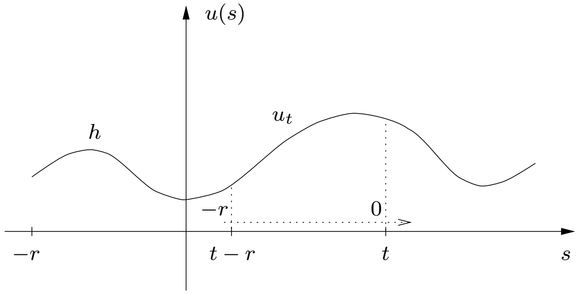



**Figure 9**

As initial value we now take a function $h : [-r, 0] \to Y$ describing the prehistory of the system, i.e., $u(t) = h(t)$ for $t \in [-r, 0]$.

The aim of this section is to show how this and much more general equations with delay can be treated within our semigroup framework.

## a. *Well-Posedness of Abstract Delay Differential Equations*

As our general setup, we associate to a Banach space $Y$ the Banach space

$$X := \mathrm{C}\big([-r, 0], Y\big)$$

of all continuous functions on $[-r, 0]$ with values in $Y$ equipped with the sup-norm. Moreover, we take a *delay operator* $\Phi \in \mathcal{L}(X, Y)$ and the generator $\big(B, D(B)\big)$ of a strongly continuous semigroup $\big(S(t)\big)_{t\geq 0}$ on $Y$. With this notation we consider the *abstract delay differential equation*

$$\text{(ADDE)} \qquad \begin{cases} \dot{u}(t) = Bu(t) + \Phi\, u_t & \text{for } t \geq 0, \\ u_0 = h \in X, \end{cases}$$

where $u_t : [-r, 0] \to Y$ is defined by (6.2). We then call a continuous function $u : [-r, \infty) \to Y$ a *(classical) solution* of (ADDE) if

(i) $u$ is right-sided differentiable at $t = 0$ and continuously differentiable for $t > 0$,

(ii) $u(t) \in D(B)$ for all $t \geq 0$, and

(iii) $u$ satisfies (ADDE).

To solve (ADDE) by our semigroup methods we introduce the corresponding *delay differential operator* $\big(A, D(A)\big)$ on $X$ defined by

$$\text{(6.3)} \qquad \begin{aligned} Af &:= f', \\ D(A) &:= \left\{ f \in \mathrm{C}^1\big([-r, 0], Y\big) : \begin{matrix} f(0) \in D(B) \text{ and} \\ f'(0) = Bf(0) + \Phi f \end{matrix} \right\}. \end{aligned}$$

For this operator we have the following generation result generalizing Proposition II.3.29.

**6.1 Theorem.** *The operator $A$ defined in* (6.3) *generates a strongly continuous semigroup $\big(T(t)\big)_{t\ge 0}$ on $X$.*

Proof. We proceed in two steps. First, we consider the operator $\widetilde{A}$ that is obtained if we take $\Phi = 0$, i.e.,

$$
\begin{aligned}
\widetilde{A}f &:= f', \\
D(\widetilde{A}) &:= \big\{ f \in \mathrm{C}^1\big([-r,0],Y\big) : f(0) \in D(B),\ f'(0) = Bf(0) \big\}.
\end{aligned}
\tag{6.4}
$$

Then $\widetilde{A}$ generates the strongly continuous semigroup $\big(\widetilde{T}(t)\big)_{t\ge 0}$ given explicitly by

$$
\big(\widetilde{T}(t)f\big)(s) = \begin{cases} f(t+s) & \text{if } t+s \le 0, \\ S(t+s)[f(0)] & \text{if } t+s > 0, \end{cases}
$$

where $\big(S(t)\big)_{t\ge 0}$ denotes the semigroup on $Y$ generated by $B$. In fact, the semigroup law and the strong continuity for $\big(\widetilde{T}(t)\big)_{t\ge 0}$ follow immediately from the corresponding property of $\big(S(t)\big)_{t\ge 0}$. In order to verify that the generator of $\big(\widetilde{T}(t)\big)_{t\ge 0}$ coincides with $\widetilde{A}$, we consider for $s \in [-r,0]$ and small $t > 0$ the difference quotient

$$
\Big(\frac{\widetilde{T}(t)f - f}{t}\Big)(s) = \begin{cases} \frac{f(t+s)-f(s)}{t} & \text{if } s < 0, \\ \frac{S(t)[f(0)]-f(0)}{t} & \text{if } s = 0. \end{cases}
\tag{6.5}
$$

Hence, its limit as $t \downarrow 0$ exists in $X$ if and only if $f$ is continuously differentiable, $f(0) \in D(B)$ and $f'(0) = Bf(0)$.

In the second step, we show that $A$ can be obtained as a Desch–Schappacher perturbation of $\widetilde{A}$; see Section III.3.a. To this end, we first note that for $\lambda \in \rho(B)$ we have

$$
A - \lambda = (\widetilde{A} - \lambda) \cdot [I - \varepsilon_\lambda \otimes R(\lambda,B)\Phi],
\tag{6.6}
$$

where $\varepsilon_\lambda \otimes R(\lambda,B) \in \mathcal{L}(Y,X)$ is defined by

$$
\big([\varepsilon_\lambda \otimes R(\lambda,B)]y\big)(s) := \mathrm{e}^{\lambda s} R(\lambda,B)y \quad \text{for } y \in Y,\ -r \le s \le 0.
$$

To verify this representation, we denote the right-hand side of equation (6.6) by $C$. Then

$$
\begin{aligned}
D(C) &= \Big\{ f \in X : f - \varepsilon_\lambda \otimes R(\lambda,B)\Phi f \in D(\widetilde{A}) \Big\} \\
&= \left\{ f \in X : \begin{array}{l} f \in \mathrm{C}^1\big([-r,0],Y\big),\ f(0) \in D(B), \text{ and} \\ f'(0) - \lambda R(\lambda,B)\Phi f = B\big(f(0) - R(\lambda,B)\Phi f\big) \end{array} \right\} \\
&= D(A),
\end{aligned}
$$

and

$$Cf = (\tfrac{d}{ds} - \lambda) \cdot [I - \varepsilon_\lambda \otimes R(\lambda, B)\Phi]f = (\tfrac{d}{ds} - \lambda)f = (A - \lambda)f$$

for all $f \in D(C) = D(A)$. We now look at the extrapolated operator $\widetilde{A}_{-1}$ defined in Section II.5.a. Then the connection between multiplicative and additive perturbations stated in Proposition III.3.18.(ii) gives

$$A = \Big(\widetilde{A}_{-1} - (\widetilde{A} - \lambda)_{-1}\big(\varepsilon_\lambda \otimes R(\lambda, B)\big)\Phi\Big)_{|X}.$$

By Corollary III.3.6, the proof is complete if we can show that the perturbing operator maps into the extrapolated Favard space associated to $\widetilde{A}_{-1}$. More precisely, we have to verify that

$$\operatorname{rg}\big((\widetilde{A} - \lambda)_{-1}(\varepsilon_\lambda \otimes R(\lambda, B))\Phi\big) \subset F_0^{\tilde{A}},$$

or, equivalently, $R := \operatorname{rg}\big((\varepsilon_\lambda \otimes R(\lambda, B))\Phi\big) \subset F_1^{\tilde{A}}$. To this end, we take $f := \varepsilon_\lambda \otimes y \in R$. Then $y \in D(B)$, and from (6.5) it follows that

$$\varlimsup_{t \downarrow 0} \Big\| \frac{\widetilde{T}(t)f - f}{t} \Big\| \le \max\big\{ \|\lambda\varepsilon_\lambda\|_\infty \cdot \|y\|,\ \|By\| \big\} < \infty,$$

i.e., $f \in F_1^{\tilde{A}}$. □

To relate the semigroup $\big(T(t)\big)_{t\ge0}$ generated by the operator $A$ from (6.3) to the abstract delay differential equation (ADDE), we need the following result, which, in the case $X = \mathbb{C}$, we already encountered in Exercise II.3.31.(2).

**6.2 Lemma.** *The semigroup $\big(T(t)\big)_{t\ge0}$ satisfies the translation property*

$$\text{(TP)} \qquad \big(T(t)f\big)(s) = \begin{cases} f(t+s) & \text{if } t+s \le 0, \\ [T(t+s)f](0) & \text{if } t+s > 0 \end{cases}$$

*for all $f \in X$.*

Proof. It suffices to show that (TP) holds for all $f \in D(A)$ and $t > 0$. To this purpose we distinguish for $s \in [-r, 0]$ the following two cases.

*Case* 1: $t + s > 0$. We have to verify that $[T(-s)g](s) = g(0)$ for $g := T(t+s)f$. To this end, we define the function

$$\varphi : [-t, 0] \to Y, \quad \varphi(\tau) := \delta_\tau[T(-\tau)g],$$

where $\delta_\tau : X \to Y$ denotes the point evaluation in $\tau \in [-t, 0]$. Then, for $\vartheta > 0$, we obtain

$$\begin{aligned}\frac{\varphi(\tau+\vartheta)-\varphi(\tau)}{\vartheta} &= \frac{[T(-\tau-\vartheta)g](\tau+\vartheta)-[T(-\tau)g](\tau)}{\vartheta} \\ &= \frac{[T(-\tau-\vartheta)g](\tau)-[T(-\tau)g](\tau)}{\vartheta} \\ &\quad + \frac{(\delta_{\tau+\vartheta}-\delta_\tau)\big(T(-\tau-\vartheta)g-T(-\tau)g\big)}{\vartheta} \\ &\quad + \frac{[T(-\tau)g](\tau+\vartheta)-[T(-\tau)g](\tau)}{\vartheta} \\ &=: D_1(\vartheta)+D_2(\vartheta)+D_3(\vartheta).\end{aligned}$$

Taking the limits as $\vartheta \to 0$ we obtain

$$\begin{aligned}&\lim_{\vartheta\to 0} D_1(\vartheta) = -[AT(-\tau)g](\tau), \\ &\lim_{\vartheta\to 0} D_2(\vartheta) = 0, \\ &\lim_{\vartheta\to 0} D_3(\vartheta) = [T(-\tau)g]'(\tau) = [AT(-\tau)g](\tau);\end{aligned}$$

hence $\varphi$ is differentiable with $\varphi' \equiv 0$. This implies that $\varphi$ is constant, and therefore

$$[T(-s)g](s) = \varphi(s) = \varphi(0) = g(0).$$

*Case* 2: $t + s \le 0$. Analogously to the first case, we define a map

$$\psi : [0,t] \to Y, \quad \psi(\tau) := \delta_{t+s-\tau}[T(\tau)f]$$

and show that it is differentiable with derivative $\psi' \equiv 0$. Thus

$$f(t+s) = \psi(0) = \psi(t) = [T(t)f](s),$$

and the proof is complete. □

We are now ready to state the main result of this subsection relating the semigroup $\big(T(t)\big)_{t\ge 0}$ generated by $A$ to the solution of (ADDE).

**6.3 Corollary.** *If $h \in D(A)$, then the function $u : [-r, \infty) \to Y$ defined by*

$$u(t) := \begin{cases} h(t) & \text{if } -r \le t \le 0, \\ [T(t)h](0) & \text{if } 0 < t \end{cases}$$

*is the unique (classical) solution of* (ADDE).

Proof. From the translation property (TP) and the definition of $u_t$ in (6.2) it follows that

$$u_t = T(t)h \tag{6.7}$$

for all $t \ge 0$. Since we assume $h \in D(A)$, we have $u_t \in D(A)$ for all $t \ge 0$, which implies that $u$ is continuous on $[-r, \infty)$ and continuously differentiable on $[0, \infty)$. Moreover, $u(t) \in D(B)$ for all $t \ge 0$ and

$$\dot u(t) = [\tfrac{d}{dt}(u_t)](0) = [Au_t](0) = (u_t)'(0) = Bu(t) + \Phi\, u_t;$$

hence $u$ is a solution of (ADDE).

In order to show uniqueness of the solution $u$, we assume $v$ to be another solution of (ADDE). Then $w := u - v$ solves the equation

$$\begin{cases} \dot{w}(t) = Bw(t) + \Phi\, w_t & \text{for } t \geq 0, \\ w_0 = 0, \end{cases}$$

where $w_t$ is defined analogously to (6.2). Now let $x(t) := w_t$ for $t \geq 0$. Then $x(t) \in \mathrm{C}^1\big([-r,0],Y\big)$ for all $t \geq 0$, and since

$$(w_t)'(0) = \dot{w}(t) = Bw(t) + \Phi\, w_t,$$

we even have $x(t) \in D(A)$. Moreover,

$$\begin{aligned} \big(\dot{x}(t)\big)(s) &= \lim_{\vartheta \to 0} \frac{w_{t+\vartheta}(s) - w_t(s)}{\vartheta} \\ &= \lim_{\vartheta \to 0} \frac{w_t(s+\vartheta) - w_t(s)}{\vartheta} \\ &= (w_t)'(s) = (Aw_t)(s) \end{aligned}$$

for all $s \in [-r,0]$. Therefore, $x(\cdot)$ solves the Cauchy problem

$$\begin{cases} \dot{x}(t) = Ax(t) & \text{for } t \geq 0, \\ x(0) = 0. \end{cases} \tag{6.8}$$

However, since $A$ is a generator, equation (6.8) has the unique solution $x(t) = w_t = 0$, which implies $u = v$ as claimed. □

**6.4 Exercises.** (1) Show that the converse to Corollary 6.3 is true. More precisely, if $u$ is a solution of (ADDE) with $u_0 \in D(A)$, then $x : \mathbb{R}_+ \to X$ defined by $x(t) := u_t$ is a solution of the abstract Cauchy problem associated to $A$ with initial value $x(0) = u_0$. In this sense, (ADDE) and (ACP) for $A$ defined by (6.3) correspond to each other.

(2) Show that for bounded $B \in \mathcal{L}(Y)$ the function $u$ defined in Corollary 6.3 is a solution of (ADDE) for every $h \in X$. Express this as a property of the corresponding semigroup.

(3) Show that the assertion in Exercise (2) is false in general for unbounded generators $B$. (Hint: Take a generator $B$ of a strongly continuous semigroup on $Y$ that is not differentiable and choose $h(s) \equiv y$ for some $y \in Y$ such that $t \mapsto S(t)y$ is not differentiable.)

## b. Regularity and Asymptotics

Having established the well-posedness of the abstract delay differential equation (ADDE), we are now interested in the asymptotic behavior of its solution. Since by the previous results the solution is given by the map

$t \mapsto u(t) = [T(t)h](0)$, its long-time behavior is determined by that of the semigroup $\big(T(t)\big)_{t\geq 0}$. However, in order to apply stability criteria such as Theorem V.1.10, we need eventual norm continuity for this semigroup. To that purpose we prove the following "*variation of parameters*"-type formula relating the semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ on $X$ and the semigroup $\mathcal{S} = \big(S(t)\big)_{t\geq 0}$ on $Y$ generated by $A$ and $B$, respectively.

**6.5 Lemma.** *With the above notation we have*

$$[T(t)f](0) = S(t)[f(0)] + \int_0^t S(t-s)\Phi T(s)f\, ds \tag{6.9}$$

*for all $t \geq 0$ and $f \in X$.*

Proof. It suffices to verify (6.9) for $f \in D(A)$ only. In this case, by Corollary 6.3, we can interpret the map $t \mapsto u(t) := [T(t)f](0)$ as the solution of the inhomogeneous Cauchy problem (compare Section 7 below)

$$\begin{cases} \dot u(t) = Bu(t) + g(t) & \text{for } t \geq 0, \\ u(0) = f(0) \end{cases}$$

for $g(t) := \Phi u_t$. Since $g(t) = \Phi T(t)f$ by (6.7), the assertion then follows from Exercise 7.10.(1). □

Formula (6.9) turns out to be quite useful in determining which regularity properties of $\big(S(t)\big)_{t\geq 0}$ are inherited by $\big(T(t)\big)_{t\geq 0}$. In particular, for immediately norm-continuous semigroups $\big(S(t)\big)_{t\geq 0}$ we obtain the following result.

**6.6 Theorem.** *If $\mathcal{S} = \big(S(t)\big)_{t\geq 0}$ is immediately norm continuous, then $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ is norm continuous for $t > r$.*

Proof. Let $t > r$ and $\vartheta \in (0,t)$. Then, from (6.9) and the translation property (TP), we obtain for $s \in [-r, 0]$ and $f \in X$ that

$$\begin{aligned} \big[\big(T(t+\vartheta) - T(t)\big)f\big]\,(s) &= \big[\big(T(t+\vartheta+s) - T(t+s)\big)f\big](0) \\ &= \big[S(t+\vartheta+s) - S(t+s)\big][f(0)] \\ &\quad + \big[(S*\Phi T)(t+\vartheta+s) - (S*\Phi T)(t+s)\big]f, \end{aligned}$$

where we used the convolution notation introduced in (1.11), Section III.1. Since $\big(S(t)\big)_{t\geq 0}$ is immediately norm continuous, it follows from part (i) in Lemma III.1.13 that the convolution $S*\Phi T$ is immediately norm continuous as well. Hence, the maps $S(\cdot)$ and $(S*\Phi T)(\cdot)$ are uniformly norm continuous

on the compact interval $[t-r,2t]$, which implies that

$$\begin{aligned}\big\|\big(T(t+\vartheta)-T(t)\big)f\big\| &= \sup_{s\in[-r,0]}\Big\|\big[S(t+\vartheta+s)-S(t+s)\big][f(0)]\\ &\qquad + \big[(S*\Phi T)(t+\vartheta+s)-(S*\Phi T)(t+s)\big]f\Big\|\\ &\le \sup_{s\in[-r,0]}\big\|S(t+\vartheta+s)-S(t+s)\big\|\cdot\|f\|\\ &\qquad + \sup_{s\in[-r,0]}\Big\|(S*\Phi T)(t+\vartheta+s)-(S*\Phi T)(t+s)\Big\|\cdot\|f\|\end{aligned}$$

converges to zero as $\vartheta\to 0$ uniformly for $\|f\|\le 1$. □

As the next step towards the description of the behavior of $\big(T(t)\big)_{t\ge 0}$, we need information on the resolvent and the spectrum of its generator $A$ in terms of $B$ and $\Phi$. To this end, we introduce for $\lambda\in\mathbb{C}$ the operators

$$\begin{aligned} H_\lambda &\in \mathcal{L}(X), & (H_\lambda f)(s) &:= \int_s^0 e^{\lambda(s-\tau)} f(\tau)\,d\tau,\\ \Phi_\lambda &\in \mathcal{L}(Y), & \Phi_\lambda y &:= \Phi(\varepsilon_\lambda\otimes y),\end{aligned}$$

where, as usual, $\varepsilon_\lambda(s) := e^{\lambda s}$ for $\lambda\in\mathbb{C}$ and $s\in[-r,0]$. With this notation we have the following result, which generalizes the proposition in Paragraph IV.2.8.

**6.7 Proposition.** *For every $\lambda\in\mathbb{C}$ one has*

$$\lambda\in\sigma(A) \qquad \textit{if and only if} \qquad \lambda\in\sigma(B+\Phi_\lambda).$$

*Moreover, for $\lambda\in\rho(A)$, the resolvent of $A$ is given by*

$$R(\lambda,A)f = \big[\varepsilon_\lambda\otimes R(\lambda,B+\Phi_\lambda)\big]\big(f(0)+\Phi H_\lambda f\big)+H_\lambda f, \quad f\in X. \tag{6.10}$$

Proof. By definition, $\lambda\in\rho(A)$ if and only if for every $g\in X$, there exists a unique solution $f\in D(A)$ of the equation

$$\lambda f - f' = g.$$

Solving this differential equation, one sees that it is satisfied if and only if

$$f = \varepsilon_\lambda\otimes y + H_\lambda g$$

for some $y\in Y$. On the other hand, $f\in D(A)$ if and only if $y\in D(B)$ and $f'(0) = Bf(0)+\Phi f$, i.e.,

$$\lambda y - g(0) = By + \Phi_\lambda y + \Phi H_\lambda g.$$

This shows that $\lambda\in\rho(A)$ if and only if for every $g\in X$ there exists a unique $y\in Y$ such that

$$(\lambda - B - \Phi_\lambda)y = S_\lambda g,$$

where $S_\lambda := \delta_0 + \Phi H_\lambda \in \mathcal{L}(X,Y)$. Hence, the proof is complete if we can show that $S_\lambda$ is surjective from $X$ to $Y$. This, however, follows from

$$S_\lambda(\varepsilon_\mu\otimes I) = I + \Phi H_\lambda(\varepsilon_\mu\otimes I)\in\mathcal{L}(X),$$

since $\|\Phi H_\lambda(\varepsilon_\mu\otimes I)\|\to 0$ as $\mu\to\infty$, and hence $S_\lambda(\varepsilon_\mu\otimes I)$ is bijective for $\mu$ sufficiently large. □

This description of the spectrum of $A$ together with Theorem 6.6 and Theorem V.1.10 characterizes uniform exponential stability of the semigroup $\big(T(t)\big)_{t\geq 0}$ and hence of the solutions $u$ of (ADDE). However, it is not easy to determine all $\lambda \in \mathbb{C}$ satisfying $\lambda \in \sigma(B+\Phi_\lambda)$ in order to obtain $\sigma(A)$. For example, if $\dim Y < \infty$, then

$$\sigma(A) = \big\{\lambda \in \mathbb{C} : \det\big(\lambda - B - \Phi(\varepsilon_\lambda \otimes I)\big) = 0\big\}.$$

Hence, we have to find the zeros of the *characteristic equation*

$$\xi(\lambda) := \det\big(\lambda - B - \Phi(\varepsilon_\lambda \otimes I)\big) = 0, \tag{6.11}$$

a task that, even for simple operators $B$ and $\Phi$, is not solvable explicitly; see Exercise 6.10.(2) and [BC63]. In Section 6.c we will show how this problem can be simplified if we make an additional positivity assumption. (Compare also Proposition 1.5.)

We continue this subsection by looking at compactness properties of the semigroup $\big(T(t)\big)_{t\geq 0}$.

**6.8 Lemma.** *If $B$ has compact resolvent, then $R(\lambda, A)T(r)$ is compact for all $\lambda \in \rho(A)$.*

PROOF. For arbitrary but fixed $\lambda$, we define the set

$$C := \big\{R(\lambda, A)T(r)f : f \in U\big\} \subset X,$$

where $U$ denotes the unit ball in $X$. Then it follows from

$$\big\|\tfrac{d}{ds}\big(R(\lambda, A)T(r)f\big)\big\| \leq \big\|AR(\lambda, A)T(r)\big\| \qquad \text{for all } f \in U$$

that the set $C$ is equicontinuous in $X$.

On the other hand, the translation property (TP) and the representation of $R(\lambda, A)$ in (6.10) imply that for arbitrary $s \in [-r, 0]$ and $f \in X$ we have

$$\begin{aligned}
[R(\lambda, A)T(r)f](s) &= [R(\lambda, A)T(r+s)f](0) \\
&= \Big(\big[\varepsilon_\lambda \otimes R(\lambda, B+\Phi_\lambda)\big]\big[\big(T(r+s)f\big)(0) + \Phi H_\lambda T(r+s)f\big]\Big)(0) \\
&= R(\lambda, B+\Phi_\lambda)(\delta_0 + \Phi H_\lambda)T(r+s)f.
\end{aligned}$$

Since by Exercise II.4.30.(2) the resolvent $R(\lambda, B+\Phi_\lambda)$ is compact, this shows that the set

$$C(s) := \big\{g(s) : g \in C\big\} \subset Y$$

is relatively compact for all $s \in [-r, 0]$. Hence, we can apply the (vector-valued) Arzelà–Ascoli theorem (see [Dug66, Chap. XII, Thm. 6.4]) and conclude that $C$ is relatively compact, which means that $R(\lambda, A)T(r)$ is a compact operator. □

If we combine this proposition with the results of Section II.4.d, we arrive at the following compactness criterion for $\big(T(t)\big)_{t\geq 0}$; see also [TW74].

**6.9 Proposition.** *If $\big(S(t)\big)_{t\geq 0}$ is immediately compact, then $\big(T(t)\big)_{t\geq 0}$ is eventually compact for $t > r$.*

PROOF. By the compactness assumption on $\big(S(t)\big)_{t\geq 0}$ and Theorem II.4.29 it follows that $B$ has compact resolvent and that $\big(S(t)\big)_{t\geq 0}$ is immediately norm continuous. Hence, by Lemma 6.8, the operator $R(\lambda, A)T(r)$ is compact for all $\lambda \in \rho(A)$. On the other hand, Theorem 6.6 implies that the semigroup $\big(T(t)\big)_{t\geq 0}$ is norm continuous for $t > r$, and thus the assertion follows from Lemma II.4.28. $\square$

Note that the above compactness assumption on $\big(S(t)\big)_{t\geq 0}$ is always satisfied if $Y$ is finite-dimensional.

**6.10 Exercises.** (1) Show that the conclusion of Theorem 6.6 is not valid for eventually norm-continuous semigroups $(S(t))_{t\geq 0}$. (Hint: Take an eventually norm continuous semigroup $(S(t))_{t\geq 0}$ and a bounded operator $\widetilde{\Phi} \in \mathcal{L}(Y)$ such that the semigroup generated by $B + \widetilde{\Phi}$ is not eventually norm continuous (cf. Example III.1.15). Then, for $\Phi := \widetilde{\Phi}\delta_0 \in \mathcal{L}(X, Y)$, the semigroup generated by $A$ is not eventually norm continuous.)

(2) Determine the spectrum of the operator $A$ associated to the scalar population model (6.1) and try to estimate the growth bound of the semigroup $(T(t))_{t\geq 0}$ generated by it. Determine the values of $b$, $d$, and $r$ such that every solution of (6.1) is exponentially stable. Compare this with Exercise 6.20.(1).

(3) Show that the semigroup $(T(t))_{t\geq 0}$ is eventually differentiable if $B$ generates an analytic semigroup. (Hint: Use the characterization in Theorem II.4.14.)

(4*) Describe the asymptotic behavior of the solutions of (ADDE) in the situation of Proposition 6.9. (Hint: Use Corollary V.3.2. Compare also [KVL92, Sec. II].)

## c. *Positivity for Delay Differential Equations*

Under appropriate assumptions, we could show in the previous subsection that the negativity of the spectral bound $\mathrm{s}(A)$ of the generator $A$ in $X$ implies stability of the solutions of the associated abstract delay differential equation (ADDE) in $Y$. However, even if $\dim Y < \infty$, it is quite difficult to determine the zeros of the characteristic equation (6.11) in order to obtain the spectral bound $\mathrm{s}(A)$; cf. Exercise 6.10.(2).

The aim of this subsection is to show how "positivity" can help in facilitating this task (compare also Section 1.c).

To this end we assume throughout that $Y$ is a Banach lattice (cf. Section 1.b), which makes $X := \mathrm{C}\big([-r, 0], Y\big)$ with the canonical order a Banach lattice as well. First, we give sufficient conditions on $B$ and $\Phi$ to ensure that $A$ generates a positive semigroup $\big(T(t)\big)_{t\geq 0}$ (cf. Definition 1.7) on this Banach space.

**6.11 Theorem.** *If $B$ generates a positive semigroup on $Y$ and the delay operator $\Phi \in \mathcal{L}(X, Y)$ is positive, then the semigroup generated by $A$ on $X$ is positive as well.*

Proof. By Theorem 1.8 it suffices to show that $R(\lambda, A)$ is positive for sufficiently large $\lambda \in \mathbb{R}$. To this end, note that $\Phi_\lambda = \Phi(\varepsilon_\lambda \otimes I) \geq 0$ for all $\lambda \in \mathbb{R}$, and, again by Theorem 1.8, that $R(\lambda, B) \geq 0$ for $\lambda$ large. Hence, using the Neumann expansion

$$R(\lambda, B + \Phi_\lambda) = R(\lambda, A) \sum_{n=0}^{\infty} \big[\Phi_\lambda R(\lambda, A)\big]^n,$$

we conclude that $R(\lambda, B + \Phi_\lambda)$ is positive for $\lambda$ large. Since the operator $H_\lambda$ in (6.10) is positive for all $\lambda \in \mathbb{R}$, this proves that $R(\lambda, A)$ is positive for $\lambda$ large, and the assertion follows. □

It is important to note that the converse of this result is not true (cf. Exercise 6.20.(2)). This is due to the fact that the boundary condition $f'(0) = Bf + \Phi f$ can be written in different ways. However, if $\Phi$ has no atomic part in zero (cf. Paragraph IV.2.8), the positivity hypotheses in Theorem 6.11 are also necessary in order to obtain the positivity of $\big(T(t)\big)_{t\geq 0}$ on $X$. For details see [Ker86, III.3].

Before we show how positivity can be used to obtain simple stability criteria for the solutions of (ADDE), we prove the following technical lemma on the operator-valued map $R : \rho \subset \mathbb{C}^2 \to \mathcal{L}(Y)$ defined by

$$\begin{aligned} &(\lambda, \mu) \mapsto R(\lambda, \mu) := R(\lambda, B + \Phi_\mu) \qquad \text{for} \\ &(\lambda, \mu) \in \rho := \big\{(r, s) \in \mathbb{C}^2 : r \in \rho(B + \Phi_s)\big\}. \end{aligned}$$

**6.12 Lemma.** *The following assertions are true.*

(i) *The set $\rho \subset \mathbb{C}^2$ is open.*

(ii) *The map $R(\cdot, \cdot)$ is analytic.*

Proof. (i) Let $(\lambda_0, \mu_0) \in \rho$ and $(\lambda, \mu) \in \mathbb{C}$. Then

$$(\lambda - B - \Phi_\mu) - (\lambda_0 - B - \Phi_{\mu_0}) = (\lambda - \lambda_0) + \Phi\big((\varepsilon_\mu - \varepsilon_{\mu_0}) \otimes I\big) =: \Delta_{\lambda,\mu}.$$

Since $\lim_{(\lambda,\mu)\to(\lambda_0,\mu_0)} \|\Delta_{\lambda,\mu}\| = 0$ and

$$(\lambda - B - \Phi_\mu) = \big(I + \Delta_{\lambda,\mu} R(\lambda_0, \mu_0)\big)(\lambda_0 - B - \Phi_{\mu_0}), \tag{6.12}$$

it follows that $(\lambda - B - \Phi_\mu)$ is invertible for $\|(\lambda, \mu) - (\lambda_0, \mu_0)\|$ sufficiently small, i.e., $(\lambda, \mu) \in \rho$.

(ii) From (6.12) it follows that for $\|(\lambda,\mu)-(\lambda_0,\mu_0)\|$ sufficiently small, we have

$$R(\lambda,\mu)=R(\lambda_0,\mu_0)\sum_{n=0}^{\infty}\bigl[\Delta_{\lambda,\mu}R(\lambda_0,\mu_0)\bigr]^n.$$

Since this series converges uniformly in small balls and the map $\mathbb{C}^2 \ni (\lambda,\mu)\mapsto \Delta_{\lambda,\mu}\in\mathcal{L}(Y)$ is analytic, we infer that $R(\cdot,\cdot)$ is analytic as well. ☐

With this lemma we will now describe the behavior of the *spectral bound function*

$$\mathrm{s}:\mathbb{R}\to\mathbb{R}\cup\{-\infty\} \qquad \text{defined by} \qquad \mathrm{s}(\lambda):=\mathrm{s}(B+\Phi_\lambda) \quad \text{for } \lambda\in\mathbb{R}.$$

**6.13 Proposition.** *Let $B$ generate a positive semigroup on $Y$ and assume that $\Phi\in\mathcal{L}(X,Y)$ is positive. Then the spectral bound function $\mathrm{s}(\cdot)$ is decreasing and continuous from the left on $\mathbb{R}$. If, in addition, $\mathrm{s}(B+\Phi_{\mu_0})$ is isolated in $\sigma(B+\Phi_{\mu_0})\cap\mathbb{R}$, then $\mathrm{s}(\cdot)$ is even continuous in $\mu_0\in\mathbb{R}$.*

Proof. For $\mu_0\le\mu_1$ we have $\Phi_{\mu_1}\le\Phi_{\mu_0}$ and therefore $\mathrm{s}(B+\Phi_{\mu_1})\le \mathrm{s}(B+\Phi_{\mu_0})$ by Corollary 1.11. This shows that the function $\mathrm{s}(\cdot)$ is decreasing.

In order to show that $\mathrm{s}(\cdot)$ is left-continuous, we assume by contradiction that

$$\mathrm{s}(\mu_0)<s^-:=\lim_{\varepsilon\downarrow 0}\mathrm{s}(\mu_0-\varepsilon)$$

for some $\mu_0\in\mathbb{R}$. Then $s^-\in\rho(B+\Phi_{\mu_0})$, and therefore $(s^-,\mu_0)\in\rho$. This contradicts the fact that $\rho\subset\mathbb{C}^2$ is open, since by Theorem 1.10 we have $\mathrm{s}(\mu_0-\varepsilon)\in\sigma(B+\Phi_{\mu_0-\varepsilon})$, i.e., $\bigl(\mathrm{s}(\mu_0-\varepsilon),\mu_0-\varepsilon\bigr)\notin\rho$ for all $\varepsilon>0$, while $\bigl(\mathrm{s}(\mu_0-\varepsilon),\mu_0-\varepsilon\bigr)\to(s^-,\mu_0)$ as $\varepsilon\downarrow 0$.

Now, assume in addition that $\mathrm{s}(B+\Phi_{\mu_0})$ is isolated in $\sigma(B+\Phi_{\mu_0})\cap\mathbb{R}$. In order to show that $\mathrm{s}(\cdot)$ is right-continuous, we assume by contradiction that

$$s^+:=\lim_{\varepsilon\downarrow 0}\mathrm{s}(\mu_0+\varepsilon)<\mathrm{s}(\mu_0). \tag{6.13}$$

Then by assumption, there exists $\lambda\in\rho(B+\Phi_{\mu_0})\cap\mathbb{R}$ satisfying

$$s^+<\lambda<\mathrm{s}(\mu_0).$$

In particular, $(\lambda,\mu_0)\in\rho$, and we conclude from Lemma 6.12.(ii) that

$$R(\lambda,\mu_0)=\lim_{\varepsilon\downarrow 0}R(\lambda,\mu_0+\varepsilon)\ge 0.$$

This contradicts Lemma 1.9, and $\mathrm{s}(\cdot)$ must be right-continuous. ☐

We mention that the spectral bound function $\mathrm{s}(\cdot)$ is always continuous if $B$ has compact resolvent, or if $\Phi_{\lambda_0}$ is compact; see Exercise 6.20.(3).

We are now well prepared to prove the following result, which allows us to estimate the spectral bound $\mathrm{s}(A)$ of the generator $A$ in terms of the operators $B$ and $\Phi$.

**6.14 Theorem.** *Let $B$ generate a positive semigroup on $Y$ and assume that $\Phi \in \mathcal{L}(X,Y)$ is positive. Then the following is true.*

(i) *If* $\mathrm{s}(B+\Phi_\lambda) < \lambda$, *then* $\mathrm{s}(A) < \lambda$.

(ii) *If* $\mathrm{s}(B+\Phi_\lambda) = \lambda$, *then* $\mathrm{s}(A) = \lambda$.

(iii) *In addition, assume that $\sigma(B) \neq \emptyset$. If $B$ has compact resolvent or if $\Phi$ is compact, then the spectral bound* $\mathrm{s}(A)$ *is the unique solution of the generalized characteristic equation*

$$\lambda = \mathrm{s}(B+\Phi_\lambda), \qquad \lambda \in \mathbb{R}. \tag{6.14}$$

*Moreover, in this case*

$$\mathrm{s}(B+\Phi_\lambda) \lesseqgtr \lambda \qquad \Longleftrightarrow \qquad \mathrm{s}(A) \lesseqgtr \lambda. \tag{6.15}$$

Proof. (i) Let $\lambda > \mathrm{s}(B+\Phi_\lambda)$. Then we obtain from the monotonicity of $\mathrm{s}(\cdot)$ that

$$\mu \geq \lambda > \mathrm{s}(B+\Phi_\lambda) \geq \mathrm{s}(B+\Phi_\mu)$$

for all $\mu \geq \lambda$. This implies $\mu \in \rho(B+\Phi_\mu)$ and therefore $\mu \in \rho(A)$ for all $\mu \geq \lambda$ by Proposition 6.7. On the other hand, Theorem 1.10 implies $\mathrm{s}(A) \in \sigma(A)$; hence $\lambda > \mathrm{s}(A)$, as claimed.

(ii) If $\lambda = \mathrm{s}(B+\Phi_\lambda)$, then again by Theorem 1.10 we have $\lambda \in \sigma(B+\Phi_\lambda)$ and therefore $\lambda \in \sigma(A)$. On the other hand, we can show as in (i) that $\mu \in \rho(A)$ for all $\mu > \lambda$, which implies $\lambda = \mathrm{s}(A)$.

(iii) If $\sigma(B) \neq \emptyset$, then by Corollary 1.11.(ii) we have $-\infty < \mathrm{s}(B) \leq \mathrm{s}(\lambda)$ for all $\lambda \in \mathbb{R}$. Moreover, by Exercise 6.20.(3) it follows that the map $\mathrm{s}(\cdot)$ is continuous and decreasing. Therefore, equation (6.14) has a unique solution $\lambda_0$, which by (ii) coincides with $\mathrm{s}(A)$. The estimates in (6.15) are then immediate (cf. Figure 7, p. 359 with $\xi(\lambda) := \mathrm{s}(B+\Phi_\lambda)$). □

In particular, under the assumptions in Theorem 6.14.(iii) we have that

$$\mathrm{s}(A) < 0 \qquad \text{if and only if} \qquad \mathrm{s}(B+\Phi_0) < 0.$$

In order to prove this equivalence without any compactness assumption, we need the following characterization of the spectral bound of $A$.

**6.15 Lemma.** *Let $B$ generate a positive semigroup on $Y$ and assume that $\Phi \in \mathcal{L}(X,Y)$ is positive. If $\sigma(B+\Phi_\lambda) \neq \emptyset$ for some $\lambda \in \mathbb{R}$, then*

$$\mathrm{s}(A) = \sup\bigl\{\lambda \in \mathbb{R} : \mathrm{s}(B+\Phi_\lambda) \geq \lambda\bigr\}. \tag{6.16}$$

*In the other case one has* $\mathrm{s}(A) = -\infty$.

Proof. If $\sigma(B+\Phi_\lambda) = \emptyset$ for all $\lambda \in \mathbb{R}$, then $\mathrm{s}(A) = -\infty$ by Theorem 6.14.(i), and the second statement follows.

Assume now $\sigma(B+\Phi_\lambda) \neq \emptyset$ for some $\lambda \in \mathbb{R}$, and denote the right-hand side of equation (6.16) by $\mu$. Then it follows from the left-continuity of $\mathrm{s}(\cdot)$ that $\mathrm{s}(B+\Phi_\mu) \geq \mu$. Accordingly, we distinguish two cases.

*Case* 1: $\mathrm{s}(B+\Phi_\mu) = \mu$. Then $\mathrm{s}(A) = \mu$ by Theorem 6.14.(ii), and (6.16) follows.

*Case* 2: $\mathrm{s}(B+\Phi_\mu) > \mu$. We first show that this implies the inclusion

$$(6.17) \qquad \big(\mu, \mathrm{s}(B+\Phi_\mu)\big] \subset \sigma(B+\Phi_\mu).$$

Assume by contradiction that there exists $r \in (\mu, \mathrm{s}(B+\Phi_\mu)] \cap \rho(B+\Phi_\mu)$. Then $(r,\mu) \in \rho$, and by the definition of $\mu$, we have $r+\varepsilon > \mu+\varepsilon > \mathrm{s}(B+\Phi_{\mu+\varepsilon})$ for all $\varepsilon > 0$. Next, we use Lemma 1.9 and Lemma 6.12 to conclude that

$$R(r, B+\Phi_\mu) = \lim_{\varepsilon \downarrow 0} R(r+\varepsilon, B+\Phi_{\mu+\varepsilon}) \geq 0.$$

Again by Lemma 1.9, this contradicts the fact that $r \leq \mathrm{s}(B+\Phi_\mu)$. Hence (6.17) is proved, and from the closedness of the spectrum we deduce $\mu \in \sigma(B+\Phi_\mu)$. Consequently, $\mu \in \sigma(A)$ by Proposition 6.7, and therefore $\mathrm{s}(A) \geq \mu$.

It remains to show that $\mathrm{s}(A) > \mu$ is not possible. To this end, we assume by contradiction that $\mathrm{s}(A) > \mu$. Then from the definition of $\mu$, we infer that

$$\mathrm{s}(B+\Phi_{\mathrm{s}(A)}) < \mathrm{s}(A), \qquad \text{and hence} \qquad \mathrm{s}(A) \in \rho(B+\Phi_{\mathrm{s}(A)}).$$

By Proposition 6.7 this implies $\mathrm{s}(A) \in \rho(A)$, contradicting Theorem 1.10. □

Under the above assumptions it is now possible to characterize delay differential operators having negative spectral bound.

**6.16 Corollary.** *Let $B$ generate a positive semigroup on $Y$ and assume that $\Phi \in \mathcal{L}(X,Y)$ is positive. Then we have for the corresponding delay differential operator $A$ on $X$ that*

$$\mathrm{s}(A) < 0 \qquad \Longleftrightarrow \qquad \mathrm{s}(B+\Phi_0) < 0.$$

Proof. We can assume that there exists $\lambda \in \mathbb{R}$ such that $\sigma(B+\Phi_\lambda) \neq \emptyset$, since otherwise $\mathrm{s}(A) = \mathrm{s}(B+\Phi_0) = -\infty$ by Lemma 6.15.

Suppose first that $\mathrm{s}(A) < 0$. Then $\mathrm{s}(B+\Phi_0) < 0$, since otherwise $\mathrm{s}(A) \geq 0$ by Lemma 6.15. This proves the "only if" part. The "if" part follows immediately from Theorem 6.14.(i), and the proof is complete. □

Finally, we combine Corollary 6.16 with the results of Chapters IV and V in order to obtain simple conditions implying the stability of the semigroup $\big(T(t)\big)_{t\geq 0}$ generated by $A$. To that purpose we use the notions *uniformly exponentially stable* (see Definitions V.1.1) and *exponentially stable* (see V.1.5).

**6.17 Corollary.** *Let $B$ generate a positive semigroup on $Y$, assume that $\Phi \in \mathcal{L}(X, Y)$ is positive, and denote by $\big(T(t)\big)_{t\geq 0}$ the semigroup generated by the corresponding delay differential operator.*

(i) *The semigroup $\big(T(t)\big)_{t\geq 0}$ is exponentially stable if and only if the spectral bound $\mathrm{s}(B + \Phi_0)$ is less than 0.*

(ii) *If $\big(S(t)\big)_{t\geq 0}$ is immediately norm continuous, then the semigroup $\big(T(t)\big)_{t\geq 0}$ is uniformly exponentially stable if and only if the spectral bound $\mathrm{s}(B + \Phi_0)$ is less than 0.*

Proof. Assertion (i) follows from Proposition 1.14 and Corollary 6.16.

(ii) If $\big(S(t)\big)_{t\geq 0}$ is immediately norm continuous, then $\big(T(t)\big)_{t\geq 0}$ is eventually norm continuous by Theorem 6.6. Hence, the assertion follows from Theorem V.1.10 and Corollary 6.16. □

**6.18 Example.** In order to illustrate some surprising consequences of Corollary 6.17.(i), we consider the Cauchy problem with delay

$$\begin{cases} \dot{u}(t) = Bu(t) + \Psi u(t - r) & \text{for } t \geq 0, \\ u_0 = h, \end{cases} \tag{6.18}$$

where $B$ generates a positive semigroup on $Y$, $\Psi \in \mathcal{L}(Y)$ is positive, and $h \in X := \mathrm{C}\big([-r, 0], Y\big)$ is the initial ("history") function. Using the above notation, we have $\Phi = \Psi\delta_{-r}$, hence $\Phi_0 = \Psi$. Then, by Corollary 6.16 and Proposition 1.14, the solution of equation (6.18) is exponentially stable for every $h \in D(A)$ if and only if the semigroup generated by $B + \Psi$ is exponentially stable. However, the semigroup generated by $B + \Psi$ is the solution semigroup of the "undelayed" Cauchy problem

$$\begin{cases} \dot{u}(t) = (B + \Psi)u(t) & \text{for } t \geq 0, \\ u(0) = x. \end{cases} \tag{6.19}$$

This observation proves the following result.

**Corollary.** *The solutions of* (6.18) *are exponentially stable for all $h \in D(A)$ and one/all $r > 0$ if and only if those of* (6.19) *are exponentially stable for all $x \in D(B)$.*

In other words, for the positive delay differential equation (6.18) the exponential stability is independent of the delay $r$. This is in sharp contrast with the general situation. In fact, even with $Y = \mathbb{C}$ there are examples of stable Cauchy problems that can be destabilized by increasing the time lag $r$. See Exercise 6.20.(5).

We close this section with an application to a diffusion equation with delay that generalizes the simple model from (6.1).

**6.19 Example.** We consider the partial differential equation with delay

$$(6.20)\quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} - d(x)\,u(t,x) & \\ \qquad\qquad + b(x)\,u(t-r,x), & t \geq 0,\ x \in [0,1], \\ \dfrac{\partial u(t,0)}{\partial x} = 0 = \dfrac{\partial u(t,1)}{\partial x}, & t \geq 0, \\ u(s,x) = h(s,x), & s \in [-r,0],\ x \in [0,1]. \end{cases}$$

This equation can be interpreted as a model for the growth of a population in $[0,1]$. In fact, $u(t,\cdot)$ is the population density at time $t$, and the term ${}^{d^2}\!/_{dx^2}\, u(t,x)$ describes the internal migration. Moreover, the continuous functions $d, b : [0,1] \to \mathbb{R}_+$ represent space-dependent death and birth rates, respectively, and $r$ is the delay due to pregnancy.

In order to rewrite (6.20) as an abstract delay differential equation of the form (ADDE), we introduce the spaces $Y := \mathrm{C}[0,1]$ and $X := \mathrm{C}\big([-r,0],Y\big)$. Moreover, we define the operators

$$\begin{aligned} &\Delta := \tfrac{d^2}{dx^2}, && D(\Delta) := \big\{y \in \mathrm{C}^2[0,1] : y'(0) = 0 = y'(1)\big\}, \\ &B := \Delta - M_d, && D(B) := D(\Delta), \\ &\Phi := M_b \delta_{-r} \in \mathcal{L}(X,Y), && \end{aligned}$$

where $M_d$ and $M_b$ are the multiplication operators induced by $d$ and $b$, respectively.

Combining the results from Exercise II.4.34.(1) and Theorem III.1.16.(i), we infer that $B$ generates an immediately compact semigroup $\big(S(t)\big)_{t\geq 0}$ on $Y$. Moreover, since $\mathrm{e}^{-tM_d}$ is positive, $\big(S(t)\big)_{t\geq 0}$ is positive by the Trotter product formula; cf. Exercise III.5.11.(1). Hence, Proposition 6.9 and Theorem 6.11 imply that $\big(T(t)\big)_{t\geq 0}$ is positive and eventually compact for $t > r$. Next, we apply Corollary IV.3.12.(i) and Example 6.18 (with $\Psi := M_b$) to conclude that

$$\omega_0(A) = \mathrm{s}(A) < \lambda \quad \Longleftrightarrow \quad \omega_0(\Delta + M_b - M_d) = \mathrm{s}(\Delta + M_b - M_d) < \lambda$$

for arbitrary $\lambda \in \mathbb{R}$. In particular, if in every $x \in [0,1]$ the death rate majorizes the birth rate, i.e., if $\delta := \inf_{x\in[0,1]}\big(d(x) - b(x)\big) > 0$, then the operator $\Delta + M_b - M_d + \delta$ is dissipative (use Example II.3.26.(i)), and hence $\omega_0(\Delta + M_b - M_d) < -\delta$. This shows that the condition $b(x) < d(x)$, $x \in [0,1]$, leads to uniformly exponentially stable solutions of (6.20) and hence to the extinction of the population, no matter whether we consider the equation with or without delay.

**6.20 Exercises.** (1) Reconsider Exercise 6.10.(2) using the results of this subsection.

(2) Show that the converse of Theorem 6.11 does not hold, i.e., find nonpositive operators $B$ and $\Phi$ such that $(T(t))_{t\geq 0}$ is positive.

(3) Show that the final conclusion of Proposition 6.13 concerning the continuity of the spectral bound function $\mathrm{s}(\cdot)$ is true if

(i) $B$ has compact resolvent, or

(ii) $\Phi_{\mu_0}$ is compact.

(Hint: For (i) use Corollary IV.1.19 and Exercise II.4.30.(2); for (ii) use the fact that $\mathrm{s}(B) < \mathrm{s}(B+\Phi_\mu)$ for all $\mu \in \mathbb{R}$ together with (6.13) and Corollary IV.2.11.)

(4) Re-prove the corollary in Paragraph IV.2.8 by using the results of this subsection. Moreover, show that $\mathrm{s}(A) < 0$ if and only if $\omega_0(A) < 0$. Hence, the condition $\|L_0\| + a < 0$ characterizes the uniform exponential stability of the semigroup $(T(t))_{t\geq 0}$ generated by $A$. (Hint: Use Proposition 6.9.)

(5*) In the situation of Example 6.18, take $Y := \mathbb{C}$ and $B, \Psi \in \mathbb{R}$ such that $0 < B < 1$ and $B + \Psi < 0$. Then the Cauchy problem (6.19) for $B + \Psi$ is uniformly exponentially stable, while there exists $r > 0$ and a history function $h$ such that the delayed Cauchy problem (6.18) is not stable. (Hint: Show that for every $0 < \lambda < B$ there exists $r > 0$ such that the map $t \mapsto \mathrm{e}^{\lambda t}$ is a solution of (6.18). See [Nag86, B.IV, Expl. 3.10] for more details. Compare also [Had78] and [Hal77, p. 107].)

## *Notes and Further Reading to Section* 6

Standard references for delay differential equations are the monographs by Hale [Hal77], Hale–Verduyn Lunel [HVL93], Diekmann et al. [DGLW95], and [Wu96]. There, as in our discussion, the state space is a space of continuous functions. For many applications, e.g., in control and approximation theory (cf. [CZ95], [NY89], or [Kap86]), it is preferable to choose the state space $Y \times \mathrm{L}^p([-r,0],Y)$ for $1 < p < \infty$, which is reflexive if $Y$ is reflexive, or even a Hilbert space if $p = 2$ and $Y$ is a Hilbert space.

Among the many papers pursuing a functional-analytic approach to delay differential equations or, more generally, to neutral equations, we quote [BHS83], [BHT90], [IKT96], [KpZ86], and [TT95]. Semigroups arising from delay differential equations with unbounded delay, their regularity properties and their asymptotic behavior, were studied, e.g., by [DBKS84] and [DBKS85] (see also [Tan97]).

Our presentation follows closely [Nag86, B-IV.3] and [Ker86]. The aspect of positivity and its influence on the stability of the solutions was first studied by Kerscher–Nagel [KN84]. Lemma 6.15, which is essential to characterize delay differential operators with negative spectral bound, is due to W. Arendt.

# 7. Semigroups for Volterra Equations

In many situations, the time change of a system is also caused by some external force. This phenomenon gives rise to so-called *inhomogeneous differential equations* and, in our context, to the following problem.

**7.1 Problem.** *Let $\big(A, D(A)\big)$ be an operator on the Banach space $X$. For a given initial value $x \in X$ and a function $f : \mathbb{R}_+ \to X$, find a function $u : \mathbb{R}_+ \to X$ satisfying*

$$\text{(iACP)} \qquad \begin{cases} \dot{u}(t) = Au(t) + f(t) & \text{for } t \geq 0, \\ u(0) = x. \end{cases}$$

In analogy to Definition II.6.1, we call this the *inhomogeneous abstract Cauchy problem* corresponding to $A$ and $f$.

In what follows we assume the homogeneous problem (i.e., with $f \equiv 0$) to be well-posed. Hence, by Theorem II.6.7, the operator $\big(A, D(A)\big)$ should generate a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. In this case, the natural candidate for the solution of (iACP) is given by the *variation of parameters formula*

$$u(t) := T(t)x + \int_0^t T(t-s)f(s)\,ds \qquad \text{for } t \geq 0. \tag{7.1}$$

A similar formula has already appeared in Chapter III in the context of perturbation problems (e.g., Corollary III.1.7, Corollary III.3.2.(i), or Corollary III.3.15.(i)).

## a. Mild and Classical Solutions

It is a simple exercise in differentiation to show that if $A$ is a bounded operator and $f$ is continuous, the function $u$ defined by (7.1) is continuously differentiable and satisfies (iACP). However, this function exists under more general hypotheses, and as in Definition II.6.3, it is useful to introduce the concept of a *mild solution*.

**7.2 Definition.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on $X$ and take $x \in X$ and $f \in \mathrm{L}^1(\mathbb{R}_+, X)$. Then the function $u(\cdot)$ defined by*

$$u(t) := T(t)x + \int_0^t T(t-s)f(s)\,ds, \quad t \geq 0, \tag{7.2}$$

*is called the mild solution of the corresponding* (iACP). *If a function $u : \mathbb{R}_+ \to X$ is continuously differentiable with $u(t) \in D(A)$ and satisfies* (iACP), *we call it a classical solution.*

It is not difficult to show that every classical solution of (iACP) is also a mild solution (see Exercise 7.10.(1)). In particular, this implies that a classical solution of (iACP) is always unique.

In order to find mild, and then classical, solutions of (iACP), we adopt the following strategy.

*Find a new state space* $\mathfrak{X}$ *and a new operator* $(\mathcal{A}, D(\mathcal{A}))$ *that generates a semigroup* $(\mathfrak{T}(t))_{t\geq 0}$ *on* $\mathfrak{X}$ *such that the (mild and classical) solutions of* (iACP) *can be obtained from this semigroup using Propositions* II.6.2 *and* II.6.4.

The fact that the data in (iACP) are the initial value $x \in X$ and the function $f : \mathbb{R}_+ \to X$ suggests that we should take as the new state space a product of $X$ with an $X$-valued function space on $\mathbb{R}_+$. For operators on such product spaces we will use the matrix notation and the matrix rules analogous to the scalar case (see [Eng97]).

We now give the precise definitions for such a construction.

Start with the generator $(A, D(A))$ of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$. Then the new space and the new semigroup will be defined in the following manner.

**7.3 Definition.** *On the Banach space*

$$\mathfrak{X} := X \times \mathrm{L}^1(\mathbb{R}_+, X),$$

*we define operators*

$$\mathfrak{T}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0, \tag{7.3}$$

*where* $(S(t))_{t\geq 0}$ *is the (left) translation semigroup on* $\mathrm{L}^1(\mathbb{R}_+, X)$ *(see Paragraph* I.4.16*) and* $R(t) : \mathrm{L}^1(\mathbb{R}_+, X) \to X$ *is defined as*

$$R(t)f := \int_0^t T(t-s)f(s)\,ds \qquad \text{for } f \in \mathrm{L}^1(\mathbb{R}_+, X). \tag{7.4}$$

We leave it to the reader to verify that these operators are well-defined and bounded on $\mathfrak{X}$. More interesting is the following property.

**7.4 Proposition.** *The family* $(\mathfrak{T}(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $\mathfrak{X}$.

Proof. Since $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are both semigroups, we can show that

$$\begin{aligned} &T(t)R(s)f + R(t)S(s) \\ &\qquad = T(t)\int_0^s T(s-r)f(r)\,dr + \int_0^t T(t-r)(S(s)f)(r)\,dr \\ &\qquad = \int_0^s T(t+s-r)f(r)\,dr + \int_s^{t+s} T(t+s-r)f(r)\,dr \\ &\qquad = R(t+s)f \qquad \text{for all } t,s \geq 0,\ f \in \mathrm{L}^1(\mathbb{R}_+, X). \end{aligned}$$

By the rules for matrix multiplication, this implies that $\big(\mathcal{T}(t)\big)_{t\ge0}$ is a semigroup. The strong continuity of $\big(\mathcal{T}(t)\big)_{t\ge0}$ follows, since

$$\lim_{t\downarrow 0} R(t)f = 0$$

for every $f \in \mathrm{L}^1(\mathbb{R}_+, X)$. □

In the next step we identify the generator of the semigroup $\big(\mathcal{T}(t)\big)_{t\ge0}$.

**7.5 Proposition.** *The generator $\big(\mathcal{A}, D(\mathcal{A})\big)$ of $\big(\mathcal{T}(t)\big)_{t\ge0}$ is given by*

$$\begin{aligned} &\mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} Ax + f(0) \\ f' \end{pmatrix} \qquad \textit{for} \\ &\begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}) := D(A) \times \mathrm{W}^{1,1}(\mathbb{R}_+, X), \end{aligned} \tag{7.5}$$

*or, using matrix notation,*

$$\mathcal{A} = \begin{pmatrix} A & \delta_0 \\ 0 & {}^{d}\!/\!_{ds} \end{pmatrix},$$

*where $\delta_0$ is the point evaluation in* 0 *and ${}^{d}\!/\!_{ds}$ denotes the generator of the (left) translation semigroup $\big(S(t)\big)_{t\ge0}$ as defined in Paragraph* II.2.10.

Proof. Rather than explicitly determining the generator $\mathcal{A}$ of $\big(\mathcal{T}(t)\big)_{t\ge0}$, we shall first compute its resolvent by using the integral representation (1.14) from Chapter II. Indeed, for $\lambda$ sufficiently large, this representation yields

$$R(\lambda, \mathcal{A}) := \int_0^\infty \mathrm{e}^{-\lambda r}\mathcal{T}(r)\,dr = \begin{pmatrix} R(\lambda, A) & Q(\lambda) \\ 0 & R(\lambda, {}^{d}\!/\!_{ds}) \end{pmatrix},$$

where

$$\begin{aligned} Q(\lambda)f :&= \int_0^\infty \mathrm{e}^{-\lambda t} R(t) f\,dt \\ &= \int_0^\infty \int_0^t \mathrm{e}^{-\lambda t} T(t-r) f(r)\,dr\,dt \\ &= \int_0^\infty \int_r^\infty \mathrm{e}^{-\lambda t} T(t-r) f(r)\,dt\,dr \\ &= \int_0^\infty \Big(\int_0^\infty \mathrm{e}^{-\lambda t} T(t)\,dt\Big) \mathrm{e}^{-\lambda r} f(r)\,dr \\ &= R(\lambda, A) \int_0^\infty \mathrm{e}^{-\lambda r} f(r)\,dr \qquad \text{for } f \in \mathrm{L}^1(\mathbb{R}_+, X). \end{aligned}$$

On the other hand, direct computation shows that the operator $\lambda - \left(\begin{smallmatrix} A & \delta_0 \\ 0 & {}^{d}\!/\!_{ds} \end{smallmatrix}\right)$ is invertible, for large $\lambda$, with inverse

$$\left[\lambda - \begin{pmatrix} A & \delta_0 \\ 0 & {}^{d}\!/\!_{ds} \end{pmatrix}\right]^{-1} = \begin{pmatrix} R(\lambda, A) & R(\lambda, A)\delta_0 R(\lambda, {}^{d}\!/\!_{ds}) \\ 0 & R(\lambda, {}^{d}\!/\!_{ds}) \end{pmatrix}. \tag{7.6}$$

Since $\big(R(\lambda, {}^{d}/_{ds})f\big)(s) = \int_s^\infty \mathrm{e}^{-\lambda(t-s)} f(t)\,dt$ (see Proposition 2 in Paragraph II.2.10), we obtain the identity

$$Q(\lambda) = R(\lambda, A)\delta_0 R(\lambda, \tfrac{d}{ds}). \tag{7.7}$$

This shows that the resolvents of $\mathcal{A}$ and $\left(\begin{smallmatrix} A & \delta_0 \\ 0 & {}^{d}/_{ds} \end{smallmatrix}\right)$ coincide. Therefore, the generator of $\big(\mathcal{T}(t)\big)_{t\geq 0}$ is given by (7.5). □

Having collected this information on the semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0}$ and its generator $\mathcal{A}$, we return to the original inhomogeneous problem. It turns out that the first coordinate of $\big(\mathcal{T}(t)\big)_{t\geq 0}$ yields the (mild and classical) solutions of (iACP). More precisely, let us write

$$\mathfrak{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} := \mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} T(t)x + R(t)f \\ S(t)f \end{pmatrix}$$

for $\binom{x}{f} \in \mathcal{X}$. Then, Proposition II.6.4 implies that $\mathfrak{u}(\cdot)$ is the *mild* solution of the abstract Cauchy problem

$$\begin{cases} \dot{\mathfrak{u}}(t) = \mathcal{A}\mathfrak{u}(t) & \text{for } t \geq 0, \\ \mathfrak{u}(0) = \binom{x}{f}. \end{cases}$$

If $\binom{x}{f} \in D(\mathcal{A})$, this implies, for the first coordinate $u_1(\cdot)$, that

$$\dot{u}_1(t) = Au_1(t) + \delta_0 u_2(t) = Au_1(t) + f(t),$$

which is precisely (iACP).

Therefore, we obtain classical solutions of (iACP) by choosing the first coordinate of classical solutions of (ACP) corresponding to $\mathcal{A}$.

**7.6 Corollary.** *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ on the Banach space $X$. If we take $x \in D(A)$ and $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, X)$, i.e., $\binom{x}{f} \in D(\mathcal{A})$, then*

$$u(t) := T(t)x + R(t)f$$

*is the unique classical solution of* (iACP).

We point out that the assumption $\binom{x}{f} \in D(\mathcal{A})$ is equivalent to the differentiability of *both* coordinates of $\mathfrak{u}(t) = \mathcal{T}(t)\binom{x}{f}$. On the other hand, we need differentiability only of the *first* coordinate $u_1(\cdot)$ in order to obtain a classical solution of (iACP). However, the above considerations indicate how we should proceed to find other sufficient conditions for the existence of classical solutions:

*Find different product spaces on which the operators $\mathcal{T}(t)$ defined in* (7.3) *induce a strongly continuous semigroup and then identify the domain of its generator.*

We discuss one such case in which the "time regularity" $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, X)$ is replaced by a "space regularity" $f \in \mathrm{C}_0\big(\mathbb{R}_+, D(A)\big)$.

As usual, we start from a strongly continuous semigroup $(T(t))_{t\ge 0}$ with generator $(A, D(A))$ on the Banach space $X$ and denote by $(T_1(t))_{t\ge 0}$ the semigroup restricted to its first Sobolev space $X_1 := (D(A), \|\cdot\|_A)$ (see Section II.5.a). In addition, we have the (left) translation semigroup $(S(t))_{t\ge 0}$ on $\mathrm{C}_0(\mathbb{R}_+, X_1)$ with generator $d/ds$. Its *extrapolated* Sobolev space (see Definition II.5.4) will be denoted by $\mathrm{C}_0^{-1}(\mathbb{R}_+, X_1)$ and the corresponding extrapolated translation semigroup by $(S_{-1}(t))_{t\ge 0}$. Using these constructions, we obtain the following new semigroups.

**7.7 Proposition.** *The operators*

$$\mathcal{T}(t) := \begin{pmatrix} T_1(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \ge 0,$$

*with* $R(t)f := \int_0^t T_1(t-s)f(s)\,ds$ *for* $f \in \mathrm{C}_0(\mathbb{R}_+, X_1)$ *form a strongly continuous semigroup on* $\mathfrak{X}_0 := X_1 \times \mathrm{C}_0(\mathbb{R}_+, X_1)$. *The corresponding extrapolated Sobolev space is*

$$\mathfrak{X}_{-1} = X \times \mathrm{C}_0^{-1}(\mathbb{R}_+, X_1),$$

*and the extrapolated semigroup* $(\mathcal{T}_{-1}(t))_{t\ge 0}$ *is given by*

$$\mathcal{T}_{-1}(t) = \begin{pmatrix} T(t) & R_{-1}(t) \\ 0 & S_{-1}(t) \end{pmatrix}, \quad t \ge 0,$$

*where the operators* $R_{-1}(t) : \mathrm{C}_0^{-1}(\mathbb{R}_+, X_1) \to X$ *are the continuous extensions of* $R(t) : \mathrm{C}(\mathbb{R}_+, X_1) \to X_1$.

Proof. The first assertion is merely Proposition 7.4 with $\mathrm{L}^1$ replaced by $\mathrm{C}_0$. Using Proposition 7.5 in combination with the proposition in Paragraph II.2.3, it follows that the generator of $(\mathcal{T}(t))_{t\ge 0}$ is precisely

$$\mathcal{A} = \begin{pmatrix} A & \delta_0 \\ 0 & d/ds \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & d/ds \end{pmatrix} + \begin{pmatrix} 0 & \delta_0 \\ 0 & 0 \end{pmatrix}$$

on $D(\mathcal{A}) = X_2 \times \mathrm{C}_0^1(\mathbb{R}_+, X_1)$. Since the point evaluation $\delta_0$ is bounded from $\mathrm{C}_0^1(\mathbb{R}_+, X_1)$ into $X_1$, we see that $\mathcal{A}$ is a bounded perturbation of the "diagonal" operator $\left(\begin{smallmatrix} A & 0 \\ 0 & d/ds \end{smallmatrix}\right)$. From Corollary III.1.4 we can then infer that the extrapolated Sobolev space $\mathfrak{X}_{-1}$ coincides with the extrapolated Sobolev space associated to $\left(\begin{smallmatrix} A & 0 \\ 0 & d/ds \end{smallmatrix}\right)$ on $\mathfrak{X}_0$. Clearly, this space is the product of $X$ (as the extrapolated space of $X_1$) and $\mathrm{C}_0^{-1}(\mathbb{R}_+, X_1)$. The remaining assertions now follow by continuous extension to the extrapolation spaces. □

Next, we consider the extrapolated semigroup $(\mathcal{T}_{-1}(t))_{t\geq 0}$ and observe that the domain of its generator is the original space $\mathcal{X}_0$. Therefore, if we take $\binom{x}{f} \in \mathcal{X}_0$, we obtain

$$t \mapsto \mathfrak{u}(t) := \mathcal{T}(t)\begin{pmatrix} x \\ f \end{pmatrix}$$

as a differentiable function in $\mathcal{X}_{-1} = X \times \mathrm{C}_0^{-1}(\mathbb{R}_+, X_1)$. In particular, its first coordinate is differentiable in $X$ and yields the classical solution of (iACP). This is our final result. For a generalization we refer to Exercise 7.10.(3).

**7.8 Corollary.** *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space $X$. If we take $x \in X_1 := D(A)$ and $f \in \mathrm{C}_0(\mathbb{R}_+, X_1)$, then*

$$u(t) := T(t)x + R(t)f$$

*is the unique classical solution of* (iACP).

Having seen these two existence results, one might ask whether such (time or space) regularity conditions on $f$ are essential in producing classical solutions. The following is an example illustrating that, for general $A$, the condition $f \in \mathrm{C}_0(\mathbb{R}_+, X)$ is not sufficient for this purpose.

**7.9 Example.** Take $(A, D(A))$ to be the generator of a uniformly exponentially stable semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and consider, for some $y \in X$, the inhomogeneous term $f$ given by

$$f(s) := T(s)y, \quad s \geq 0.$$

Then $f \in \mathrm{C}_0(\mathbb{R}_+, X)$ and $u(t) = R(t)f = \int_0^t T(t-s)T(s)y\,ds = tT(t)y$ is the mild solution of (iACP) corresponding to $x = 0$. If, e.g., $A$ generates a group and $y \notin D(A)$, then this mild solution is differentiable only at $t = 0$.

**7.10 Exercises.** (1) Let $A$ generate a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and assume that $f \in \mathrm{L}^1(\mathbb{R}_+, X)$. Show that every classical solution $u : \mathbb{R}_+ \to X$ of the corresponding inhomogeneous problem (iACP) satisfies the variation of parameters formula (7.1). In particular, (iACP) admits at most one classical solution. (Hint: For $t > 0$ consider the differentiable function $v : [0, t] \to X$ defined by $v(s) := T(t-s)u(s)$.)

(2) Let $A$ generate a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and assume that $f \in \mathrm{L}^1(\mathbb{R}_+, X) \cap \mathrm{C}(\mathbb{R}_+, X)$ and take $x \in X$. Show that the mild solution $u : \mathbb{R}_+ \to X$ of the corresponding inhomogeneous Cauchy problem (iACP) is a classical solution if and only if $u(\cdot) \in \mathrm{C}(\mathbb{R}_+, X_1)$.

(3) Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $A$ on a Banach space $X$, and take $\alpha, \beta \in (0, 1)$.

(i) Consider the semigroup $(\mathfrak{T}(t))_{t\geq 0}$ on the Banach space

$$\mathfrak{X}_0 := X_\alpha \times \mathrm{C}_0(\mathbb{R}_+, X_\alpha)$$

(see Proposition 7.7). Show that the abstract Hölder space of order $\beta$ corresponding to $(\mathfrak{T}(t))_{t\geq 0}$ is given by

$$\mathfrak{X}_\beta = (X_\alpha)_\beta \times \mathrm{h}^\beta(\mathbb{R}_+, X_\alpha).$$

(ii) Assume that $\alpha + \beta > 1$. Show that for $x \in (X_\alpha)_\beta$ and $f \in \mathrm{h}^\beta(\mathbb{R}_+, X_\alpha)$ the mild solution $u(\cdot)$ of the corresponding inhomogeneous abstract Cauchy problem is already a classical solution. (Hint: Show that $u(\cdot) \in \mathrm{C}(\mathbb{R}_+, (X_\alpha)_\beta)$ and use Proposition II.5.35 to conclude that $u(\cdot) \in \mathrm{C}(\mathbb{R}_+, X_1)$.)

## b. Optimal Regularity

In this subsection the underlying idea remains the same, but more sophisticated techniques are employed in order to obtain classical solutions of (iACP). In the first part we study inhomogeneities that take values in the extrapolated Favard class $F_0$. Hence, results from Section II.5.b will be used extensively. In the second part, we require $A$ to generate an analytic semigroup in order to prove a so-called *optimal (or maximal) regularity* result in the sense that $\dot{u}$ and $Au$ have the same regularity as $f$.

We start from a strongly continuous semigroup $\big(T_0(t)\big)_{t\geq 0}$ with generator $A_0$ on a Banach space $X_0$ and construct the induced semigroups $\big(T_n(t)\big)_{t\geq 0}$ on the corresponding Sobolev spaces $X_n$, $n \in \mathbb{Z}$ (cf. Section II.5.a). Furthermore, we insert the Favard spaces $F_n$ from Section II.5.b into this tower of spaces. We already know from Definition 7.3 that the operators $R(t)$ map $\mathrm{L}^1(\mathbb{R}_+, X_n)$ into $X_n$. In the following lemma we extend these operators to $\mathrm{L}^1(\mathbb{R}_+, F_n) \to X_n$.

**7.11 Lemma.** *Under the previously discussed assumptions, define*

$$R(t)f := \int_0^t T_{-1}(t-s)f(s)\,ds \qquad \text{for } t \geq 0 \text{ and } f \in \mathrm{L}^1(\mathbb{R}_+, F_0).$$

*Then the following properties hold.*

(i) $R(t) : \mathrm{L}^1(\mathbb{R}_+, F_0) \to X_0$.

(ii) $\|R(t)\| \leq M$ *for* $0 \leq t \leq 1$ *and for some* $M > 0$.

(iii) $\lim_{t\downarrow 0} \|R(t)f\| = 0$ *for all* $f \in \mathrm{L}^1(\mathbb{R}_+, F_0)$.

The proof is very similar to the proof of Corollary III.3.6 and is therefore left as Exercise 7.18.

This lemma immediately implies the first part of the following proposition.

**7.12 Proposition.** *Under the above assumptions, define*

$$\mathcal{X} := X_0 \times \mathrm{L}^1(\mathbb{R}_+, F_0) \qquad \text{and} \qquad \mathcal{T}(t) := \begin{pmatrix} T_0(t) & R(t) \\ 0 & S(t) \end{pmatrix} \quad \text{for } t \geq 0.$$

*Then $\big(\mathcal{T}(t)\big)_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}$, and its generator is*

$$\mathcal{A} := \begin{pmatrix} A_{-1} & \delta_0 \\ 0 & {}^d\!/_{ds} \end{pmatrix} \quad \text{with domain}$$

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in F_1 \times \mathrm{W}^{1,1}(\mathbb{R}_+, F_0) : A_{-1}x + f(0) \in X_0 \right\}.$$

Proof. It remains to prove the characterization of the generator and the coupling condition in $D(\mathcal{A})$, which may seem strange at first glance. We start from the space

$$\mathcal{Y} := X_{-1} \times \mathrm{L}^1(\mathbb{R}_+, X_{-1})$$

on which the operators $\widetilde{\mathcal{T}}(t) := \left(\begin{smallmatrix} T_{-1}(t) & R(t) \\ 0 & S(t) \end{smallmatrix}\right)$ form a strongly continuous semigroup by Proposition 7.4. Its generator is given by the matrix operator $\left(\begin{smallmatrix} A_{-1} & \delta_0 \\ 0 & {}^d\!/_{ds} \end{smallmatrix}\right)$ with domain $X_0 \times \mathrm{W}^{1,1}(\mathbb{R}_+, X_{-1})$ (use Proposition 7.5). Our space $\mathcal{X}$ is now a continuously embedded subspace of $\mathcal{Y}$, and $\big(\mathcal{T}(t)\big)_{t\geq 0}$ is the restricted semigroup of $\big(\widetilde{\mathcal{T}}(t)\big)_{t\geq 0}$. Therefore, Proposition II.2.3 tells us that its generator is just the part of $\left(\begin{smallmatrix} A_{-1} & \delta_0 \\ 0 & {}^d\!/_{ds} \end{smallmatrix}\right)$ in $\mathcal{X}$. Writing this explicitly yields the above statement. □

As before, we use this semigroup to obtain classical solutions of (iACP).

**7.13 Corollary.** *Using the above notation and for given $x \in F_1$, $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, F_0)$ such that*

$$A_{-1}x + f(0) \in X_0, \tag{7.8}$$

*there exists a unique classical solution of* (iACP).

We close this discussion with two comments.

First, if $\big(A, D(A)\big)$ is a Hille–Yosida operator (see Definition II.3.22) on the Banach space $X$, it follows that

$$\overline{D(A)} =: X_0 \subset X \subset F_0$$

(see Exercise II.5.23.(3)). Therefore, Corollary 7.13 holds for every $x \in D(A)$ and $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, X)$ satisfying $Ax + f(0) \in X_0$.

Second, we recall that for the translation semigroup on $\mathrm{C}_{\mathrm{ub}}(\mathbb{R})$ the extrapolated Favard space becomes $F_0 = \mathrm{L}^\infty(\mathbb{R})$ (see Example II.5.22). Therefore, Corollary 7.13 extends Corollary 7.6 considerably.

Now we shall extend the results from Section 7.a in a different direction. We derive solutions $u$ of (iACP) such that, e.g., if $f$ is continuous, then $\dot{u}$ and $Au$ are continuous as well. While this is always true for bounded $A$, it does not hold in general (see Example 7.9). Therefore, we need further conditions on the generator $A$. The standing assumption will be that $A$ generates an analytic semigroup, but even in this case the above regularity property does not hold in general.

**7.14 Example.** Let $X := \ell^2(\mathbb{N})$ and consider the analytic multiplication semigroup $\mathcal{T} = \big(T(t)\big)_{t\geq 0}$ generated by the operator $A(x_n)_{n\in\mathbb{N}} := (-nx_n)_{n\in\mathbb{N}}$ with maximal domain.

Take a nonzero function $0 \leq g \in \mathrm{C}(\mathbb{R}_+)$ with $\operatorname{supp} g \subset [{}^1\!/_2, 1]$. Moreover, let $e_n$ be the $n$th unit vector in $X$, and define $f_n(t) := g\big(2^n(1-t)\big) \cdot e_{2^n}$ and $s_N := \sum_{n=1}^N f_n$. Since the functions $f_n$ have disjoint supports, we have $\|s_N\|_\infty \leq \|g\|_\infty$. A short computation shows that

$$AR(1)f_n = \int_{1/2}^1 \mathrm{e}^{-s} g(s)\, ds \cdot e_{2^n} =: c \cdot e_{2^n},$$

and hence $\|AR(\cdot)s_N\|_\infty \geq \|AR(1)s_N\|_X = c\sqrt{N}$. Thus, there exists $f \in \mathrm{C}_0(\mathbb{R}_+, X)$ such that $u(t) := R(t)f$ is not a classical solution of (iACP). This can be expressed by saying that the operator $A$ does not have optimal regularity.

As disappointing as this example might be, if we use the Favard and abstract Hölder spaces from Section II.5.b, we obtain regularity results that are "optimal" with respect to these spaces. To this end, we will use the characterization of the spaces $F_\alpha$ and $X_\alpha$ given in Proposition II.5.13 for generators of analytic semigroups.

We start from with the product space

$$\mathcal{X} := X_0 \times \mathrm{C}_0(\mathbb{R}_+, X_0)$$

and the semigroup thereon defined by (cf. Definition 7.3)

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0.$$

As in the proof of Proposition 7.7, we conclude that the extrapolated Sobolev space is

$$\mathcal{X}_{-1} := X_{-1} \times \mathrm{C}_0^{-1}(\mathbb{R}_+, X_0),$$

while the extrapolated operators are given by

$$\mathcal{T}_{-1}(t) := \begin{pmatrix} T_{-1}(t) & R_{-1}(t) \\ 0 & S_{-1}(t) \end{pmatrix}, \quad t \geq 0.$$

We will use the above intermediate Favard and Hölder spaces to find new $(\mathcal{T}(t))$-invariant subspaces of $\mathcal{X}_{-1}$ leading to classical solutions of (iACP). In order to do so, we seek regularizing properties of the operators

$$R_{-1}(t) : \mathrm{C}_0^{-1}(\mathbb{R}_+, X_0) \to X_{-1}.$$

For each $0 < \alpha < 1$, the space $\mathrm{C}_0(\mathbb{R}_+, F_\alpha)$ is a subspace of $\mathrm{C}_0(\mathbb{R}_+, X_0)$. Since (the continuous extension of) $(I - {}^d\!/\!_{ds})$ is a bijection from $\mathrm{C}_0(\mathbb{R}_+, X_0)$ onto $\mathrm{C}_0^{-1}(\mathbb{R}_+, X_0)$ with inverse $R(1, {}^d\!/\!_{ds})$, we can restrict the operators $R_{-1}(t)$ to the subspace

$$\mathrm{C}_0^{-1}(\mathbb{R}_+, F_\alpha) := (I - \tfrac{d}{ds})\mathrm{C}_0(\mathbb{R}_+, F_\alpha)$$

and then obtain the following estimate.

**7.15 Lemma.** *Let $A$ be the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on $X$ with $\mathrm{s}(A) < 0$. For each $0 < \alpha < 1$, the operators $R_{-1}(t)$ map $\mathrm{C}_0^{-1}(\mathbb{R}_+, F_\alpha)$ into $F_\alpha$, and there exists a constant $M$ such that*

$$\|R_{-1}(t)\|_{\mathcal{L}(\mathrm{C}_0^{-1}(\mathbb{R}_+,F_\alpha),F_\alpha)} \leq M. \tag{7.9}$$

Proof. Since $\mathrm{C}_0^\infty(\mathbb{R}_+, F_\alpha)$ is dense in $\mathrm{C}_0^{-1}(\mathbb{R}_+, F_\alpha)$, it suffices to show that

$$\|R_{-1}(t)g\|_{F_\alpha} = \|R(t)g\|_{F_\alpha} \leq M\,\|g\|_{\mathrm{C}_0^{-1}(\mathbb{R}_+,F_\alpha)} = M\,\big\|R(1, \tfrac{d}{ds})g\big\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}$$

for each $g \in \mathrm{C}_0^\infty(\mathbb{R}_+, F_\alpha)$ and $t \geq 0$. Setting $g := f - f'$ for some $f \in \mathrm{C}_0^\infty(\mathbb{R}_+, F_\alpha)$, this becomes

$$\|R(t)f - R(t)f'\|_{F_\alpha} \leq M\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)} \qquad \text{for } f \in \mathrm{C}_0^\infty(\mathbb{R}_+, F_\alpha).$$

Since we always have $\|R(t)f\|_{F_\alpha} \leq c\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}$ for some constant $c$, we need to show only that

$$\|R(t)f'\|_{F_\alpha} \leq m\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)} \tag{7.10}$$

for each $f \in \mathrm{C}_0^\infty(\mathbb{R}_+, F_\alpha)$ and some constant $m$.

First, observe that

$$R(t)f' = \int_0^t T(t-s)f'(s)\,ds = \int_0^t AT(t-s)f(s)\,ds - T(t)f(0) + f(t)$$

for $t \geq 0$. Since we have $\|f(t)\|_{F_\alpha} \leq \|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}$ and $\|T(t)f(0)\|_{F_\alpha} \leq m_0\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}$, it remains to find a constant $\widetilde{m}$ such that

$$\left\|\int_0^t AT(t-s)f(s)\,ds\right\|_{F_\alpha} \leq \widetilde{m}\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}\,. \tag{7.11}$$

We use Theorem II.4.6.(c) and the Favard norm $\|\cdot\|_{F_\alpha}$ for analytic semigroups introduced in Proposition II.5.13 in order to estimate

$$\begin{aligned}\left\|A^2T(r)f(s)\right\| &= \|AT(r/2)AT(r/2)f(s)\| \le m_1\tfrac{2}{r}\,\|AT(r/2)f(s)\| \\ &= m_1\tfrac{2}{r}\left(\tfrac{r}{2}\right)^{\alpha-1}\left\|\left(\tfrac{r}{2}\right)^{1-\alpha}AT(r/2)f(s)\right\| \\ &\le m_1 2^{2-\alpha}r^{\alpha-2}\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}\end{aligned}$$

for all $r,s\ge 0$ and an appropriate constant $m_1$. Therefore, we obtain

$$\begin{aligned}\left\|r^{1-\alpha}AT(r)\int_0^t AT(t-s)f(s)\,ds\right\| &= \left\|r^{1-\alpha}\int_0^t A^2T(t+r-s)f(s)\,ds\right\| \\ &\le m_1 2^{2-\alpha}r^{1-\alpha}\int_0^t (t+r-s)^{\alpha-2}\,ds\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)} \\ &= \frac{m_1 2^{2-\alpha}}{1-\alpha}\,r^{1-\alpha}r^{\alpha-1}\Big(1-\left(\tfrac{t+r}{r}\right)^{\alpha-1}\Big)\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)} \\ &\le \frac{m_1 2^{2-\alpha}}{1-\alpha}\,\|f\|_{\mathrm{C}_0(\mathbb{R}_+,F_\alpha)}.\end{aligned}$$

Since this last constant is independent of $t$, we obtain a uniform estimate for the norm of $R_{-1}(t)$ as an operator from $\mathrm{C}_0(\mathbb{R}_+,F_\alpha)$ into $F_\alpha$. ☐

This lemma immediately implies that the operators

$$\mathcal{T}_{-1}(t) := \begin{pmatrix} T(t) & R_{-1}(t) \\ 0 & S_{-1}(t)\end{pmatrix}, \qquad t\ge 0,$$

form a uniformly bounded semigroup on $F_\alpha\times \mathrm{C}_0^{-1}(\mathbb{R}_+,F_\alpha)$. However, this semigroup need not be strongly continuous. To obtain strong continuity, we must restrict our operators to the abstract Hölder spaces $X_\alpha$.

**7.16 Proposition.** *Let $A$ generate an analytic semigroup $\big(T(t)\big)_{t\ge0}$ on $X$ with $\mathrm{s}(A)<0$. For $0<\alpha<1$, the operators*

$$\mathcal{T}_{\alpha}(t) := \begin{pmatrix} T(t) & R_{-1}(t) \\ 0 & S_{-1}(t)\end{pmatrix}, \qquad t\ge 0,$$

*form a strongly continuous semigroup on*

$$\mathcal{Y}_\alpha := X_\alpha\times \mathrm{C}_0^{-1}(\mathbb{R}_+,X_\alpha),$$

*whose generator is given by*

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & d/ds\end{pmatrix} \qquad \text{on} \qquad D(\mathcal{A}) := X_{\alpha+1}\times \mathrm{C}_0(\mathbb{R}_+,X_\alpha).$$

PROOF. By Theorem II.5.15.(ii), $(T(t))_{t\geq 0}$ induces a strongly continuous semigroup on $X_\alpha$, while the translation semigroup $(S(t))_{t\geq 0}$ can be continuously extended to $\mathrm{C}_0^{-1}(\mathbb{R}_+, X_\alpha)$. Since $X_\alpha$ is the $\|\cdot\|_{F_\alpha}$-closure of $X_1$ (use Proposition II.5.14) and since $R(t)$ maps $\mathrm{C}_0^\infty(\mathbb{R}_+, X)$, hence $\mathrm{C}_0^\infty(\mathbb{R}_+, X_\alpha)$, into $X_1$, we conclude from Lemma 7.15 that the operators $R_{-1}(t)$ map $\mathrm{C}_0^{-1}(\mathbb{R}_+, X_\alpha)$ into $X_\alpha$. The estimate (7.9) then implies that the operators $\mathcal{T}_\alpha(t)$ are uniformly bounded. Finally, we obtain strong continuity of $(\mathcal{T}_\alpha(t))_{t\geq 0}$, since $\lim_{t\downarrow 0} \|R(t)f\|_1 = 0$ for every $f \in \mathrm{C}_0^\infty(\mathbb{R}_+, X)$ implies

$$\lim_{t\downarrow 0} \|R_{-1}(t)f\|_{F_\alpha} = 0$$

for every $f \in \mathrm{C}_0^{-1}(\mathbb{R}_+, X_\alpha)$. The generator of $(\mathcal{T}_\alpha(t))_{t\geq 0}$ is then found by taking the part of the generator of $(\mathcal{T}_{-1}(t))_{t\geq 0}$ in $\mathcal{Y}_\alpha$ (use Proposition II.2.3). □

Interpreting this semigroup result in terms of the inhomogeneous Cauchy problem, we obtain *optimal space regularity* in the following sense.

**7.17 Corollary.** *Let $A$ generate an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $X$ and let $0 < \alpha < 1$. If one takes $x \in X_{\alpha+1}$ and $f \in \mathrm{C}_0(\mathbb{R}_+, X_\alpha)$, then* (iACP) *has a unique classical solution $u \in \mathrm{C}_0^1(\mathbb{R}_+, X_\alpha) \cap \mathrm{C}(\mathbb{R}_+, X_{\alpha+1})$.*

The proof is obtained by taking as solution $u$ the first coordinate of

$$t \mapsto \mathcal{T}_{-1}(t)\binom{x}{f}$$

for $\binom{x}{f} \in D(\mathcal{A})$.

**7.18 Exercise.** Prove Lemma 7.11. (Hint: For $f \in \mathrm{L}^1(\mathbb{R}_+, F_0)$ take an approximating sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{C}_c^1(\mathbb{R}_+, F_0)$ and use arguments analogous to those used in the proof of Corollary III.3.6 (with $Bf_n$ replaced by $f_n$) to prove (i). Assertion (ii) follows from an estimate analogous to (3.18) in Chapter III. Finally, due to the estimate in (ii) it suffices to verify (iii) on the dense subspace $\mathrm{C}_c^1(\mathbb{R}_+, F_0)$.)

## c. *Integro-Differential Equations*

We now return to the situation studied in Section 7.a and assume $(A, D(A))$ to be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the Banach space $X$. We then know from Propositions 7.4 and 7.5 that the operator

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & {}^{d}/_{ds} \end{pmatrix} \qquad \text{with} \qquad D(\mathcal{A}) := D(A) \times \mathrm{W}^{1,1}(\mathbb{R}_+, X)$$

generates the strongly continuous semigroup $\big(\mathcal{T}(t)\big)_{t\geq 0} := \left(\begin{smallmatrix} T(t) & R(t) \\ 0 & S(t) \end{smallmatrix}\right)_{t\geq 0}$ on $\mathcal{X} := X \times \mathrm{L}^1(\mathbb{R}_+, X)$. The operator $\big(\mathcal{A}, D(\mathcal{A})\big)$ will now be perturbed in such a way that it generates a new semigroup $\big(\mathcal{S}(t)\big)_{t\geq 0}$. From this semigroup we will obtain solutions of a *Volterra integro-differential equation*. To this end, we introduce the following operators.

**7.19 Definition.** *Take $a(\cdot) \in \mathrm{W}^{1,1}(\mathbb{R}_+, \mathbb{C})$ and define*

$$\begin{aligned} Bx &:= a(\cdot)Ax && \textit{for} \quad x \in D(A), \\ \mathcal{B}\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) &:= \left(\begin{smallmatrix} 0 \\ Bx \end{smallmatrix}\right) && \textit{for} \quad \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{B}) := D(\mathcal{A}), \\ \mathcal{C}\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) &:= \left(\begin{smallmatrix} Ax + f(0) \\ Bx + f' \end{smallmatrix}\right) && \textit{for} \quad \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{C}) := D(\mathcal{A}). \end{aligned}$$

As before, we will write these operators in matrix form as

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \qquad \text{and} \qquad \mathcal{C} = \begin{pmatrix} A & \delta_0 \\ 0 & {}^{d}/_{ds} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} = \mathcal{A} + \mathcal{B}.$$

Since $\|Bx\|_{\mathrm{W}^{1,1}(\mathbb{R}_+,X)} = \big(\int_0^\infty |a(s)|\, ds + \int_0^\infty |a'(s)|\, ds\big)\, \|Ax\| \leq c\, \|x\|_A$ for $x \in D(A)$, we see that $B$ is bounded from $D(A)$ into $\mathrm{W}^{1,1}(\mathbb{R}_+, X)$ (for the respective graph norms). From this we can immediately make a simple observation that will be the key to the subsequent argument.

**7.20 Lemma.** *The operator $\mathcal{B}$ maps $D(\mathcal{A})$ into $D(\mathcal{A})$ and is bounded with respect to the graph norm.*

We now apply Corollary III.1.5 to the operator $\mathcal{C} = \mathcal{A} + \mathcal{B}$.

**7.21 Proposition.** *The operator $\mathcal{C}$ with domain $D(\mathcal{C}) := D(\mathcal{A})$ generates a strongly continuous semigroup $\big(\mathcal{S}(t)\big)_{t\geq 0}$ on $\mathcal{X}$. For each $x \in D(A)$ and $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, X)$, we have*

$$\mathcal{S}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) = \mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) + \int_0^t \mathcal{T}(t-s)\mathcal{C}\mathcal{S}(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) ds. \tag{7.12}$$

Proof. As noted already above, the generation property follows from Corollary III.1.5. In addition, formula (7.12) is the variation of parameters formula (IE) from Exercise III.1.17.(4). □

Having obtained the semigroup $\big(\mathcal{S}(t)\big)_{t\geq 0}$, we know that its orbits $t \mapsto \mathcal{S}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ are differentiable for $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$. It is again the first coordinate of these orbits that satisfies an interesting equation.

**7.22 Corollary.** *Under the above assumptions and for each $x \in D(A)$ and $f \in \mathrm{W}^{1,1}(\mathbb{R}_+, X)$, there exists a unique (classical) solution $u \in \mathrm{C}^1(\mathbb{R}_+, X) \cap \mathrm{C}\big(\mathbb{R}_+, D(A)\big)$ satisfying the Volterra integro-differential equation*

$$\text{(IDE)} \qquad \begin{cases} \dot{u}(t) = Au(t) + \int_0^t a(t-s)Au(s)\,ds + f(t), \quad t \geq 0, \\ u(0) = x. \end{cases}$$

Proof. For $\binom{x}{f} \in D(\mathcal{A})$ we write $\binom{u(t)}{F(t)} := \mathcal{S}(t)\binom{x}{f} \in X \times \mathrm{L}^1(\mathbb{R}_+, X)$. By general semigroup theory (see Proposition II.6.2), it follows that $u$ and $F$ are continuously differentiable and that $u$ satisfies

$$(7.13) \qquad \dot{u}(t) = Au(t) + F(t)(0) \qquad \text{for } t \geq 0.$$

On the other hand, (7.12) becomes

$$\begin{aligned} &\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} \\ &\quad = \begin{pmatrix} T(t)x + R(t)f \\ S(t)f \end{pmatrix} + \int_0^t \begin{pmatrix} T(t-s) & R(t-s) \\ 0 & S(t-s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} u(s) \\ F(s) \end{pmatrix} ds \\ &\quad = \begin{pmatrix} T(t)x + R(t)f \\ S(t)f \end{pmatrix} + \int_0^t \begin{pmatrix} R(t-s)Bu(s) \\ S(t-s)Bu(s) \end{pmatrix} ds. \end{aligned}$$

In particular,

$$(7.14) \qquad F(t) = S(t)f + \int_0^t a(\cdot + t - s)Au(s)\,ds,$$

whence

$$(7.15) \qquad F(t)(0) = f(t) + \int_0^t a(t-s)Au(s)\,ds \qquad \text{for } t \geq 0.$$

After substituting (7.15) into (7.13), we obtain (IDE). The proof of the uniqueness of the solution $u$ is left as Exercise 7.26.(3). $\square$

Having determined solutions of (iACP) and (IDE), we now attempt to investigate the qualitative behavior of these solutions. However, we will remain within our semigroup setting and study the qualitative behavior of the semigroups $\big(\mathcal{T}(t)\big)_{t\geq 0}$ and $\big(\mathcal{S}(t)\big)_{t\geq 0}$ instead. Due to the results from Chapter V, we know that in order to achieve this goal it is essential to determine the spectra of the generators $\mathcal{A}$ and $\mathcal{C}$. In the case where $\mathcal{A} = \begin{pmatrix} A & \delta_0 \\ 0 & {}^{d}\!/_{ds} \end{pmatrix}$, this is quite simple.

**7.23 Lemma.** *With the above definitions, one has*

$$\sigma(\mathcal{A}) \subset \sigma(A) \cup \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}. \tag{7.16}$$

Proof. Let $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$. Then we have seen in (7.6) that the resolvent of $\mathcal{A}$ is

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & R(\lambda, A)\delta_0 R(\lambda, {}^{d}\!/\!_{ds}) \\ 0 & R(\lambda, {}^{d}\!/\!_{ds}) \end{pmatrix} \in \mathcal{L}(\mathfrak{X}). \tag{7.17}$$

□

From (7.7) we recall that for $\lambda \in \rho(A)$ with $\operatorname{Re}\lambda > 0$, we have

$$R(\lambda, A)\delta_0 R(\lambda, \tfrac{d}{ds})f = R(\lambda, A)\Big(\int_0^\infty \mathrm{e}^{-\lambda\tau} f(\tau)\, d\tau\Big) \qquad \text{for } f \in \mathrm{L}^1(\mathbb{R}_+, X).$$

Since $\mathcal{C}$ is a perturbation of $\mathcal{A}$ by the relatively bounded operator $\mathcal{B}$, we try to obtain the resolvent of $\mathcal{C}$ from the resolvent of $\mathcal{A}$. In fact, since $\mathcal{B} \in \mathcal{L}\big(X_1^{\mathcal{A}}\big)$, we can write

$$\lambda - \mathcal{C} = \lambda - \mathcal{A} - \mathcal{B} = (\lambda - \mathcal{A})[1 - R(\lambda, \mathcal{A})\mathcal{B}]$$

as long as $\lambda \in \rho(\mathcal{A})$. For such $\lambda$, the product on the right-hand side is an invertible operator if and only if its second factor is invertible (in $D(\mathcal{A})$). This simple observation yields the following lemma.

**7.24 Lemma.** *For* $\lambda \in \rho(A)$ *with* $\operatorname{Re}\lambda > 0$, *one has*

$$\lambda \in \sigma(\mathcal{C}) \iff 1 \in \sigma\big(R(\lambda, \mathcal{A})\mathcal{B}\big). \tag{7.18}$$

This equivalence is useful, since we can compute $R(\lambda, \mathcal{A})\mathcal{B}$. In fact, for $\lambda \in \rho(A)$ and $\operatorname{Re}\lambda > 0$, it follows from (7.17) that

$$\begin{aligned} R(\lambda, \mathcal{A})\mathcal{B} &= \begin{pmatrix} R(\lambda, A) & R(\lambda, A)\delta_0 R(\lambda, {}^{d}\!/\!_{ds}) \\ 0 & R(\lambda, {}^{d}\!/\!_{ds}) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \\ &= \begin{pmatrix} R(\lambda, A)\delta_0 R(\lambda, {}^{d}\!/\!_{ds})B & 0 \\ R(\lambda, {}^{d}\!/\!_{ds})B & 0 \end{pmatrix}, \end{aligned}$$

which is a bounded operator on $D(A) \times \mathrm{W}^{1,1}(\mathbb{R}_+, X)$. Observe that this is a triangular, bounded operator matrix on $D(\mathcal{A})$, and its spectrum is the union of the spectra of its diagonal entries. Therefore, (7.18) is equivalent to the condition

$$1 \in \sigma\big(R(\lambda, A)\delta_0 R(\lambda, \tfrac{d}{ds})B\big) \qquad \text{in } D(A). \tag{7.19}$$

This operator, defined on $D(A)$, can be computed using (7.7), and we obtain

$$R(\lambda, A)\delta_0 R(\lambda, \tfrac{d}{ds})Bx = \Big(\int_0^\infty \mathrm{e}^{-\lambda\tau} a(\tau)\, d\tau\Big) R(\lambda, A)Ax$$

for $x \in D(A)$. Since $R(\lambda, A)A \subset AR(\lambda, A) = \lambda R(\lambda, A) - I$, the above operator has a unique continuous extension $Q(\lambda)$ to $X$, and $\big(1 - Q(\lambda)\big)$ is invertible in $X$ if and only if it is invertible in $D(A)$. So, (7.19) becomes

$$1 \in \sigma\Big(\big(\int_0^\infty \mathrm{e}^{-\lambda\tau} a(\tau)\, d\tau\big) AR(\lambda, A)\Big),$$

or

$$\left(\int_0^\infty \mathrm{e}^{-\lambda\tau} a(\tau)\, d\tau\right)^{-1} \in \sigma\big(AR(\lambda, A)\big). \tag{7.20}$$

In the final step, we use the Spectral Mapping Theorem for the Resolvent IV.1.13, and obtain (for unbounded $A$)

$$\sigma\big(AR(\lambda, A)\big) = \Big\{\frac{\mu}{\lambda - \mu} : \mu \in \sigma(A)\Big\} \cup \{-1\}.$$

This yields the following *characteristic equation* for the spectrum of $\mathcal{C}$.

**7.25 Proposition.** *Under the above assumptions and for* $\lambda \in \rho(A)$, $\operatorname{Re}\lambda > 0$, *one has*

$$\lambda \in \sigma(\mathcal{C}) \iff 1 = \int_0^\infty \mathrm{e}^{-\lambda\tau} a(\tau)\, d\tau\, \frac{\mu}{\lambda - \mu} \quad \text{for some} \quad \mu \in \sigma(A) \cup \{\infty\}.$$

At this point we leave it to the reader to, e.g., estimate the spectral bound $\mathrm{s}(\mathcal{C})$ and then draw conclusions, using results from Chapter IV and Chapter V, on the asymptotic behavior of the solutions of (IDE).

**7.26 Exercises.** (1) Show that a function $u \in \mathrm{C}(\mathbb{R}_+, X)$ is a mild solution of (iACP) with $f \in \mathrm{L}^1(\mathbb{R}_+, X)$ if and only if $\int_0^t u(s)\, ds \in D(A)$ and

$$u(t) = x + A\int_0^t u(s)\, ds + \int_0^t f(s)\, ds \qquad \text{for } t \geq 0.$$

Compare this to Definition II.6.3.

(2) Show that in Definition 7.2 and in the subsequent results the space $\mathrm{L}^1(\mathbb{R}_+, X)$ can be replaced by a Banach space $\mathrm{E}(\mathbb{R}_+, X)$ satisfying the following properties.

- (i) $\mathrm{E}(\mathbb{R}_+, X)$ is a subspace of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+, X)$.
- (ii) $\mathrm{E}(\mathbb{R}_+, X)$ is left translation-invariant.
- (iii) $\mathrm{E}(\mathbb{R}_+, X)$ is a space on which the left translation semigroup is strongly continuous.

Find concrete examples of such spaces.

(3) Show that the solution $u$ of (IDE) obtained in Corollary 7.22 is unique. (Hint: Show that for a solution $u$ of (IDE) and $F(t)$ defined by (7.14) the map $t \mapsto \binom{u(t)}{F(t)}$ is a solution of the abstract Cauchy problem with initial value $\binom{x}{f}$ associated to $\mathcal{C}$. Use now Proposition II.6.2 to obtain uniqueness of $u$.)

## *Notes and Further Reading to Section 7*

The existence of mild and classical solutions of (iACP) was already known to Phillips [Phi53] and the results stated in Corollary 7.6 and Corollary 7.8 are proved in most books on semigroup theory (e.g., [Gol85, Sec. II.1] or [Paz83, Sec. 4.2]). The idea to reduce inhomogeneous Cauchy problems and Volterra equations to an abstract Cauchy problem on a product space goes back (at least) to Miller [Mil74]. It has been applied by many authors in various forms (e.g., Chen, Grimmer, Desch–Schappacher; see [CG80], [DS85], [DGS88]), but the use of extrapolation spaces is taken from [NS93]. For more information on Volterra integro-differential equations see the monograph by Prüss [Prü93].

The result stated in Corollary 7.13 is valid also for Hille–Yosida operators and was proved first in [DPS85], [DPS87], and then, with our method, in [NS93]. Example 7.14 is taken from [CHA$^+$87, Exer. 6.19].

The optimal space regularity from Corollary 7.17 is due to [DPG79]. An analogous time regularity result is in [Sin85], from where the estimates from Lemma 7.15 are taken. For a detailed treatment of this subject we refer to [Lun95].

Finally, we refer to [Eng97] for a systematic treatment of (unbounded) operator matrices including the characterization of their spectra as in Lemma 7.24.

# 8. Semigroups for Control Theory

Control theory in infinite-dimensional spaces is a relatively new field and started blooming only after a well-developed semigroup theory was at hand. We therefore present a short introduction to this field and discuss some typical applications to the control of the heat and the wave equations. For more information we refer to the recent monographs [BDPDM93], [CZ95], and [Zab92].

In the following, we will study "controlled" abstract Cauchy problems of the form

$$\text{(cACP)} \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \qquad \text{for } t \geq 0, \\ x(0) = x_0. \end{cases}$$

Here, we assume that the *system operator* $A$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on the *state Banach space* $X$, $B$ is a bounded *control operator* from the *control Banach space* $U$ to $X$, $u : \mathbb{R}_+ \to U$ is a locally integrable *control function* (also called the *input*), $C$ is a bounded *observation operator* from $X$ to the *observation Banach space* $Y$, the function $y : \mathbb{R}_+ \to Y$ is the *observation* (or *output*) of the system, and $x_0 \in X$ is its *initial state*.

We denote the *abstract control system* associated to the controlled Cauchy problem (cACP) by $\Sigma(A, B, C)$. If there is no observation operator $C$ or no control operator $B$, we will write $\Sigma(A, B, -)$ and $\Sigma(A, -, C)$, respectively. Moreover, it will be convenient to write $x(t; x_0, u)$ for the state of the control system $\Sigma(A, B, C)$ at time $t$ for the initial value $x_0$ and the control function $u$.

Given a control system $\Sigma(A, B, C)$, we address the following problems.

**8.1 Problems.** (i) For given states $x_0$, $x_1$ in $X$ and time $t > 0$, find a control $u_0$ such that the system $\Sigma(A, B, C)$ at time $t$ reaches $x_1$, i.e., such that $x(t; x_0, u_0) = x_1$.

(ii) Recover the initial state $x_0$ of (cACP) from the knowledge of the observation $y(\cdot)$ on some time interval $[0, t]$.

(iii) Find a *feedback operator* $F$ from $X$ to $U$ such that (cACP) with *feedback control* $u(\cdot) := Fx(\cdot)$ is stable, i.e., such that $A + BF$ generates a stable semigroup.

These three problems correspond to the concepts of

(i) *controllability*,

(ii) *observability*, and

(iii) *stabilizability*

of $\Sigma(A, B, C)$, respectively, and will first be illustrated by two concrete examples.

**8.2 Example.** (*Heat Equation*). We consider a hot bar of length one that is insulated at its endpoints $s = 0, 1$. We assume that the bar can be heated around some point $s_0 \in (0, 1)$ and that we can measure its average temperature around some other point $s_1 \in (0, 1)$. Denote by $x(s, t)$ the temperature at position $s \in [0, 1]$ and time $t \geq 0$ and by $x_0(\cdot)$ the initial temperature profile. If we now rescale all physical constants to one, then this model can be described by the equations

$$\text{(HE)} \qquad \begin{cases} \dfrac{\partial x(s,t)}{\partial t} = \dfrac{\partial^2 x(s,t)}{\partial s^2} + b(s)u(t) & \text{for } t \geq 0,\ s \in [0,1], \\ \dfrac{\partial x(0,t)}{\partial s} = 0 = \dfrac{\partial x(1,t)}{\partial s} & \text{for } t \geq 0, \\ x(s,0) = x_0(s) & \text{for } s \in [0,1], \\ y(t) = \displaystyle\int_0^1 c(s)x(s,t)\, ds & \text{for } t \geq 0. \end{cases}$$

Here $b$ and $c$ represent the "shaping" functions around the "control point" $s_0$ and the "sensing point" $s_1$, respectively, i.e., we may take

$$\text{(8.1)} \qquad \begin{aligned} b(s) &:= \frac{1}{2\varepsilon_0} \mathbb{1}_{[s_0 - \varepsilon_0, s_0 + \varepsilon_0]}(s), \\ c(s) &:= \frac{1}{2\varepsilon_1} \mathbb{1}_{[s_1 - \varepsilon_1, s_1 + \varepsilon_1]}(s) \end{aligned}$$

for $\varepsilon_0$, $\varepsilon_1 > 0$, where $\mathbb{1}_J$ denotes the characteristic function of a subset $J \subset [0, 1]$.

In order to transform (HE) into an abstract control problem of the form (cACP), we choose the state space $X := \mathrm{L}^2[0,1]$, the control space $U := \mathbb{C}$, and the observation space $Y := \mathbb{C}$, and define the operators

$$
\begin{aligned}
&A := \tfrac{d^2}{ds^2}, && D(A) := \bigl\{x \in \mathrm{H}^2[0,1] : x'(0) = 0 = x'(1)\bigr\},\\
&B \in \mathcal{L}(U,X), && Bu := b(\cdot)u,\\
&C \in \mathcal{L}(X,Y), && Cx := \int_0^1 c(s)x(s)\,ds.
\end{aligned}
\tag{8.2}
$$

Then $A$ is self-adjoint with $\sigma(A) \subset (-\infty, 0]$ (see Exercise II.4.12.(12)), hence generates an analytic semigroup $\bigl(T(t)\bigr)_{t\geq 0}$ on $X$. Moreover, the norm of the state $x(t) := x(\cdot,t) \in X$ can be interpreted as the energy of the system at time $t$.

The questions of controllability, observability and stabilizability raised above read now as follows.

(i) *Is it possible to steer an arbitrary initial temperature profile by a suitable control function $u_0$ (approximately) to a given profile $x_1$, e.g., to $x_1 \equiv 0$?*

(ii) *Is it possible to determine the initial temperature profile $x_0$ by measuring the temperature over some time interval around some point $s_1$?*

(iii) *Is it possible to find a feedback operator $F \in \mathcal{L}(X,U)$ such that the energy of the feedback system governed by $A+BF$ converges to zero as $t \to \infty$?*

**8.3 Example.** (*Wave Equation*). We now consider a vibrating string of length one clipped at the endpoints $s = 0, 1$. We assume that we can apply an external force around some point $s_0 \in (0,1)$ and that we can measure the average displacement around $s_1 \in (0,1)$. Denote by $x(s,t)$ the vertical displacement from the zero stage at position $s \in [0,1]$ and time $t \geq 0$, and by $x_0(\cdot)$, $x_1(\cdot)$ the initial displacement and velocity profiles. Then this model can be described by the equations

$$
\text{(WE)}\qquad
\begin{cases}
\dfrac{\partial^2 x(s,t)}{\partial t^2} = \dfrac{\partial^2 x(s,t)}{\partial s^2} + b(s)u(t) & \text{for } t \geq 0,\ s \in [0,1],\\[2mm]
x(0,t) = 0 = x(1,t) & \text{for } t \geq 0,\\[2mm]
x(s,0) = x_0(s), \quad \dfrac{\partial x(s,0)}{\partial s} = x_1(s) & \text{for } s \in [0,1],\\[2mm]
y(t) = \displaystyle\int_0^1 c(s)x(s,t)\,ds & \text{for } t \geq 0.
\end{cases}
$$

We again rescaled all physical constants to one and took the shaping functions $b$ and $c$ from (8.1) around the control point $s_0$ and the sensing point $s_1$, respectively.

In order to transform (WE) into an abstract control problem of the form (cACP), we first rewrite it as a controlled abstract second-order Cauchy problem of the form

$$(\mathrm{cACP}_2) \qquad \begin{cases} \ddot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \qquad \text{for } t \geq 0, \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases}$$

Here, $A$ denotes the Laplace operator

$$A := \tfrac{d^2}{ds^2}, \qquad D(A) := \bigl\{x \in \mathrm{H}^2[0,1] : x(0) = 0 = x(1)\bigr\}$$

on the space $X := \mathrm{L}^2[0,1]$, $U$ and $Y$ are one-dimensional spaces, and $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$ are defined as in (8.2). Following the ideas of Section 3, we now transform the second-order problem ($\mathrm{cACP}_2$) into a first-order system.

To this end, we first observe that $-A$ is self-adjoint and positive definite on $X$. Hence, there exists a unique positive definite square root $(-A)^{1/2}$ with domain $D\bigl((-A)^{1/2}\bigr) = \mathrm{H}^1_0[0,1] := \bigl\{x \in \mathrm{H}^1[0,1] : x(0) = 0 = x(1)\bigr\}$; see Exercise II.5.36.(4). We then introduce the Hilbert space $\mathcal{X} := D\bigl((-A)^{1/2}\bigr) \times X$ with the inner product

$$\left( \binom{v_1}{v_2} \,\middle|\, \binom{w_1}{w_2} \right) := \bigl((-A)^{1/2} v_1 \,\big|\, (-A)^{1/2} w_1\bigr) + (v_2 \,|\, w_2),$$

where $(\cdot\,|\,\cdot)$ denotes the inner product in $X$. With this notation, ($\mathrm{cACP}_2$) is described by the control system $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C})$ on the state space $\mathcal{X}$ with control space $U$, observation space $Y$, initial value $\mathfrak{x}_0 := \binom{x_0}{x_1}$, and

$$(8.3) \qquad \begin{aligned} \mathcal{A} &:= \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}, & D(\mathcal{A}) &:= D(A) \times D\bigl((-A)^{1/2}\bigr), \\ \mathcal{B} &:= \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathcal{L}(U, \mathcal{X}), & \mathcal{C} &:= (C, 0) \in \mathcal{L}(\mathcal{X}, Y). \end{aligned}$$

It is now easy to verify that $\pm\mathcal{A}$ are dissipative and that $\mathcal{A}$ is invertible. Hence, by the Lumer–Phillips Theorem II.3.15, $\mathcal{A}$ generates a group of contractions, i.e., a unitary group $\bigl(\mathcal{T}(t)\bigr)_{t\in\mathbb{R}}$ on $\mathcal{X}$.

As in the previous example, we can interpret the norm of $\mathfrak{x}(t) \in \mathcal{X}$, i.e., of the solution of the control problem (cACP) associated with $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C})$ at time $t \geq 0$, as the energy of the system at time $t$.

The questions of controllability, observability, and stabilizability raised above now read as follows.

(i) *Is it possible to steer an arbitrary initial displacement/velocity profile by a suitable control $u_0$ (approximately) to a given profile, e.g., to the rest position?*

(ii) *Is it possible to determine the initial profile of the string by measuring the displacement over some time interval around some point $s_1$?*

(iii) *Is it possible to find a feedback operator such that the energy of the resulting feedback system tends to zero?*

The questions (i)–(iii) raised in Examples 8.2 and 8.3 will be the guidelines for the theory we are going to develop now.

## a. Controllability

We first recall from Section 7.a that the solution $x(\cdot)$ of (cACP), if it exists, is given by the variation of parameters formula

$$\begin{aligned} x(t) = x(t;x_0,u) &= T(t)x_0 + \int_0^t T(t-r)Bu(r)\,dr \\ &= x(t;x_0,0) + x(t;0,u), \quad t \geq 0, \end{aligned}$$

cf. Problem 7.1. Since for unbounded $A$ classical solutions of (cACP) remain in $D(A) \neq X$ for all times, we cannot steer the associated control system $\Sigma(A,B,-)$ to every given state $x_1 \in X$. For this reason, we base the concept of controllability on the notion of mild solutions introduced in Definition 7.2. To this end, fix some $p \geq 1$ and define for a control system $\Sigma(A,B,-)$ with state space $X$ and control space $U$ the *controllability map* $\mathcal{B}_t \in \mathcal{L}\big(\mathrm{L}^p([0,t],U),X\big)$ by

$$\mathcal{B}_t[u(\cdot)] := \int_0^t T(t-r)Bu(r)\,dr, \qquad u(\cdot) \in \mathrm{L}^p\big([0,t],U\big),$$

where $\big(T(t)\big)_{t\geq 0}$ denotes the semigroup generated by $A$ on the Banach space $X$.

**8.4 Definition.** *The system* $\Sigma(A,B,-)$ *is called exactly p-controllable on* $[0,t]$ *if* $\operatorname{rg} \mathcal{B}_t = X$.

Hence, $\Sigma(A,B,-)$ is exactly $p$-controllable on $[0,t]$ if every state $x_1$ can be reached from the initial state 0 by some suitable control $u_0 \in \mathrm{L}^p\big([0,t],U\big)$. However, since for given states $x_0$, $x_1 \in X$, we can find a control $u_1 \in \mathrm{L}^p\big([0,t],U\big)$ such that $x(t;0,u_1) = x_1 - x(t;x_0,0)$, i.e., $x(t;x_0,u_1) = x_1$, this is equivalent to the fact that every state $x_1$ can be reached from every initial state $x_0$.

From the following proposition we can see that in infinite dimensions the concept of exact controllability is too strong.

**8.5 Proposition.** *If the control operator* $B \in \mathcal{L}(U,X)$ *for a given control system* $\Sigma(A,B,-)$ *is compact, then the controllability operator* $\mathcal{B}_t \in \mathcal{L}\big(\mathrm{L}^p([0,t],U),X\big)$ *is compact for all* $p \geq 1$ *and* $t > 0$. *In particular, if* $\dim X = \infty$, *then* $\Sigma(A,B,-)$ *is never exactly p-controllable on* $[0,t]$.

PROOF. In order to show that $\mathcal{B}_t$ is compact for all $t > 0$ and $p > 1$, take $n \in \mathbb{N}$ and put $s_k := {}^{kt}\!/_{n}$, $k = 0, 1, \ldots, n$. Next, define operators $K_n \in \mathcal{L}\big(\mathrm{L}^p([0,t],U), X\big)$ by

$$K_n[u(\cdot)] := \sum_{k=1}^{n} T(s_k) B \int_{s_{k-1}}^{s_k} u(s)\, ds, \qquad u(\cdot) \in \mathrm{L}^p\big([0,t],U\big).$$

Since $B$ is compact, $K_n$ is compact as well. We now show that $\lim_{n\to\infty} K_n = K$ with respect to the operator norm, where $K$ is given by

$$K[u(\cdot)] := \int_0^t T(s) B u(s)\, ds, \qquad u(\cdot) \in \mathrm{L}^p\big([0,t],U\big).$$

In fact, by Proposition A.3 there exists for given $\varepsilon > 0$ an integer $n_0 \in \mathbb{N}$ such that $\|[T(s) - T(s_k)]B\| < \varepsilon$ for all $s \in [s_{k-1}, s_k]$, $k = 0, 1, \ldots, n$, whenever $n \geq n_0$. Using this, we conclude that

$$\begin{aligned}
\|K_n u(\cdot) - K u(\cdot)\| &= \Big\| \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} [T(s) - T(s_k)] B u(s)\, ds \Big\| \\
&\leq \sum_{k=1}^{n} \int_{s_{k-1}}^{s_k} \big\|[T(s) - T(s_k)]B\big\| \cdot \|u(s)\|\, ds \\
&\leq \varepsilon \int_0^t \|u(s)\|\, ds \leq \varepsilon t^{1/q} \|u\|_{\mathrm{L}^p} \qquad \text{for all } n \geq n_0
\end{aligned}$$

and for ${}^1\!/_p + {}^1\!/_q = 1$. This shows that the operator $K$ is compact; hence $\mathcal{B}_t$ is compact as well. In particular, if $\mathcal{B}_t$ is surjective, the induced map $\widehat{\mathcal{B}_t} : {}^{\mathrm{L}^p([0,t],U)}\!/_{\ker \mathcal{B}_t} \to X$ is compact and invertible. Since this implies $\dim X < \infty$, a control system $\Sigma(A, B, -)$ with infinite-dimensional state space $X$ is never exactly $p$-controllable for a compact control operator $B$. □

In many applications, e.g., in Examples 8.2 and 8.3, the control space $U$ is finite-dimensional, and the above result implies that these systems will never be exactly controllable. For this reason, we introduce the following weaker concepts.

**8.6 Definition.** *For fixed* $1 \leq p < \infty$, *the system* $\Sigma(A, B, -)$ *is called*

(i) *approximately* $p$*-controllable on* $[0,t]$ *if* $\overline{\operatorname{rg} \mathcal{B}_t} = X$; *it is approximately* $p$*-controllable if* $\overline{\bigcup_{t>0} \operatorname{rg} \mathcal{B}_t} = X$;

(ii) *exactly* $p$*-null controllable on* $[0,t]$ *if* $\operatorname{rg} \mathcal{B}_t \supset \operatorname{rg} T(t)$.

Hence, $\Sigma(A, B, -)$ is approximately $p$-controllable on $[0,t]$ if for all $x_0$, $x_1 \in X$ and all $\varepsilon > 0$ there exists $u_\varepsilon \in \mathrm{L}^p\big([0,t],U\big)$ such that $\|x_1 - x(t; x_0, u_\varepsilon)\| < \varepsilon$. On the other hand, $\Sigma(A, B, -)$ is exactly $p$-null controllable if every initial value can be steered to zero by means of a suitable control function $u$.

We are now interested in conditions characterizing these controllability concepts for a given system $\Sigma(A, B, -)$. The idea is to apply Lemma B.13 to the controllability operator $\mathcal{B}_t$. To do so we calculate its adjoint assuming that $X$ and $U$ are both reflexive. Then, for $1 < p < \infty$, the space $\mathrm{L}^p([0,t],U)$ is reflexive as well, and its dual is $\mathrm{L}^q([0,t],U')$ with $1/p + 1/q = 1$, see [DU77, Chap. IV, Thm.1].

**8.7 Lemma.** *If $X$ and $U$ are reflexive Banach spaces, then the adjoint of the controllability operator $\mathcal{B}_t$ is given by*

$$\mathcal{B}_t' \in \mathcal{L}\Big(X', \mathrm{L}^q\big([0,t],U'\big)\Big), \qquad \mathcal{B}_t' x' := B'T(t-\cdot)'x' \quad \textit{for } x' \in X'.$$

The simple proof of this result is left as Exercise 8.16.(10).

Applying Lemma B.13 to $I$ and $\mathcal{B}_t$, or $T(t)$ and $\mathcal{B}_t$, respectively, we easily obtain the following characterizations of the above controllability conditions.

**8.8 Theorem.**[4] *Let $\Sigma(A, B, -)$ be a control system with reflexive state space $X$ and reflexive control space $U$, and assume that $1 < p < \infty$.*

(i) *The following conditions are equivalent.*

  (a) *$\Sigma(A, B, -)$ is exactly $p$-controllable on $[0,t]$.*

  (b) *There exists $\gamma > 0$ such that*

$$\gamma\, \|B'T'(\cdot)x'\|_{\mathrm{L}^q([0,t],U')} \geq \|x'\|_{X'} \quad \textit{for all } x' \in X',$$

  *where $1/p + 1/q = 1$.*

(ii) *The following conditions are equivalent.*

  (a) *$\Sigma(A, B, -)$ is approximately $p$-controllable on $[0,t]$.*

  (b) *$\bigcap_{s=0}^{t} \ker B'T(s)' = \{0\}$.*

(iii) *The following conditions are equivalent.*

  (a) *$\Sigma(A, B, -)$ is exactly $p$-null controllable on $[0,t]$.*

  (b) *There exists $\gamma > 0$ such that*

$$\gamma\, \|B'T'(\cdot)x'\|_{\mathrm{L}^q([0,t],U')} \geq \|T(t)'x'\|_{X'} \quad \textit{for all } x' \in X',$$

  *where $1/p + 1/q = 1$.*

---

[4] In the case that $X$ is a Hilbert space, this and the following results referring to Banach space adjoints "$'$" are also valid for Hilbert space adjoints "$*$".

**8.9 Example.** (*Heat Equation*). We continue the discussion of the controlled heat equation from Example 8.2. To this end, we first observe that the resolvent of $A$ is compact, and hence $\sigma(A) = P\sigma(A)$. Moreover, there exists an orthonormal basis $\{e_n : n \in \mathbb{N}_0\}$ of eigenvectors of $A$ given by

$$e_n(s) := \begin{cases} 1 & \text{if } n = 0, \\ \sqrt{2}\cos(n\pi s) & \text{if } n \in \mathbb{N}, \end{cases}$$

with corresponding eigenvalues $\lambda_n := -\pi^2 n^2$ for $n \in \mathbb{N}_0$. If we identify $x \in X$ with the sequence $\big((x \,|\, e_n)\big)_{n\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$ of its Fourier coefficients, we can interpret $A$ either as the multiplication operator $M_{(\lambda_n)}$ on $\ell^2(\mathbb{N}_0)$, or, equivalently, as

$$\begin{aligned} A &= \sum_{n=0}^{\infty} \lambda_n (\,\cdot\,|\, e_n)\, e_n, \\ D(A) &= \big\{x \in X : \big(\lambda_n (x \,|\, e_n)\big)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}_0)\big\}. \end{aligned} \tag{8.4}$$

The semigroup $\big(T(t)\big)_{t\geq 0}$ generated by $A$ consists of the multiplication operators $M_{(\mathrm{e}^{\lambda_n t})}$ on $\ell^2(\mathbb{N}_0)$, or

$$T(t) = \sum_{n=0}^{\infty} \mathrm{e}^{\lambda_n t} (\,\cdot\,|\, e_n)\, e_n, \tag{8.5}$$

while the resolvent of $A$, for $\lambda \in \rho(A) = \mathbb{C} \setminus \{\lambda_n : n \in \mathbb{N}_0\}$, is the operator $M_{(\frac{1}{\lambda-\lambda_n})}$ on $\ell^2(\mathbb{N}_0)$, or

$$R(\lambda, A) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} (\,\cdot\,|\, e_n)\, e_n. \tag{8.6}$$

Using these facts, we now show that the controlled heat equation is not exactly 2-controllable, even if we replace the control operator in (8.2) by an arbitrary bounded operator $B$ on some (possibly infinite-dimensional) control space $U$. More generally, we prove the following result.

**Proposition.** *If $A$ is a self-adjoint generator with compact resolvent on an infinite-dimensional Hilbert space $X$, then the control system $\Sigma(A, B, -)$ is not exactly 2-controllable on arbitrary $[0, t]$ for an arbitrary control Hilbert space $U$ and an arbitrary control operator $B \in \mathcal{L}(U, X)$.*

Proof. As above, the assumptions imply that the semigroup generated by $A$ is given by (8.5), where $P\sigma(A) = \{\lambda_n : n \in \mathbb{N}_0\} \subset \mathbb{R}$ and $\{e_n : n \in \mathbb{N}_0\}$ is an orthonormal basis consisting of the corresponding eigenvectors. Moreover, the compactness of $R(\lambda, A)$ implies $\lim_{n\to\infty} \lambda_n = -\infty$ and therefore

$$\lim_{n\to\infty} \|T(t)e_n\| = \lim_{n\to\infty} \|\mathrm{e}^{t\lambda_n} e_n\| = 0 \qquad \text{for all } t > 0.$$

Using the fact that $T(t)$ is self-adjoint, Lebesgue's dominated convergence theorem then implies

$$\lim_{n\to\infty} \int_0^t \|B^*T^*(s)e_n\|^2\, ds = 0.$$

Hence condition (b) from Theorem 8.8.(i) is not be satisfied for arbitrary $\gamma > 0$, independent of $B \in \mathcal{L}(U, X)$ and $t > 0$. □

Despite this negative result, one can show that under the assumptions of the previous proposition the control system $\Sigma(A, B, -)$ is exactly 2-null controllable, provided that $B \in \mathcal{L}(U, X)$ is surjective; cf. Exercise 8.16.(5).

**8.10 Example.** (*Wave Equation*). For the controlled wave equation from Example 8.3 the situation is quite different. In fact, we have the following general result.

**Proposition.** *Let $A$ be a self-adjoint, negative definite operator on a Hilbert space $X$ with compact resolvent. If the control operator $B \in \mathcal{L}(U, X)$ defined on the Hilbert space $U$ is surjective, then the control system $\Sigma(\mathcal{A}, \mathcal{B}, -)$ defined in* (8.3) *is exactly* 2*-controllable on arbitrary* $[0, t]$.

Proof. Using the notation from Example 8.3, we first observe that the operator

$$\mathcal{S} := \begin{pmatrix} (-A)^{-1/2} & \mathrm{i}(-A)^{-1/2} \\ \mathrm{i}I & I \end{pmatrix}$$

from the Hilbert space $\widetilde{\mathcal{X}} := X \times X$ to $\mathcal{X} := D\big((-A)^{1/2}\big) \times X$ is bounded and invertible with bounded inverse

$$\mathcal{S}^{-1} := \frac{1}{2} \begin{pmatrix} (-A)^{1/2} & -\mathrm{i}I \\ -\mathrm{i}(-A)^{1/2} & I \end{pmatrix}.$$

By Exercise 8.16.(7), it therefore suffices to show that the similar system $\Sigma(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, -)$ on the state space $\widetilde{\mathcal{X}}$ is exactly 2-controllable on $[0, t]$, where

$$\widetilde{\mathcal{A}} := \mathcal{S}^{-1}\mathcal{A}\mathcal{S} = \begin{pmatrix} \mathrm{i}(-A)^{1/2} & 0 \\ 0 & -\mathrm{i}(-A)^{1/2} \end{pmatrix} \quad \text{with domain}$$

$$D(\widetilde{\mathcal{A}}) := D\big((-A)^{1/2}\big) \times D\big((-A)^{1/2}\big), \quad \text{and}$$

$$\widetilde{\mathcal{B}} := \mathcal{S}^{-1}\mathcal{B} = \begin{pmatrix} -\mathrm{i}B \\ B \end{pmatrix} \in \mathcal{L}(U, \widetilde{\mathcal{X}}).$$

In order to do so, we use Theorem 8.8 and calculate the unitary group $\big(\widetilde{\mathcal{T}}(t)\big)_{t\geq 0}$ generated by $\widetilde{\mathcal{A}}$ as

$$\widetilde{\mathcal{T}}(t) = \begin{pmatrix} U(t) & 0 \\ 0 & U(t)^* \end{pmatrix}.$$

Using the same arguments as in Example 8.9, one can show that $\big(U(t)\big)_{t\in\mathbb{R}}$ is given by

$$U(t) = \sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}\mu_n t}(\,\cdot\,|\,e_n)\,e_n \tag{8.7}$$

for the eigenvalues $\mu_n > 0$ of $(-A)^{1/2}$ and the corresponding eigenvectors $e_n$ forming an orthonormal basis of $X$. Now $\widetilde{\mathcal{B}}^* = (\mathrm{i}B^*, B^*) \in \mathcal{L}(\widetilde{\mathfrak{X}}, U)$, and since $B$ is surjective, by Lemma B.13.(ii) there exists a constant $\widetilde{\gamma} > 0$ such that

$$\begin{aligned}\left\|\widetilde{\mathcal{B}}^*\widetilde{\mathfrak{T}}(s)^*\tbinom{x}{y}\right\|^2 &= \big\|B^*\big(\mathrm{i}U^*(s)x + U(s)y\big)\big\|^2\\ &\geq \widetilde{\gamma}\cdot\big\|\mathrm{i}U^*(s)x + U(s)y\big\|^2\\ &= \widetilde{\gamma}\cdot\Big[\|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\big(U(2s)y|\mathrm{i}x\big)\Big] \quad \text{for all } x, y \in X.\end{aligned}$$

Next, we calculate

$$\begin{aligned}J(t) := \int_0^t U(2s)\,ds &= \int_0^t \sum_{n=1}^{\infty} \mathrm{e}^{2\mathrm{i}\mu_n s}(\,\cdot\,|\,e_n)\,e_n\,ds\\ &= t\sum_{n=1}^{\infty}\frac{\mathrm{e}^{2\mathrm{i}\mu_n t} - 1}{2\mathrm{i}\mu_n t}(\,\cdot\,|\,e_n)\,e_n;\end{aligned}$$

hence

$$\|J(t)\| \leq t\cdot\sup_{r\geq r_0}\left|\frac{\mathrm{e}^{\mathrm{i}r} - 1}{r}\right|$$

for $r_0 := \inf\{2\mu_n t : n \in \mathbb{N}\} > 0$. However,

$$\left|\frac{\mathrm{e}^{\mathrm{i}r} - 1}{r}\right|^2 = \frac{\sin^2(r/2)}{(r/2)^2} \leq \sup_{s\geq r_0}\frac{\sin^2(s/2)}{(s/2)^2} =: \delta < 1 \quad \text{for all } r \geq r_0,$$

and we finally obtain

$$\begin{aligned}\int_0^t \left\|\widetilde{\mathcal{B}}^*\widetilde{\mathfrak{T}}(s)^*\tbinom{x}{y}\right\|^2 ds &\geq \widetilde{\gamma}\Big[t\left(\|x\|^2 + \|y\|^2\right) - 2\delta t\|x\|\cdot\|y\|\Big]\\ &= \widetilde{\gamma}t(1-\delta)\left(\|x\|^2 + \|y\|^2\right) + \widetilde{\gamma}t\delta\left(\|x\|^2 - 2\|x\|\cdot\|y\| + \|y\|^2\right)\\ &\geq \widetilde{\gamma}t(1-\delta)\cdot\left\|\tbinom{x}{y}\right\|^2 \qquad \text{for all } \tbinom{x}{y} \in \widetilde{\mathfrak{X}}.\end{aligned}$$

The assertion now follows from Theorem 8.8.(i) with $\gamma := \frac{1}{\widetilde{\gamma}t(1-\delta)} > 0$. $\square$

While in the previous examples it was possible to verify the conditions of Theorem 8.8, there are many cases in which the semigroup $\big(T(t)\big)_{t\geq 0}$ governing the control system $\Sigma(A,B,-)$ is not known explicitly. For this reason, it is desirable to have characterizations of the above controllability concepts involving the resolvent operators $R(\lambda,A)$ or, even better, the generator $A$ instead. Before we proceed, recall from Section IV.2.b that $\rho_+(A)$ denotes the connected component of $\rho(A)$ that is unbounded to the right. Moreover, by Corollary B.12, we have $\rho(A)=\rho(A')$ and therefore $\rho_+(A)=\rho_+(A')$ as well.

**8.11 Corollary.** *Let $\Sigma(A,B,-)$ be a control system with reflexive state space $X$ and reflexive control space $U$. Then, for every $1<p<\infty$, the following conditions are equivalent.*

(a) *$\Sigma(A,B,-)$ is approximately $p$-controllable.*

(b) $\bigcap_{t\geq 0}\ker B'T(t)'=\{0\}$.

(c) $\bigcap_{n\geq 0}\ker B'R(\lambda_0,A')^n=\{0\}$ *for some/all* $\lambda_0\in\rho_+(A)$.

(d) $\bigcap_{\lambda\in\Lambda}\ker B'R(\lambda,A')=\{0\}$ *for some/all subsets* $\Lambda\subseteq\rho_+(A)$ *having an accumulation point in* $\rho_+(A)$.

Proof. The equivalence of (a) and (b) follows as in Theorem 8.8.(ii).

To verify that (b) ⇒ (c), we assume that $x'\in\bigcap_{n\geq 0}\ker B'R(\lambda_0,A')^n$ for some $\lambda_0\in\rho_+(A)$. Then the function

$$f_{x'}:\rho_+(A)\to X',\qquad f_{x'}(\lambda):=B'R(\lambda,A')x' \tag{8.8}$$

is analytic and satisfies $f_{x'}^{(n)}(\lambda_0)=(-1)^n n!B'R(\lambda_0,A')^{n+1}x'=0$ for all $n\in\mathbb{N}_0$. We conclude that $f_{x'}\equiv 0$ and therefore $f_{x'}^{(n)}(\lambda)=0$ for all $n\in\mathbb{N}_0$ and all $\lambda\in\rho_+(A)$. This proves $x'\in\bigcap_{n\geq 0}\ker B'R(\lambda,A')^n$ for all $\lambda\in\rho_+(A)$. The Post–Widder inversion formula in Corollary III.5.5 implies

$$B'T(t)'x'=\lim_{n\to\infty}B'\left({}^t\!/\!_nR({}^n\!/\!_t,A')\right)^n x'=0\qquad\text{for all } t>0.$$

By assumption, this is possible only for $x'=0$, which proves (c).

To show that (c) ⇒ (d), we take $x'\in\bigcap_{\lambda\in\Lambda}\ker B'R(\lambda,A')$. Then the function $f_{x'}$ in (8.8) restricted to $\Lambda$ is zero, hence, by analyticity, zero on $\rho_+(A)$. This shows that $x'\in\bigcap_{n\geq 1}\ker B'R(\lambda_0,A')^n$ for arbitrary $\lambda_0\in\rho_+(A)$. Moreover, by Lemma II.3.4.(i) we have $B'x'=\lim_{\lambda\to\infty}\lambda f_{x'}(\lambda)=0$. Hence, $x'\in\bigcap_{n\geq 0}\ker B'R(\lambda_0,A')^n=\{0\}$ and therefore $x'=0$.

Finally, we prove (d) ⇒ (b). To this end, we choose $x'\in\bigcap_{t\geq 0}\ker B'T(t)'$ and obtain from the integral representation of the resolvent in part (i) of Theorem II.1.10 that

$$B'R(\lambda,A')x'=\int_0^\infty \mathrm{e}^{-\lambda s}B'T(s)'x'\,ds=0\qquad\text{for all }\lambda>\omega_0\,.$$

Hence, (d) is satisfied for $\Lambda=(\omega_0,\infty)$, and we conclude that $x'=0$. □

In particular, the previous result implies that the notion of approximate $p$-controllability is, in fact, independent of the value of $p \in (1,\infty)$.

If the semigroup $(T(t))_{t\geq 0}$ governing the control system $\Sigma(A,B,-)$ has some additional properties, we can strengthen the conclusion on approximate controllability.

**8.12 Corollary.** *Let $\Sigma := \Sigma(A,B,-)$ be an approximate $p$-controllable system with reflexive state and control space.*

(i) *If the semigroup $(T(t))_{t\geq 0}$ generated by $A$ is analytic, then $\Sigma$ is approximately $p$-controllable on each $[0,t]$.*

(ii) *If the semigroup $(T(t))_{t\geq 0}$ generated by $A$ is periodic with period $\tau$, then $\Sigma$ is approximately $p$-controllable on $[0,t]$ for all $t \geq \tau$.*

PROOF. (i) Assume that $x' \in \cap_{s=0}^{t} \ker B'T(s)'$. Then the function $g_{x'} : \mathbb{R}_+ \to X'$, $g_{x'}(s) := B'T(s)'x'$ can be extended analytically to a sector containing $\mathbb{R}_+$ and vanishes on $[0,t]$. Hence, by analyticity, $g_{x'} \equiv 0$, and we conclude by Corollary 8.11.(b) that $x' = 0$. The assertion then follows from Theorem 8.8.(ii).

(ii) follows immediately from Theorem 8.8.(ii) and Corollary 8.11.(b) if one observes that $\cap_{t\geq 0} \ker B'T(t)' = \cap_{s=0}^{t} \ker B'T(s)'$ for all $t \geq \tau$. □

We consider next a rather special, but quite important, case. In fact, we will assume that the state space $X$ of $\Sigma(A,B,-)$ is a Hilbert space and that the generator $A$ is given by the "multiplication" operator

$$A = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{r_n} (\,\cdot\,|\, e_{n,k})\, e_{n,k}. \tag{8.9}$$

Here $\lambda_n$ are the distinct eigenvalues of $A$, which we assume to be isolated in $\mathbb{C}$, and $\{e_{n,k} : n \in \mathbb{N},\ k = 1,\dots,r_n\}$ is an orthonormal basis of $X$ consisting of the corresponding eigenvectors. Moreover, we assume the control space $U$ to be finite-dimensional. Under these assumptions, we arrive at the following simple criterion.

**8.13 Corollary.** *Let $\Sigma(A,B,-)$ be a control system on a Hilbert space $X$, where $A$ is given by (8.9). If $U := \mathbb{C}^m$ and $B := (b_1,\dots,b_m) \in \mathcal{L}(U,X)$ for some $b_1,\dots,b_m \in X$, then $\Sigma(A,B,-)$ is approximately 2-controllable if and only if* $\operatorname{rank} B_n = r_n$ *for all $n \in \mathbb{N}$, where*

$$B_n := \begin{pmatrix} (e_{n,1}\,|\,b_1) & \cdots & (e_{n,r_n}\,|\,b_1) \\ \vdots & & \vdots \\ (e_{n,1}\,|\,b_m) & \cdots & (e_{n,r_n}\,|\,b_m) \end{pmatrix}_{m\times r_n}.$$

PROOF. We are going to apply Corollary 8.11 and therefore start by calculating the (Hilbert) adjoints[5]

$$A^* = \sum_{n=1}^{\infty} \bar{\lambda}_n \sum_{k=1}^{r_n} (\,\cdot\,|\,e_{n,k})\, e_{n,k}, \qquad D(A^*) = D(A),$$
$$B^* = \big((\,\cdot\,|\,b_1), \dots, (\,\cdot\,|\,b_m)\big)^t \in \mathcal{L}(X, \mathbb{C}^m).$$

Using these representations, we obtain (cf. (8.6))

$$\begin{aligned} B^* R(\lambda, A^*)x &= B^* \sum_{n=1}^{\infty} \frac{1}{\lambda - \bar{\lambda}_n} \sum_{k=1}^{r_n} (x\,|\,e_{n,k})\, e_{n,k} \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda - \bar{\lambda}_n}\, B_n \Phi_n(x) \end{aligned} \tag{8.10}$$

for all $\lambda \in \rho(A^*)$ and $x \in X$, where

$$\Phi_n(x) := \big((x\,|\,e_{n,1}), \dots, (x\,|\,e_{n,r_n})\big)^t \in \mathbb{C}^{r_n}.$$

By assumption, the set $\{\lambda_n : n \in \mathbb{N}\}$ consists of isolated points only; hence we can consider in (8.10) the limit $\rho_+(A^*) \ni \lambda \to \bar{\lambda}_n$ for fixed $n \in \mathbb{N}$ and infer that

$$\left.\begin{array}{l} B^* R(\lambda, A^*)x = 0 \\ \text{for all } \lambda \in \rho_+(A^*) \end{array}\right\} \iff \left\{\begin{array}{l} B_n \Phi_n(x) = 0 \\ \text{for all } n \in \mathbb{N}. \end{array}\right. \tag{8.11}$$

On the other hand, the orthonormal system $\{e_{n,k} : n \in \mathbb{N},\ 1 \le k \le r_n\}$ is complete in $X$, and therefore

$$x = 0 \iff \Phi_n(x) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{8.12}$$

Finally, for $B_n : \mathbb{C}^{r_n} \to \mathbb{C}^m$ we have $r_n = \dim \ker B_n + \operatorname{rank} B_n$. This shows that

$$\operatorname{rank} B_n < r_n \iff \ker B_n \ne \{0\}. \tag{8.13}$$

After these preparations, assume that $\Sigma(A, B, -)$ is not approximately 2-controllable and take, using Corollary 8.11, $0 \ne x \in X$ with $B^* R(\lambda, A^*)x = 0$ for all $\lambda \in \rho_+(A^*)$. Then by (8.12) we can choose $l \in \mathbb{N}$ such that $\Phi_l(x) \ne 0$. However, from (8.11) we know that $\Phi_l(x) \in \ker B_l$, and therefore $\operatorname{rank} B_l < r_l$ by (8.13).

Conversely, if $\operatorname{rank} B_l < r_l$ for some $l \in \mathbb{N}$, we find, by (8.13), a vector $0 \ne (\beta_1, \dots, \beta_{r_l})^t \in \ker B_l$. For $x := \sum_{k=1}^{r_l} \beta_k e_{l_k} \ne 0$, this implies

$$B_n \Phi_n(x) = \begin{cases} B_n 0 = 0 & \text{if } n \ne l, \\ B_l(\beta_1, \dots, \beta_{r_l})^t = 0 & \text{if } n = l. \end{cases}$$

From (8.11) and Corollary 8.11, we conclude that $\Sigma(A, B, -)$ is not 2-approximately controllable. □

[5] Here we denote by $(\beta_1, \dots, \beta_m)^t$ the transpose of the vector $(\beta_1, \dots, \beta_m)$.

In particular, this result implies that the number of controls necessary for approximate 2-controllability has to be at least equal to the highest multiplicity of the eigenvalues of $A$.

We are now in a position to examine the approximate 2-controllability of the examples above.

**8.14 Example.** (*Heat Equation*). We continue the discussion of the controlled heat equation from Examples 8.2 and 8.9. Recall from Corollary II.4.7 that a self-adjoint generator always generates an analytic semigroup. Hence, we obtain from Corollary 8.13 and Corollary 8.12.(i) that $\Sigma(A, B, -)$ for $B \in \mathcal{L}(\mathbb{C}, X)$ defined by (8.2) is approximately 2-controllable (on $[0, t]$ for some/all $t > 0$) if and only if we choose $\varepsilon_0$ and $s_0$ such that $0 \neq (e_0 \,|\, b) = 1$ and

$$\begin{aligned} 0 \neq (e_n \,|\, b) &= \frac{\sqrt{2}}{2\varepsilon_0} \int_{s_0-\varepsilon_0}^{s_0+\varepsilon_0} \cos(n\pi s)\, ds \\ &= \sqrt{2}\, \frac{\sin\bigl(n\pi(s_0+\varepsilon_0)\bigr) - \sin\bigl(n\pi(s_0-\varepsilon_0)\bigr)}{2n\pi\varepsilon_0} \\ &= \sqrt{2}\, \frac{\cos(n\pi s_0)\sin(n\pi\varepsilon_0)}{n\pi\varepsilon_0} \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

To have approximate 2-controllability it is therefore important not to place the control in a zero of an eigenvector $e_n$ of $A$. Moreover, it is interesting to observe that for the limit case $\varepsilon_0 \downarrow 0$ of a point control in $s = s_0$, we obtain (formally) approximate 2-controllability if $\cos(n\pi s_0) \neq 0$ for all $n \in \mathbb{N}$.

**8.15 Example.** (*Wave Equation*). In the same manner, we can treat the controlled wave equation from Examples 8.3 and 8.10 with a control operator $B \in \mathcal{L}(\mathbb{C}, X)$ of the form (8.2). Here, the system operator $\mathcal{A}$ of the associated control system $\Sigma(\mathcal{A}, \mathcal{B}, -)$ generates a 2-periodic group (use (8.7) for $\mu_n = n\pi$) and the eigenvectors $e_n$ of $\mathcal{A}$ are given by

$$e_n(s) = \frac{1}{in\pi} \begin{pmatrix} \sin(n\pi s) \\ in\pi \sin(n\pi s) \end{pmatrix} \qquad \text{for all } n \in \mathbb{Z} \setminus \{0\}. \tag{8.14}$$

Hence, from Corollary 8.13 and Corollary 8.12.(i), we obtain that $\Sigma(\mathcal{A}, \mathcal{B}, -)$ is approximately 2-controllable (on $[0, t]$ for some/all $t \geq 2$) if and only if

$$\frac{\sin(n\pi s_0)\sin(n\pi\varepsilon_0)}{n\pi\varepsilon_0} \neq 0 \qquad \text{for all } n \in \mathbb{N}.$$

As in the previous example, it is necessary for approximate 2-controllability not to place the control point $s_0$ in a zero of an eigenmode of $\mathcal{A}$. Moreover, for the limit case of a point control in $s_0$, we obtain (formally) approximate 2-controllability if $\sin(n\pi s_0) \neq 0$ for all $n \in \mathbb{N}$.

**8.16 Exercises.** (1) Show that for all $F \in \mathcal{L}(X,U)$, $\mu \in \mathbb{C}$, and $p \geq 1$ the control system $\Sigma(A,B,-)$ is exactly (approximately) $p$-controllable on $[0,t]$ if and only if $\Sigma(A+BF+\mu,B,-)$ is exactly (approximately) $p$-controllable on $[0,t]$.

(2) If $\Sigma(A,B,-)$ is a control system on the state space $X$ of finite dimension $n$, then $\operatorname{rg}\mathcal{B}_t = \operatorname{rg}(B,AB,\dots,A^{n-1}B)$ for all $t>0$ and $p \geq 1$.

(3) If $\dim X = n < \infty$, then for the system $\Sigma := \Sigma(A,B,-)$ the following conditions are equivalent.

(a) $\Sigma$ is exactly/approximately $p$-controllable on $[0,t]$ for some/all $p \geq 1$, $t > 0$.

(b) $\operatorname{rank}(B,AB,\dots,A^{n-1}B) = n$.

(c) $\operatorname{rg}(\lambda - A, B) = X$ for all $\lambda \in \mathbb{C}$ or, equivalently, for all $\lambda \in \sigma(A)$.

(4) Show that the closure of the *reachability space* $\mathcal{R} := \bigcup_{t\geq 0} \operatorname{rg}\mathcal{B}_t$ of a control system $\Sigma(A,B,-)$ is the smallest closed, $(T(t))_{t\geq 0}$-invariant subspace of $X$ containing $\operatorname{rg} B$.

(5) Show that the heat equation in Example 8.2 with a surjective control operator $B \in \mathcal{L}(U,X)$ on a Hilbert space $U$ is exactly 2-null controllable on $[0,t]$ for all $t > 0$.

(6) The propositions in Example 8.10 and, for $A$ unbounded, in Example 8.9 remain true without the assumption that $A$ has compact resolvent. (Hint: Use Theorem I.4.9.)

(7) Let $S \in \mathcal{L}(\widetilde{X},X)$ be an invertible operator between two Banach spaces $X$ and $\widetilde{X}$. Then for every generator $A$ on $X$ and every control operator $B \in \mathcal{L}(U,X)$ on some control space $U$, the system $\Sigma(A,B,-)$ is exactly $p$-controllable if and only if the *similar control system* $\Sigma(\widetilde{A},\widetilde{B})$ on the state space $\widetilde{X}$ is exactly $p$-controllable, where $\widetilde{A} := S^{-1}AS$ and $\widetilde{B} := S^{-1}B$. Corresponding assertions are valid for approximate $p$-controllability and exact $p$-null controllability, respectively.

(8) Show that a control system $\Sigma(A,B,-)$ on an infinite-dimensional state space $X$ governed by an immediately compact semigroup $(T(t))_{t\geq 0}$ is never exactly $p$-controllable. (Hint: Consider the operators $T(\varepsilon)\mathcal{B}_t$ for $\varepsilon \downarrow 0$ in order to show that $\mathcal{B}_t$ is compact for all $t > 0$.)

(9) Show that for every $p \in (1,\infty)$ the system $\Sigma(A,B,-)$ with reflexive state and control spaces is approximately $p$-controllable if and only if the "bounded" control system $\Sigma(R(\lambda_0,A),B,-)$ is approximately $p$-controllable for some/all $\lambda_0 \in \rho_+(A)$. (Hint: Use Corollary 8.11.)

(10) Prove Lemma 8.7.

(11) Let $A$ be the generator of the left translation semigroup on $X := \mathrm{L}^p(\mathbb{R}_+)$, $1 < p < \infty$ (cf. Paragraph II.2.10), and take $U := X$ and $B := M_{\mathbb{1}_{[t_0,\infty)}}$ for some $t_0 > 0$. Show that $\Sigma := \Sigma(A,B,-)$ is exactly $p$-controllable on $[0,t]$ if and only if $t > t_0$. For which values of $t > 0$ is $\Sigma$ approximately $p$-controllable?

## *b. Observability*

We now turn to Problem 8.1.(ii), i.e., the observability of a control system. For fixed $q \geq 1$, $t > 0$ and a given control system $\Sigma(A,-,C)$ with state space $X$ and observation space $Y$, we introduce the *observability map*

$$\mathcal{C}_t : X \to \mathrm{L}^q\big([0,t],Y\big), \qquad \mathcal{C}_t x := CT(\cdot)x \quad \text{for } x \in X.$$

**8.17 Definition.** *The control system* $\Sigma(A,-,C)$ *is exactly* $q$*-observable on* $[0,t]$ *if there exists* $\gamma > 0$ *such that*

$$\gamma \, \|\mathcal{C}_t x\| \geq \|x\| \qquad \text{for all } x \in X.$$

*If* $\ker \mathcal{C}_t = \{0\}$*, then* $\Sigma(A,-,C)$ *is called approximately* $q$*-observable on* $[0,t]$*. Finally, if* $\bigcap_{t>0} \ker \mathcal{C}_t = \{0\}$*, then* $\Sigma(A,-,C)$ *is called approximately* $q$*-observable.*

Hence, $\Sigma(A,-,C)$ is approximately $q$-observable if the knowledge of the output $y(\cdot)$ uniquely determines the initial state. Moreover, $\Sigma(A,-,C)$ is exactly $q$-observable if, in addition, the operator mapping the output to the initial state is continuous.

Next, we relate the observability of a control system $\Sigma(A,-,C)$ with reflexive state space $X$ and reflexive observation space $Y$ to the controllability of the dual system

$$\Sigma'(A,-,C) := \Sigma(A',C',-) \tag{8.15}$$

on the state space $X'$ and with control space $Y'$. In fact, the following result follows immediately from Theorem 8.8 and the definition of observability.

**8.18 Theorem.** *Let* $X$, $Y$ *be reflexive Banach spaces and let* $t > 0$, $q > 1$*. Then for* $p > 1$ *and* ${}^1/_p + {}^1/_q = 1$ *the following assertions are true.*

(i) $\Sigma(A,-,C)$ *is exactly* $q$*-observable on* $[0,t]$ *if and only if* $\Sigma'(A,-,C)$ *is exactly* $p$*-controllable on* $[0,t]$*.*

(ii) $\Sigma(A,-,C)$ *is approximately* $q$*-observable on* $[0,t]$ *if and only if the system* $\Sigma'(A,-,C)$ *is approximately* $p$*-controllable on* $[0,t]$*.*

(iii) $\Sigma(A,-,C)$ *is approximately* $q$*-observable if and only if* $\Sigma'(A,-,C)$ *is approximately* $p$*-controllable.*

As an immediate consequence of this result, we obtain from the controllability characterizations in Section 8.a necessary and sufficient criteria for the observability of a control system $\Sigma(A,-,C)$. We state only the result corresponding to Corollary 8.13 and leave the reformulation of Corollary 8.11 and Corollary 8.12 as Exercise 8.22.(3).

**8.19 Corollary.** *Let* $\Sigma(A,-,C)$ *be a control system on a Hilbert space* $X$ *with* $A$ *given by* (8.9)*. If* $Y = \mathbb{C}^l$ *and* $C = \big((\cdot\,|\,c_1),\dots,(\cdot\,|\,c_l)\big)$ *for some* $c_1,\dots,c_l \in X$*, then* $\Sigma(A,-,C)$ *is approximately* 2*-observable if and only if* $\operatorname{rank} C_n = r_n$*, where*

$$C_n := \begin{pmatrix} (e_{n,1}\,|\,c_1) & \cdots & (e_{n,r_n}\,|\,c_1) \\ \vdots & & \vdots \\ (e_{n,1}\,|\,c_l) & \cdots & (e_{n,r_n}\,|\,c_l) \end{pmatrix}_{l\times r_n}.$$

With this result it is easy to characterize the approximate 2-observability of the controlled heat and wave equations, respectively.

**8.20 Example.** (*Heat Equation*). By the same calculations as in Example 8.14, we obtain from Corollary 8.19 that the controlled heat equation is approximately 2-observable (on $[0,t]$ for some/all $t>0$) if and only if

$$\frac{\cos(n\pi s_1)\sin(n\pi\varepsilon_1)}{n\pi\varepsilon_1}\neq 0 \qquad \text{for all } n\in\mathbb{N}.$$

Again, the remark in Example 8.14 for the limit case $\varepsilon_1\downarrow 0$ applies, and one should not place the observation point $s_1$ into a zero of some eigenfunction of $A$.

**8.21 Example.** (*Wave Equation*). For the controlled wave equation, we obtain from Corollary 8.19 by essentially the same calculations as in Example 8.15 that it is approximately 2-observable (on $[0,t]$ for some/all $t\geq 2$) if and only if

$$\frac{\sin(n\pi s_1)\sin(n\pi\varepsilon_1)}{n\pi\varepsilon_1}\neq 0 \qquad \text{for all } n\in\mathbb{N}.$$

As in Example 8.15, this shows that placing the observation in a zero of an eigenfunction will result in a loss of approximate 2-observability.

**8.22 Exercises.** (1) If $\dim X=n<\infty$, then for the system $\Sigma:=\Sigma(A,-,C)$ the following conditions are equivalent.

(a) $\Sigma$ is exactly/approximately $q$-observable on $[0,t]$ for some/all $q\geq 1$, $t>0$.
(b) $\operatorname{rank}(C',A'C',\dots,(A')^{n-1}C')=n$.
(c) $\ker\binom{\lambda-A}{C}=\{0\}$ for all $\lambda\in\mathbb{C}$/for all $\lambda\in\sigma(A)$.

(2) Show that the *nonobservable subspace* $\mathcal{N}:=\bigcap_{t\geq 0}\ker\mathcal{C}_t$ of a control system $\Sigma(A,-,C)$ is the largest closed, $(T(t))_{t\geq 0}$-invariant subspace of $X$ contained in the kernel $\ker C$.

(3) Give a characterization of approximate 2-observability of a control system $\Sigma(A,-,C)$ by combining Corollary 8.11 and Theorem 8.18. How can this result be improved by means of Corollary 8.12?

## c. Stabilizability and Detectability

While in control problem (i) we search for controls $u(\cdot)$ steering the initial value $x_0$ towards a given state $x_1$, in many applications one is interested only in the design of a feedback control such that the resulting controlled system is asymptotically stable in the following sense; cf. Definition V.1.1.

**8.23 Definition.** *The control system $\Sigma(A,B,-)$ with state space $X$, control space $U$, and control operator $B\in\mathcal{L}(U,X)$ is called $\beta$-exponentially stabilizable for some $\beta\in\mathbb{R}$ if there exists a feedback operator $F\in\mathcal{L}(X,U)$ such that the growth bound $\omega_0(A+BF)$ is less than $\beta$. Moreover, if $\Sigma(A,B,-)$ is 0-exponentially stabilizable, then it is called exponentially stabilizable.*

It can be shown, see Exercise 8.30.(3), that exact null controllability implies exponential stabilizability, while approximate controllability is not sufficient to obtain this conclusion; see Exercise 8.30.(4).

Using the spectral decomposition from Proposition IV.1.16, we now characterize the stabilizability of a control system $\Sigma(A, B, -)$ in the following way. For $\beta \in \mathbb{R}$, we define

$$\begin{aligned} \mathbb{C}_\beta^- &:= \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < \beta\}, &\quad \sigma_\beta^-(A) &:= \sigma(A) \cap \mathbb{C}_\beta^-, \\ \mathbb{C}_\beta^+ &:= \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \beta\}, &\quad \sigma_\beta^+(A) &:= \sigma(A) \cap \overline{\mathbb{C}_\beta^+}. \end{aligned}$$

Then, if $\sigma_\beta^-(A)$ is closed and $\sigma_\beta^+(A)$ is bounded, we can perform the spectral decomposition from Proposition IV.1.16. This yields the spectral projections $P_\beta^+$ and $P_\beta^- = I - P_\beta^+ \in \mathcal{L}(X)$ such that

$$\begin{aligned} X = X_\beta^- \oplus X_\beta^+ &= \operatorname{rg} P_\beta^- \oplus \operatorname{rg} P_\beta^+, & D(A) &= D(A_\beta^-) \oplus X_\beta^+, \\ A_\beta^- &= A_{|X_\beta^-}, & A_\beta^+ &= A_{|X_\beta^+} \in \mathcal{L}(X_\beta^+), \\ T_\beta^-(t) &= T(t)_{|X_\beta^-} \in \mathcal{L}(X_\beta^-), & T_\beta^+(t) &= T(t)_{|X_\beta^+} \in \mathcal{L}(X_\beta^+). \end{aligned}$$

Here, $A_\beta^-$ and $A_\beta^+$ are the generators of $\big(T_\beta^-(t)\big)_{t\ge 0}$ and $\big(T_\beta^+(t)\big)_{t\ge 0}$, respectively (see Proposition II.2.3), satisfying

$$\sigma(A_\beta^-) = \sigma_\beta^-(A), \qquad \sigma(A_\beta^+) = \sigma_\beta^+(A).$$

Moreover, we can write $B \in \mathcal{L}(U, X)$ as

$$B = \begin{pmatrix} B_\beta^- \\ B_\beta^+ \end{pmatrix} \in \mathcal{L}(U, X_\beta^- \oplus X_\beta^+) = \mathcal{L}(U, X).$$

We recall from Theorem V.3.7 that this decomposition can always be carried out if $\omega_{\mathrm{ess}}(A) < \beta$. In this case, $X_\beta^+$ will be finite-dimensional and therefore $\omega_{\mathrm{ess}}(A) = \omega_{\mathrm{ess}}(A_\beta^-)$, which, by Corollary IV.2.11, implies that

$$\omega_0(A_\beta^-) = \max\big\{\omega_{\mathrm{ess}}(A_\beta^-),\, \mathrm{s}(A_\beta^-)\big\} < \beta. \tag{8.16}$$

We now show that $\beta$-exponential stabilizability implies, for a compact control operator, the existence of such a spectral decomposition.

**8.24 Theorem.** *For the control system $\Sigma(A, B, -)$ with compact control operator $B \in \mathcal{L}(U, X)$ and for $\beta \in \mathbb{R}$, the following assertions are equivalent.*

(a) *$\Sigma(A, B, -)$ is $\beta$-exponentially stabilizable.*

(b) *$\omega_{\mathrm{ess}}(A) < \beta$ and the finite-dimensional system $\Sigma(A_\beta^+, B_\beta^+, -)$ is controllable.*

Proof. (a) ⇒ (b). By assumption there exists $F \in \mathcal{L}(X,U)$ such that $\omega_0(A+BF) < \beta$. Since $B \in \mathcal{L}(U,X)$ is compact, $BF \in \mathcal{L}(X)$ is compact as well, and from Proposition IV.2.12 and equation (2.6) in Corollary IV.2.11 we obtain that

$$\omega_{\mathrm{ess}}(A) = \omega_{\mathrm{ess}}(A+BF) \le \omega_0(A+BF) < \beta,$$

which shows the first assertion. Moreover, we can perform the above spectral decomposition and obtain the control system $\Sigma(A_\beta^+, B_\beta^+, -)$ on the finite-dimensional state space $X_\beta^+$ and with control space $U$. Assume now that $\Sigma(A_\beta^+, B_\beta^+, -)$ is not controllable. Then we can find $0 \neq z' \in (X_\beta^+)'$ such that

$$\left\langle \int_0^t T_\beta^+(t-s)B_\beta^+ u(s)\, ds, z' \right\rangle = 0 \quad \text{for all } t>0 \text{ and all } u \in \mathrm{L}^1\big([0,t],U\big).$$

If we denote by $\big(S_F(t)\big)_{t\ge 0}$ the semigroup generated by $A+BF$, then we obtain from the variation of parameters formula in Corollary III.1.7

$$P_\beta^+ S_F(t)z = T_\beta^+(t)z + \int_0^t T_\beta^+(t-s)B_\beta^+\,[FS_F(s)z]\, ds \qquad \text{for all } z \in X_\beta^+.$$

Since $\omega_0(A+BF) < \beta$, these facts imply

$$\lim_{t\to\infty} \left\langle \mathrm{e}^{-\beta t} T_\beta^+(t)z, z' \right\rangle = \lim_{t\to\infty} \left\langle P_\beta^+ \mathrm{e}^{-\beta t} S_F(t)z, z' \right\rangle = 0 \tag{8.17}$$

for all $z \in X_\beta^+$. On the other hand, $X_\beta^+$ is finite-dimensional, and the spectrum of the generator $A_\beta^+ - \beta$ of $\big(\mathrm{e}^{-\beta t}T_\beta^+(t)\big)_{t\ge 0}$ satisfies $\sigma(A_\beta^+ - \beta) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \ge 0\}$. Together with (8.17), this gives $z' = 0$, contradicting the assumption of $z' \neq 0$.

(b) ⇒ (a). Since $\Sigma(A_\beta^+, B_\beta^+, -)$ is controllable, by Exercise 8.30.(2) there exists a feedback operator $F_\beta^+ \in \mathcal{L}(X_\beta^+, U)$ such that $\omega_0(A_\beta^+ + B_\beta^+ F_\beta^+) < \beta$. We now put $F := (0, F_\beta^+) \in \mathcal{L}(X_\beta^- \oplus X_\beta^+, U) = \mathcal{L}(X,U)$ and obtain

$$A + BF = \begin{pmatrix} A_\beta^- & B_\beta^- F_\beta^+ \\ 0 & A_\beta^+ + B_\beta^+ F_\beta^+ \end{pmatrix}.$$

This and (8.16) imply

$$\omega_0(A+BF) = \max\Big\{\omega_0(A_\beta^-),\, \omega_0(A_\beta^+ + B_\beta^+ F_\beta^+)\Big\} < \beta,$$

as claimed. □

Before discussing our examples, we will introduce the concept being “dual” to stabilizability.

**8.25 Definition.** *The control system $\Sigma(A,-,C)$ with state space $X$, observation space $Y$, and observation operator $C \in \mathcal{L}(X,Y)$ is called $\beta$-exponentially detectable for $\beta \in \mathbb{R}$ if there exists an output injection operator $L \in \mathcal{L}(Y,X)$ such that the growth bound $\omega_0(A+LC)$ is less than $\beta$. If $\Sigma(A,-,C)$ is 0-exponentially detectable, then it is called exponentially detectable.*

From this definition it is immediately clear that if $A'$ generates a strongly continuous semigroup on the dual space $X'$, the system $\Sigma(A,-,C)$ is ($\beta$-) exponentially detectable if and only if the dual system $\Sigma'(A,-,C)$ from (8.15) is ($\beta$-) exponentially stabilizable.

We leave it to the reader to give a characterization of exponentially detectable systems $\Sigma(A,-,C)$ analogous to Theorem 8.24. Instead, we return to our two standard examples.

**8.26 Example.** (*Heat equation*). Using the notation in (8.4), we have for the system $\Sigma(A,B,C)$ describing the heat equation from Example 8.2

$$X_\beta^+ = \operatorname{lin}\{e_n : 0 \le n \le n_\beta\}, \qquad P_\beta^+ = (P_\beta^+)^* = \sum_{0 \le n \le n_\beta} (\,\cdot\,|\,e_n)\,e_n$$

for each $\beta \le 0$ and $n_\beta := \sqrt{-\beta}/\pi$. Therefore,

$$A_\beta^+ = \sum_{0 \le n \le n_\beta} -n^2\pi^2(\,\cdot\,|\,e_n)\,e_n \in \mathcal{L}(X_\beta^+),$$
$$B_\beta^+ = \sum_{0 \le n \le n_\beta} (b\,|\,e_n)\,e_n \in \mathcal{L}(\mathbb{C}, X_\beta^+).$$

Since the generator $A$ satisfies $\omega_{\text{ess}}(A) = -\infty$, we obtain from Theorem 8.24 and Corollary 8.13 applied to $\Sigma(A_\beta^+, B_\beta^+, -)$ that the heat equation (HE) in Example 8.2 is $\beta$-exponentially stabilizable if and only if (cf. the calculations in Example 8.14)

$$\cos(n\pi s_0)\sin(n\pi\varepsilon_0) \ne 0 \quad \text{for all } n \in \mathbb{N} \text{ satisfying } \lambda_n = -n^2\pi^2 \ge \beta.$$

Similarly, it follows that $\Sigma(A,B,C)$ is $\beta$-exponentially detectable if and only if

$$\cos(n\pi s_1)\sin(n\pi\varepsilon_1) \ne 0 \quad \text{for all } n \in \mathbb{N} \text{ satisfying } \lambda_n = -n^2\pi^2 \ge \beta.$$

As Theorem 8.24 shows, a control system $\Sigma(A,B,-)$ having a generator $A$ with $\omega_{\text{ess}}(A) = 0$ is never exponentially stabilizable if its control operator $B$ is compact. In particular, we conclude that the wave equation from Example 8.3 is not exponentially stabilizable.

For this reason, we introduce the following weaker concept; compare also Definition V.1.1.(c).

**8.27 Definition.** *The control system $\Sigma(A, B, -)$ with state space $X$, control space $U$, and control operator $B \in \mathcal{L}(U, X)$ is called strongly stabilizable if there exists a feedback operator $F \in \mathcal{L}(X, U)$ such that the semigroup $\big(S_F(t)\big)_{t\geq 0}$ generated by $A + BF$ is strongly stable, i.e., if*

$$\lim_{t\to\infty} S_F(t)x = 0 \qquad \text{for all } x \in X.$$

Our next result gives a sufficient condition for strong stabilizability for an important class of control systems.

**8.28 Theorem.** *Let $\Sigma(A, B, -)$ be a control system where the state space $X$ and the control space $U$ are Hilbert spaces. Moreover, assume that $A$ has compact resolvent and generates a contraction semigroup. Then $F := -B^*$ yields a strongly stabilizing feedback control, i.e., the semigroup generated by $A - BB^*$ is strongly stable if and only if*

$$\ker(\mu - A^*) \cap \ker B^* = \{0\} \qquad \text{for all } \mu \in \mathrm{i}\mathbb{R} \cap P\sigma(A^*). \tag{8.18}$$

*Condition* (8.18) *is in particular satisfied if $\Sigma(A, B, -)$ is approximately 2-controllable.*

Proof. If there exists $0 \neq x \in \ker(\mu - A^*) \cap \ker B^*$, then we have $x \in \ker(\mu - A^* + BB^*)$, and $A - BB^*$ is not strongly stable by the remark preceding Lemma V.2.20.

For the converse implication, we have to show that $A - BB^*$ generates a strongly stable semigroup on $X$ if (8.18) is true. Since $BB^*$ is self-adjoint and positive semidefinite, $A - BB^*$ is dissipative, hence generates a contraction semigroup. Moreover, for $\lambda > \|BB^*\|$ we have

$$R(\lambda, A - BB^*) = \big(I + R(\lambda, A)BB^*\big)^{-1} R(\lambda, A),$$

and therefore $A - BB^*$ has compact resolvent. In particular, we obtain $\sigma(A - BB^*) = P\sigma(A - BB^*)$; hence by Theorem V.2.21 it suffices to verify that

$$P\sigma(A^* - BB^*) \cap \mathrm{i}\mathbb{R} = \emptyset. \tag{8.19}$$

To this end, we assume $(A^* - BB^*)x = \mu x$ for some $\mu \in \mathrm{i}\mathbb{R}$. Then

$$(A^*x \,|\, x) - \|B^*x\|^2 = \mu \cdot \|x\|^2,$$

and from $\mathrm{Re}(A^*x \,|\, x) \leq 0$ we obtain $x \in \ker B^*$ and hence $x \in \ker(\mu - A^*)$. By assumption, this is possible only for $x = 0$, and (8.19) follows.

Assume now that $\Sigma(A, B, -)$ is approximately 2-controllable and take some $x \in \ker(\mu - A^*) \cap \ker B^*$. Then, by Theorem IV.1.13.(ii), we have

$$B^* R(\lambda, A^*)x = B^* \frac{x}{\lambda - \mu} = 0 \qquad \text{for all } \lambda \in \rho(A^*).$$

However, by Corollary 8.11, we have $\bigcap_{\lambda \in \rho_+(A^*)} \ker B^* R(\lambda, A^*) = \{0\}$, and therefore (8.18) is satisfied. □

**8.29 Example.** (*Wave equation*). In Example 8.15, we already characterized the approximate $p$-controllability of the wave equation from Example 8.3. If $s_0 = s_1$ and $\varepsilon_0 = \varepsilon_1$, then $\mathcal{C} = \mathcal{B}^*$, and we obtain from the previous result that the feedback system governed by $\mathcal{A} - \mathcal{B}\mathcal{C}$ is strongly stable if

$$\frac{\sin(n\pi s_0)\sin(n\pi\varepsilon_0)}{n\pi\varepsilon_0} \neq 0 \qquad \text{for all } n \in \mathbb{N}.$$

**8.30 Exercises.** (1) If $X$ is finite-dimensional, then $\Sigma(A, B, -)$ is $\beta$-exponentially stabilizable if and only if $\operatorname{rg}(\lambda - A, B) = X$ for all $\lambda \in \overline{\mathbb{C}^+_\beta}$/for all $\lambda \in \sigma^+_\beta(A)$.

(2) If $X$ is finite-dimensional, then $\Sigma(A, B, -)$ is $\beta$-exponentially stabilizable for all $\beta \in \mathbb{R}$/some $\beta < \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ if and only if it is controllable.

(3*) Show that if $X$ and $U$ are Hilbert spaces, exact null controllability of a control system $\Sigma(A, B, -)$ implies its exponential stabilizability. (Hint: See [Zab92, Part IV, Thm. 3.3].)

(4) On $X := \ell^2(\mathbb{N})$ take the multiplication operator $A := M_{({}^1\!/_n)}$, let $U := \mathbb{C}$, and choose some $B = b = (b_n) \in \mathcal{L}(U, X) \cong X$ satisfying $b_n \neq 0$ for all $n \in \mathbb{N}$. Show that $\Sigma(A, B, -)$ is approximately 2-controllable on $[0, t]$ for all $t > 0$ but not exponentially stabilizable. (Hint: Use Corollary 8.13 and Corollary 8.12.(i) to prove the first claim. The second assertion follows from the fact that $A + BF$ is compact for all $F \in \mathcal{L}(X, U)$.)

(5) Let $A$, $B$, and $C$ be as in Corollary 8.13 and Corollary 8.19. Then the following assertions are true.

- (i) $\Sigma(A, B, -)$ is $\beta$-exponentially stabilizable if and only if $\mathbb{N}_\beta := \{n \in \mathbb{N} : \operatorname{Re}\lambda_n \geq \beta\}$ is finite and $\operatorname{rank} B_n = r_n$ for all $n \in \mathbb{N}_\beta$.
- (ii) $\Sigma(A, -, C)$ is $\beta$-exponentially detectable if and only if $\mathbb{N}_\beta$ is finite and $\operatorname{rank} C_n = r_n$ for all $n \in \mathbb{N}_\beta$.

## *d. Transfer Functions and Stability*

While all our previous considerations took place in the "time domain," we now give a "frequency domain" description of the control system $\Sigma(A, B, C)$. This is obtained by applying the Laplace transform to the differential equation (cACP). Indeed, at least formally and for the initial value $x_0 = 0$, one obtains in this way the equation

$$\mathcal{L}\, y(\lambda) = CR(\lambda, A)B\ \mathcal{L}\, u(\lambda)$$

for all $\lambda$ with sufficiently large real part, where $\mathcal{L}$ denotes the Laplace transform.

This motivates the following important notion.

**8.31 Definition.** *For a control system $\Sigma(A,B,C)$ the analytic map*

$$G : \rho(A) \to \mathcal{L}(U,Y), \quad G(\lambda) := CR(\lambda,A)B$$

*is called the transfer function of $\Sigma(A,B,C)$. If there exists a set $\Lambda \subset \sigma(A)$ of isolated points such that each $\lambda \in \Lambda$ is a removable singularity of $G$, we call the unique analytic extension of $G$ to $\rho(A) \cup \Lambda$ an extended transfer function.*

The transfer function can be interpreted as the Laplace transform of the *impulse response function*

$$H : [0,\infty) \to \mathcal{L}(U,Y), \quad H(t) := CT(t)B,$$

i.e., of the output of the system $\Sigma(A,B,C)$ for the input "$u(\cdot) = \delta_0(\cdot)$" and the initial value $x_0 = 0$, where $\delta_0(\cdot)$ denotes the Dirac function.

**8.32 Example.** (*Heat equation*). From the representations of $\big(T(t)\big)_{t\geq 0}$ and $R(\lambda,A)$ in (8.5) and (8.6), respectively, we immediately obtain the following formulas for the impulse response and transfer functions for the heat equation of Example 8.2.

$$H(t) = 1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi s_0)\sin(n\pi\varepsilon_0)\cos(n\pi s_1)\sin(n\pi\varepsilon_1)}{\varepsilon_0\varepsilon_1(n\pi)^2}\,\mathrm{e}^{-(n\pi)^2 t}, \ t \geq 0,$$

$$G(\lambda) = \frac{1}{\lambda} + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi s_0)\sin(n\pi\varepsilon_0)\cos(n\pi s_1)\sin(n\pi\varepsilon_1)}{\varepsilon_0\varepsilon_1(n\pi)^2\big(\lambda + (n\pi)^2\big)}, \ \lambda \neq -(n\pi)^2.$$

We just mention that by solving the linear, second-order ordinary differential equation

$$\begin{cases} \lambda y(s) - y''(s) = x(s), \\ x'(0) = x'(1) = 0, \end{cases}$$

one obtains an integral representation for $R(\lambda,A)$, hence a more explicit representation of the transfer function. Similarly, every representation of $\big(T(t)\big)_{t\geq 0}$ yields a corresponding formula for the impulse response function.

**8.33 Example.** (*Wave equation*). By the same arguments as in the previous example, we calculate, using (8.14), the impulse response and transfer functions for the wave equation from Example 8.3 as

$$H(t) = 2\sum_{n=1}^{\infty} \frac{\sin(n\pi s_0)\sin(n\pi\varepsilon_0)\sin(n\pi s_1)\sin(n\pi\varepsilon_1)}{\varepsilon_0\varepsilon_1(n\pi)^3}\,\sin(n\pi t), \ t \geq 0,$$

$$G(\lambda) = 2\sum_{n=1}^{\infty} \frac{\sin(n\pi s_0)\sin(n\pi\varepsilon_0)\sin(n\pi s_1)\sin(n\pi\varepsilon_1)}{\varepsilon_0\varepsilon_1(n\pi)^2\big(\lambda^2 + (n\pi)^2\big)}, \ \lambda \neq -(n\pi)^2,$$

respectively.

There is a rich interplay between the time- and the frequency-domain descriptions of control systems. Here, we just give one example linking the "internal" and the "external" stability of a control system.

**8.34 Definition.** *The control system $\Sigma(A, B, C)$ is called*

(i) *internally stable if the semigroup $\big(T(t)\big)_{t\geq 0}$ generated by $A$ is uniformly exponentially stable;*

(ii) *input–output stable if there exist constants $M \geq 0$, $w < 0$ such that $\|CT(t)B\| \leq M\mathrm{e}^{wt}$ for all $t \geq 0$;*

(iii) *externally stable if there exists an extended transfer function $G$ that is analytic and bounded on $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

It is easy to see that internal stability always implies external and input–output stability, and that the converse implications are, in general, false. However, based on Theorem V.1.11, we can prove the following relation between these stability concepts for control systems.

**8.35 Theorem.** *Suppose $\Sigma(A, B, C)$ is an exponentially stabilizable and exponentially detectable control system on a Hilbert space $X$. If $B$ and $C$ are compact, then the following assertions are equivalent.*

(a) *$\Sigma(A, B, C)$ is internally stable.*

(b) *$\Sigma(A, B, C)$ is input–output stable.*

(c) *$\Sigma(A, B, C)$ is externally stable.*

Proof. As already mentioned, the implication (a) ⇒ (b) is trivial. To verify (b) ⇒ (c), it suffices to observe that the Laplace transform $\mathcal{L}(CT(\cdot)B)$ is an extension of the transfer function $G$, which is analytic and bounded on $\mathbb{C}^+$.

Finally, we show that (c) ⇒ (a). Since $\Sigma(A, B, C)$ is exponentially stabilizable, we can find an operator $B \in \mathcal{L}(X, U)$ such that the semigroup generated by $A + BF$ is uniformly exponentially stable. Moreover, since $\Sigma(A, B, C)$ is exponentially detectable, we can find an operator $L \in \mathcal{L}(Y, X)$ such that the semigroup generated by $A + LC$ is uniformly exponentially stable. Then

$$\begin{aligned} R(\lambda, A) = {} & R(\lambda, A + LC)\,(\lambda - A - LC - BF)\,R(\lambda, A + BF) \\ & + R(\lambda, A + LC)\,LC\,R(\lambda, A)\,BF\,R(\lambda, A + BF) \end{aligned}$$

for all $\lambda \in \mathbb{C}^+ \cap \rho(A)$. Since by assumption

$$\sup_{\lambda \in \mathbb{C}^+ \cap \rho(A)} \|CR(\lambda, A)B\| < \infty,$$

we infer

$$\sup_{\lambda \in \mathbb{C}^+ \cap \rho(A)} \|R(\lambda, A)\| < \infty.$$

Hence, Theorem V.1.11 implies that $\big(T(t)\big)_{t\geq 0}$ is exponentially stable. □

This result provides a criterion for uniform exponential stability of a strongly continuous semigroup. Here, we just give one sample result.

**8.36 Theorem.** *Let $A$ be given by formula* (8.4) *for an orthonormal basis* $\{e_n : n \in \mathbb{N}\}$ *of a Hilbert space $X$ and a sequence* $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ *satisfying* $\sup_{n\in\mathbb{N}} \operatorname{Re}\lambda_n \le 0$. *Moreover, assume that* $\omega_{\mathrm{ess}}(A) < 0$ *and that* $\lambda_k \neq \lambda_l$ *for all* $\lambda_k, \lambda_l \in \mathrm{i}\mathbb{R}$, $k \neq l$. *If* $b \in X$ *satisfies* $(b \,|\, e_k) \neq 0$ *for all* $k \in \mathbb{N}$ *with* $\lambda_k \in \mathrm{i}\mathbb{R}$, *then the semigroup generated by* $A - rBB^*$ *for* $B := b \in \mathcal{L}(\mathbb{C}, X)$ *is uniformly exponentially stable for all* $r > 0$.

Proof. By Theorem 8.35, it suffices to show that $\Sigma(A - rBB^*, B, B^*)$ is exponentially stabilizable and detectable, and that its transfer function is bounded on $\mathbb{C}^+$. To that purpose, we consider the linear subspace $X^+ := \operatorname{lin}\{e_k : \lambda_k \in \mathrm{i}\mathbb{R}\}$. By Corollary 8.13, the system $\Sigma(A^+, B^+, (B^*)^+)$ is controllable, while Theorem 8.24 implies that $\Sigma(A, B, B^*)$ is exponentially stabilizable. Therefore, $\Sigma(A - rBB^*, B, B^*)$ is exponentially stabilizable. The fact that $\Sigma(A - rBB^*, B, B^*)$ is exponentially detectable is proved analogously.

We now put $g(\lambda) := B^*R(\lambda, A)B$ and $g_r(\lambda) := B^*R(\lambda, A - rBB^*)B$. Then from

$$B^*R(\lambda, A - rBB^*)B - B^*R(\lambda, A)B = -rB^*R(\lambda, A - rBB^*)BB^*R(\lambda, A)B$$

it follows that $g_r(\lambda) - g(\lambda) = -rg_r(\lambda)g(\lambda)$. This implies

$$g_r(\lambda) = \frac{g(\lambda)}{1 + rg(\lambda)}.$$

Since $g(\lambda) \in \mathbb{C}^+$ for all $\lambda \in \mathbb{C}^+$, we obtain $|g_r(\lambda)| \le {}^1/_r$ for all $\lambda \in \mathbb{C}^+$, and the proof is complete. □

**8.37 Example.** (*Heat equation*). From the previous theorem and the calculations in Example 8.15 it follows that the control system $\Sigma(A, B, C)$ of Example 8.2 with $b = c$ (that is, $C = B^*$) and control $u(\cdot) = -ry(\cdot)$ is uniformly exponentially stable for arbitrary $B = b \neq 0$ and $r > 0$.

**8.38 Exercise.** Prove that the conditions in Theorem 8.35 are not equivalent without assuming exponential stabilizability/exponential detectability.

## *Notes and Further Reading to Section* 8

Control of infinite-dimensional systems became a vast area, and we refer to specialized monographs like [Ahm91], [BDPDM93], [CP78], [CZ95], and [Zab92] or the survey articles [PZ81] and [Rus78] for further reading and references.

In our presentation we are guided by the standard Examples 8.2 and 8.3, which are taken from [CP78] and [CZ95]. An early investigation of Theorems 8.24 and 8.28 was done by Triggiani in [Tri75]; see also [Tri89]. A nonautonomous Banach space version of Theorem 8.35 appears in [CLR97]. However, examples can be given that show that only the equivalence (a) $\iff$ (b) holds without the Hilbert space structure; see [CLMSR99]. For the use of abstract Sobolev spaces (cf. Section II.5.a) in the treatment of control problems with unbounded control and/or observation operator see [Reb93], [Reb95], [Wei89], and [Wei91].

# 9. Semigroups for Nonautonomous Cauchy Problems

**(by Roland Schnaubelt)**

In this section we investigate a quite natural generalization of the (autonomous) Cauchy problem (ACP). We replace the fixed operator $A$ by operators $A(t)$ depending on a (time) parameter $t \in \mathbb{R}$ and consider the *nonautonomous abstract Cauchy problem*

$$\text{(nACP)} \qquad \begin{cases} \dot u(t) = A(t)u(t) & \text{for } t,s \in \mathbb{R},\ t \geq s, \\ u(s) = x \end{cases}$$

on a Banach space $X$. As we will see below, this problem is much more difficult than the autonomous case. We first discuss some basic properties of (nACP). Then, we present results on the existence and asymptotic behavior of solutions of (nACP), which will be obtained via the semigroup theory developed in this book. To that purpose, we introduce the so-called *evolution semigroup* associated with (nACP) as our basic tool.

## *a. Cauchy Problems and Evolution Families*

As in the autonomous case, we define well-posedness of (nACP) by

*"existence + uniqueness + continuous dependence on the data"*;

see Definition II.6.8. However, we have to observe that the solvability of (nACP) may depend heavily on the initial time $s$; cf. [Nic96, Expl. 3.2].

**9.1 Definition.** *Let* $\big(A(t), D(A(t))\big)$, $t \in \mathbb{R}$, *be linear operators on the Banach space* $X$ *and take* $s \in \mathbb{R}$ *and* $x \in D\big(A(s)\big)$. *Then a (classical) solution of* (nACP) *is a function* $u(\cdot\,; s,x) = u \in \mathrm{C}^1([s,\infty),X)$ *such that* $u(t) \in D\big(A(t)\big)$ *and* $u$ *satisfies* (nACP) *for* $t \geq s$.
*The Cauchy problem* (nACP) *is called well-posed (on spaces* $Y_t$*) if there are dense subspaces* $Y_s \subseteq D\big(A(s)\big)$, $s \in \mathbb{R}$, *of* $X$ *such that for* $s \in \mathbb{R}$ *and* $x \in Y_s$ *there is a unique solution* $t \mapsto u(t;s,x) \in Y_t$ *of* (nACP). *In addition, for* $s_n \to s$ *and* $Y_{s_n} \ni x_n \to x \in Y_s$, *we have* $\widetilde u(t;s_n,x_n) \to \widetilde u(t;s,x)$ *uniformly for* $t$ *in compact intervals in* $\mathbb{R}$, *where we set* $\widetilde u(t;s,x) := u(t;s,x)$ *for* $t \geq s$ *and* $\widetilde u(t;s,x) := x$ *for* $t < s$.

The solutions of the autonomous problem (ACP) are given by a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ as solutions of the functional equation (FE). In the present situation, the functional equation (FE) has to be replaced by the following concept.

**9.2 Definition.** *A family of bounded operators $\big(U(t,s)\big)_{t,s\in\mathbb{R},t\geq s}$ on a Banach space $X$ is called a (strongly continuous) evolution family if*

(i) *$U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = I$ for $t \geq r \geq s$ and $t,r,s \in \mathbb{R}$ and*

(ii) *the mapping $\{(\tau,\sigma) \in \mathbb{R}^2 : \tau \geq \sigma\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.*

*We say that $\big(U(t,s)\big)_{t\geq s}$ solves the Cauchy problem* (nACP) *(on spaces $Y_t$) if there are dense subspaces $Y_s$, $s \in \mathbb{R}$, of $X$ such that $U(t,s)Y_s \subseteq Y_t \subseteq D\big(A(t)\big)$ for $t \geq s$ and the function $t \mapsto U(t,s)x$ is a solution of* (nACP) *for $s \in \mathbb{R}$ and $x \in Y_s$.*

Evolution families are also called *evolution systems, evolution operators, evolution processes, propagators*, or *fundamental solutions*. Notice that a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ gives rise to the evolution family $U(t,s) := T(t-s)$.

In contrast to semigroups, it is possible that the mapping $t \mapsto U(t,s)x$ is differentiable only for $x = 0$ (compare Theorem II.1.4). (Take, for instance, $X = \mathbb{C}$ and $U(t,s) := {}^{p(t)}/_{p(s)}$ for a nowhere differentiable function $p$ such that $p, {}^1/_p \in \mathrm{C_b}(\mathbb{R})$.) Nevertheless, we can state an analogue of Theorem II.6.7.

**9.3 Proposition.** *The Cauchy problem* (nACP) *is well-posed on $Y_t$ if and only if there is an evolution family solving* (nACP) *on $Y_t$.*

Since we do not need this result in the following, we omit its proof. The details can be found in [Nic96, §3.2]. We note, however, that the implication " $\Rightarrow$ " is shown in the same way as (d) $\Rightarrow$ (a) in Theorem II.6.7.

Generation Theorem II.3.8 provides a characterization of well-posedness of (ACP) in terms of properties of the operator $A$. There is no analogue of this result in the time-dependent situation. In fact, the following examples indicate that it seems to be rather difficult to find necessary conditions for the well-posedness of (nACP).

**9.4 Examples.** (i) Even if each operator $\big(A(t), D\big(A(t)\big)\big)$ is a generator for each $t \in \mathbb{R}$ and if (nACP) is well-posed, it may happen that (nACP) cannot be solved on all of $D\big(A(t)\big)$; see, e.g., [Nic96, Expl. 3.5].

(ii) There are well-posed Cauchy problems for generators $A(t)$ such that the intersection $\bigcap_{t\in\mathbb{R}} D\big(A(t)\big)$ equals $\{0\}$; see, e.g., [Fat83, Expl. 7.3.2].

(iii) The Cauchy problem on $X := \mathrm{C_b}(\mathbb{R})$ given by $A(t)f := f'(t)\mathbb{1}$ for $f \in Y := \mathrm{C}^1_\mathrm{b}(\mathbb{R})$ is solved by $U(t,s)f := f + \big(f(t) - f(s)\big)\mathbb{1}$ on $Y$, but $A(t)$ is not closable in $X$; see [Hah95].

However, there exist several sufficient conditions for well-posedness that are well documented in, for instance, the monographs [Ama95], [Fat83], [Gol85], [Kre71], [Lun95], [Paz83], [Tan79], [Tan97]. We present two main results in simplified form.

**9.5 The Hyperbolic Case.** Let $A(t)$, $t \in \mathbb{R}$, be generators of contraction semigroups satisfying $D\big(A(t)\big) \equiv Y$ and $A(\cdot)x \in \mathrm{C}^1(\mathbb{R}, X)$ for $x \in Y$. Then (nACP) is well-posed on $Y$.

This result is due to Kato [Kat53], who proved a more general version in [Kat70].

**9.6 The Parabolic Case.** Let $A(t)$ be generators of bounded analytic semigroups of the same type $(M, \delta)$ such that $A(t)$ is invertible and

$$\|A(t)R\big(\lambda, A(t)\big)\,\big(A(t)^{-1} - A(s)^{-1}\big)\| \le L\,|t-s|^{\mu} \cdot |\lambda|^{-\nu}$$

for $\lambda \in \Sigma_{\pi/2+\delta}$, $t, s \in \mathbb{R}$, and constants $L \ge 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$.

This condition was introduced (in a somewhat more general form) by Acquistapace and Terreni, [AT87b]. It is shown in [Acq88, Thm. 2.3] that in this case (nACP) is well-posed on $D\big(A(t)\big)$. Moreover, the solving evolution family enjoys additional regularity properties similar to those of analytic semigroups. More precisely, $U(t,s)$ maps $X$ into the domain of the fractional power $\big(-A(t)\big)^{\alpha}$ (see Section II.5.c), $t \mapsto \big(-A(t)\big)^{\alpha}U(t,s)$ is continuous, and

$$\Big\|\big(-A(t)\big)^{\alpha}U(t,s)\Big\| \le C(t-s)^{-\alpha}\mathrm{e}^{w(t-s)} \tag{9.1}$$

for $t > s$, $0 \le \alpha \le 1$, and constants $C \ge 0$ and $w \in \mathbb{R}$, see [FY94, Thm. 2.3] and [Acq88], [AT87b], [Yag90].

However, we point out that there are many situations where one can solve (nACP) only in a "weaker" sense; see, e.g., [Tan79, §5.5]. Moreover, Example 9.21 will show that well-posedness in the above sense is not preserved under bounded perturbations. Therefore, we will study evolution families without assuming differentiability properties.

We now turn our attention to asymptotic properties of evolution families. First, as in Definition I.5.6 we define the *(exponential) growth bound* of an evolution family $\big(U(t,s)\big)_{t\ge s}$ by

$$\omega_0(U) := \inf\Big\{w \in \mathbb{R} : \exists\, M_w \ge 1 \text{ with } \|U(t,s)\| \le M_w\,\mathrm{e}^{w(t-s)} \text{ for } t \ge s\Big\}.$$

Notice that this number coincides with $\omega_0 = \omega_0(\mathcal{T})$ if $U(t,s) = T(t-s)$ for a semigroup $\mathcal{T} = \big(T(t)\big)_{t\ge0}$. An evolution family is called *(uniformly) exponentially stable* if $\omega_0(U) < 0$ and *exponentially bounded* if $\omega_0(U) < +\infty$. It might be surprising that there are evolution families that are *not* exponentially bounded. For instance, $U(t,s) := \mathrm{e}^{t^2-s^2}$ on $X = \mathbb{C}$. In a manner similar to the proof of Proposition I.5.5 we can prove that

(i) $\omega_0(U) < \infty$ if and only if there are constants $M_0 \ge 0$, $t_0 > 0$ such that $\|U(s+t,s)\| \le M_0$ for $0 \le t \le t_0$ and $s \in \mathbb{R}$, and

(ii) $\omega_0(U) < w$ if and only if $\omega_0(U) < \infty$ and there are constants $M_1, t_1 > 0$ such that ${}^{\log M_1}/_{t_1} < w$ and $\|U(s+t_1,s)\| \le M_1$ for all $s \in \mathbb{R}$.

The following examples show that in contrast to Proposition V.1.7, the growth bound is not determined by the spectral radius or the norm of a single operator $U(t,s)$.

**9.7 Examples.** (i) Let $\big(T_l(t)\big)_{t\geq 0}$ be the nilpotent left translation semigroup on $X := \mathrm{L}^1[0,1]$; see Paragraph I.4.17. Set $U(t,s) := \mathrm{e}^{t^2-s^2} T_l(t-s)$. Then $U(s+t,s) = 0$ for $t \geq 1$ and $s \in \mathbb{R}$, but it is easy to see that $\|U(s+{}^1\!/\!_2, s)\| = \mathrm{e}^{s+{}^1\!/\!_4}$, and so $\omega_0(U) = +\infty$.
(ii) In [Sch99, §5] an example is constructed in which $\sigma\big(U(t,s)\big) = \{0\}$ for $t > s$, but $t \mapsto \|U(t,s)\|$ grows faster than any exponential function as $t \to \infty$.

Further, one is interested in exponential estimates on "invariant" subspaces; cf. [Cop78], [DK74], [Hen81], and Section V.1.c. Throughout, we set $Q := I - P$ for a projection $P \in \mathcal{L}(X)$.

**9.8 Definition.** *An evolution family $\big(U(t,s)\big)_{t\geq s}$ on a Banach space $X$ is called hyperbolic (or has exponential dichotomy) if there are projections $P(t)$, $t \in \mathbb{R}$, and constants $N, \delta > 0$ such that $P(\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$ and*

(i) *$U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$,*
(ii) *the restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) := U_Q(t,s)^{-1}$),*
(iii) *$\|U(t,s)P(s)\| \leq N\mathrm{e}^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq N\mathrm{e}^{-\delta(t-s)}$ for all $t \geq s$.*

We remark that properties (i)–(iii) for the projections $P(t)$ already imply $P(\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$; cf. [NRS98, Lem. 4.2].

In Chapters IV and V we have seen that in many situations the spectrum of a generator determines exponential stability or hyperbolicity of the semigroup. The following examples, cf. [Cop78, p. 3], show that this fails in the time-dependent situation even for $X = \mathbb{C}^2$. In particular, $\omega_0(U)$ can be strictly greater or smaller than the (constant) spectral bound $\mathrm{s}\big(A(t)\big)$.

**9.9 Example.** Let $X := \mathbb{C}$ and define $A(t) := D(-t)A_0D(t)$, $B(t) := D(-t)B_0D(t)$ for $t \in \mathbb{R}$, where

$$D(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \qquad A_0 := \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}, \qquad B_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding Cauchy problems are solved by

$$U(t,s) := D(-t) \exp\left[(t-s)\begin{pmatrix} -1 & -4 \\ -1 & -1 \end{pmatrix}\right] D(s)$$

and

$$V(t,s) := D(-t) \exp\left[(t-s)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right] D(s),$$

respectively. Thus, $\omega_0(U) = 1$ and $\omega_0(V) = 0 = \omega_0(V^{-1})$. On the other hand, $\sigma\big(A(t)\big) = \{-1\}$ and $\sigma\big(B(t)\big) = \{-1, 1\}$ for all $t \in \mathbb{R}$.

## *b. Evolution Semigroups*

In order to deal with some of the problems mentioned in the previous section, we now introduce a semigroup approach to nonautonomous Cauchy problems. Throughout, let $\big(U(t,s)\big)_{t\geq s}$ be a strongly continuous evolution family on a Banach space $X$ such that $\|U(t,s)\| \leq M\mathrm{e}^{w(t-s)}$ for $t \geq s$ and constants $M \geq 1$ and $w \in \mathbb{R}$. We then define on the space $E := \mathrm{C}_0(\mathbb{R}, X)$, endowed with the sup-norm $\|\cdot\|_\infty$, bounded operators $T(t)$, $t \geq 0$, by setting

$$(9.2) \qquad \big(T(t)f\big)(s) := U(s, s-t)f(s-t) \qquad \text{for } s \in \mathbb{R} \text{ and } f \in E.$$

We first note the following basic property.

**9.10 Lemma.** *Equation* (9.2) *defines a strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$ *on* $E$.

Proof. Clearly, $\big(T(t)\big)_{t\geq 0}$ is a semigroup of bounded operators on $E$ with $\|T(t)\| \leq M\mathrm{e}^{wt}$. For $f \in \mathrm{C}_c(\mathbb{R}, X)$, it is easy to see that $T(t)f \to f$ in $E$ as $t \to 0$. Since $\mathrm{C}_c(\mathbb{R}, X)$ is dense in $E$, Proposition I.5.3 implies strong continuity of $\big(T(t)\big)_{t\geq 0}$. ☐

**9.11 Definition.** *Let* $\big(U(t,s)\big)_{t\geq s}$ *be a strongly continuous, exponentially bounded evolution family on a Banach space* $X$*. The strongly continuous semigroup* $\big(T(t)\big)_{t\geq 0}$ *on* $E = \mathrm{C}_0(\mathbb{R}, X)$ *defined in* (9.2) *is called an evolution semigroup. Its generator is denoted by* $\big(G, D(G)\big)$.

Recall that for bounded operators $A(t)$ evolution semigroups were already introduced in Example III.5.9. For the special case $U(t,s) \equiv I$, we designate by $\big(T_r(t)\big)_{t\in\mathbb{R}}$ the group of right translations on $E$ (and $\mathrm{C}_0(\mathbb{R})$) with generator $G_0 f = -f'$ and $D(G_0) = \{f \in \mathrm{C}^1(\mathbb{R}, X) : f, f' \in E\}$; see Exercise I.4.19.(5). Further, for a family of linear operators $A(t)$, $t \in \mathbb{R}$, we define the multiplication operator $\mathcal{A} := A(\cdot)$ on $E$ with maximal domain as in Paragraph III.4.13.

**9.12 Remark.** (i) There is a one-to-one correspondence between evolution semigroups on $E$ and exponentially bounded evolution families on $X$.

(ii) It is possible to define an evolution semigroup for intervals $I$ instead of $\mathbb{R}$ and on spaces $\mathrm{L}^p(I, X)$ for $1 \leq p < \infty$. This is useful in certain situations, see, e.g., [RRSV99] or [Sch99], but not needed in what follows.

(iii) It is shown in [Nic97, Thm. 2.9] that $\big(U(t,s)\big)_{t\geq s}$ solves a Cauchy problem (nACP) if and only if there is an invariant core $D \subseteq \mathrm{C}_0^1(\mathbb{R}, X) \cap D(\mathcal{A})$ of $G$ such that $Gf = G_0 f + \mathcal{A}f$ for $f \in D$; see also [Lum85b]. In particular, it is possible to solve (nACP) by means of results on the sum of $G_0$ and $\mathcal{A}$; see [DPG75], [DPI76], [MP97], [MR99], [Nic97], and the references therein.

The following characterization of bounded multiplication operators on $E$ will lead to a characterization of evolution semigroups.

**9.13 Proposition.** *A bounded operator $\mathcal{M}$ on $E = \mathrm{C}_0(\mathbb{R}, X)$ is of the form $(\mathcal{M}f)(s) = M(s)f(s)$ for an operator family $M(\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$ if (and only if) $\mathcal{M}(\varphi f) = \varphi \mathcal{M} f$ for $f \in E$ and $\varphi \in \mathrm{C_c}(\mathbb{R})$. Moreover, $\|\mathcal{M}\|_{\mathcal{L}(E)} = \sup_{t\in\mathbb{R}} \|M(t)\|_{\mathcal{L}(X)}$.*

Proof. For $\varepsilon > 0$ and $t \in \mathbb{R}$, choose a continuous function $\varphi_\varepsilon : \mathbb{R} \to [0,1]$ with $\varphi_\varepsilon(t) = 1$ and $\operatorname{supp} \varphi_\varepsilon \subseteq [t-\varepsilon, t+\varepsilon]$. Then, by assumption,

$$\|(\mathcal{M}f)(t)\| = \|(\mathcal{M}\varphi_\varepsilon f)(t)\| \le \|\mathcal{M}\| \cdot \|\varphi_\varepsilon f\|_\infty \le \|\mathcal{M}\| \sup_{|t-s|\le\varepsilon} \|f(s)\|$$

for every $f \in E$. Therefore, $f(t) = 0$ implies $(\mathcal{M}f)(t) = 0$. So we can define linear operators $M(t)$ on $X$ by setting $M(t)x := (\mathcal{M}f)(t)$ for some $f \in E$ with $f(t) = x$. Clearly, $\sup_{t\in\mathbb{R}} \|M(t)\| = \|\mathcal{M}\|$ and $\mathcal{M} = M(\cdot)$. For $x \in X$ and $t \in \mathbb{R}$, take $f \in E$ with $f(t) = x$ in a neighborhood $J$ of $t$. Since $M(\cdot)x = \mathcal{M}f(\cdot)$ on $J$ and $\mathcal{M}f$ is continuous, $M(\cdot)$ is a strongly continuous operator function. □

**9.14 Theorem.** *Let $\big(T(t)\big)_{t\ge0}$ be a strongly continuous semigroup on $E = \mathrm{C}_0(\mathbb{R}, X)$ with generator $\big(G, D(G)\big)$. Then the following assertions are equivalent.*

(a) *$\big(T(t)\big)_{t\ge0}$ is an evolution semigroup.*

(b) *$T(t)\,(\varphi f) = \big(T_r(t)\varphi\big)\, T(t)f$ for all $\varphi \in \mathrm{C_c}(\mathbb{R})$, $f \in E$, and $t \ge 0$.*

(c) *For $f \in D(G)$ and $\varphi \in \mathrm{C}^1_c(\mathbb{R})$, we have $\varphi f \in D(G)$ and $G(\varphi f) = \varphi\, Gf - \varphi' f$.*

Proof. (a) ⇒ (c). For $f \in D(G)$ and $\varphi \in \mathrm{C}^1_c(\mathbb{R})$, the difference quotient

$$\tfrac{1}{t}\big(T(t)(\varphi f) - \varphi f\big) = \tfrac{1}{t}\big(T_r(t)\varphi - \varphi\big)\, T(t)f + \varphi\, \tfrac{1}{t}\big(T(t)f - f\big)$$

converges to $-\varphi' f + \varphi\, Gf$ in $E$ as $t \downarrow 0$.

(c) ⇒ (b). Set $u(t) := \big(T_r(t)\varphi\big)\, T(t)f$ for $\varphi \in \mathrm{C}^1_\mathrm{c}(\mathbb{R})$, $f \in D(G)$, and $t \ge 0$. By assumption, $u(t) \in D(G)$ and

$$Gu(t) = -\big(T_r(t)\varphi'\big)\, T(t)f + \big(T_r(t)\varphi\big)\, GT(t)f = \tfrac{d}{dt}u(t).$$

Since $u(0) = \varphi f$ and $G$ generates $\big(T(t)\big)_{t\ge0}$, this gives $u(t) = T(t)(\varphi f)$. Now (b) follows by an obvious approximation argument.

(b) $\Rightarrow$ (a). Set $\mathcal{M}_t = T(t)T_r(-t)$ for $t \geq 0$. The assumption and an application of Proposition 9.13 show that $\mathcal{M}_t = M(t,\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$. Define $U(t,s) := M(t-s,t)$ for $t \geq s$. Then

$$\big(T(t)f\big)(s) = \big(\mathcal{M}_t T_r(t) f\big)(s) = M(t,s)f(s-t) = U(s,s-t)f(s-t)$$

for $f \in E$, $t \geq 0$, and $s \in \mathbb{R}$. Thus, strong continuity and exponential boundedness of $U(\cdot,\cdot)$ in $X$ follow easily from the respective properties of $T(\cdot)$ in $E$. Finally,

$$\begin{aligned} U(s,s-t)x &= \big(T(t)f\big)(s) = \big(T(r)T(t-r)f\big)(s) \\ &= U(s,s-r)\,\big(T(t-r)f\big)(s-r) = U(s,s-r)U(s-r,s-t)x \end{aligned}$$

for $x \in X$, $s \in \mathbb{R}$, $t \geq r \geq 0$, and $f \in E$ with $f(s-t) = x$. □

Theorem 9.14 and its proof are essentially due to Evans [Eva76], Lumer [Lum85a], Neidhardt [Nei81], and Paquet [Paq79], in differing situations. We will use this characterization in the next section in order to derive perturbation results for evolution families from the perturbation theory of semigroups. This idea goes back to Howland [How74], who has introduced evolution semigroups in an $\mathrm{L}^2$-setting. Variants of Theorem 9.14 for evolution semigroups on $\mathrm{L}^p(I,X)$ can be found in [Eva76], [How74], [Nei81], [RRS96], [RRSV99].

In recent years, evolution semigroups have attracted renewed interest since it was discovered that their spectra characterize the hyperbolicity of the underlying evolution family. At first, we show the *spectral mapping theorem* (SMT) for evolution semigroups.

**9.15 Theorem.** *Let $\big(T(t)\big)_{t\geq 0}$ be an evolution semigroup on $E = \mathrm{C}_0(\mathbb{R}, X)$ with generator $G$. Then $\sigma\big(T(t)\big)$ is rotationally invariant for $t > 0$ and $\sigma(G)$ is invariant under translations along the imaginary axis. Moreover, $\big(T(t)\big)_{t\geq 0}$ satisfies the spectral mapping theorem*

$$\text{(SMT)} \qquad \sigma\big(T(t)\big) \setminus \{0\} = \exp\big(t\sigma(G)\big), \qquad t \geq 0.$$

Proof. (1) We define an isomorphism on $E$ by $M_\mu f(s) := \mathrm{e}^{\mathrm{i}\mu s} f(s)$ for $\mu \in \mathbb{R}$. Clearly, $M_\mu T(t) M_{-\mu} = \mathrm{e}^{\mathrm{i}\mu t} T(t)$ for $t \geq 0$. This establishes the asserted symmetry properties due to the results in Paragraphs II.2.1 and II.2.2.

(2) Recall that the spectral mapping theorem follows from the inclusion $A\sigma\big(T(t)\big) \setminus \{0\} \subseteq \mathrm{e}^{t\sigma(G)}$; see Theorem IV.3.6 and Theorem IV.3.7. Since the rescaled semigroup $\big(\mathrm{e}^{\lambda t}T(t)\big)_{t\geq 0}$ is again an evolution semigroup, it suffices to prove the following:

$$1 \in A\sigma\big(T(t_0)\big) \text{ for some } t_0 > 0 \text{ implies } 0 \in \sigma(G).$$

So assume that $1 \in A\sigma\big(T(t_0)\big)$. For each $n \in \mathbb{N}$, there exists $f_n \in \mathrm{C}_0(\mathbb{R}, X)$ such that $\|f_n\|_\infty = 1$ and $\|f_n - T(kt_0)f_n\|_\infty < 1/2$ for all $k = 0, 1, \dots, 2n$. Hence,

$$\frac{1}{2} < \sup_{s\in\mathbb{R}} \|U(s, s - kt_0)f_n(s - kt_0)\| \le 2 \tag{9.3}$$

for $k = 0, 1, \dots, 2n$. For each $n$, take $s_n \in \mathbb{R}$ such that $\|U(s_n, s_n - nt_0)x_n\| \ge 1/2$ for $x_n := f_n(s_n - nt_0)$. Let $I_n := [s_n - nt_0, s_n + nt_0]$ and choose $\alpha_n \in \mathrm{C}^1(\mathbb{R})$ such that $\alpha_n(s_n) = 1$, $0 \le \alpha_n \le 1$, $\operatorname{supp}\alpha_n \subseteq I_n$, and $\|\alpha_n'\|_\infty \le 2/nt_0$. Define

$$g_n(s) := \begin{cases} \alpha_n(s)U(s, s_n - nt_0)x_n, & s \ge s_n - nt_0, \\ 0, & s < s_n - nt_0, \end{cases}$$

for $n \in \mathbb{N}$. Then $g_n \in E$, $\|g_n\|_\infty \ge \|g_n(s_n)\| \ge 1/2$, and

$$T(t)g_n(s) = \alpha_n(s-t)U(s, s-t)U(s-t, s_n-nt_0)x_n = \alpha_n(s-t)U(s, s_n-nt_0)x_n$$

for $s - t \ge s_n - nt_0$ and $T(t)g_n(s) = 0$ for $s - t < s_n - nt_0$. Therefore, $g_n \in D(G)$ and

$$Gg_n(s) = \begin{cases} -\alpha_n'(s)U(s, s_n - nt_0)x_n, & s \ge s_n - nt_0, \\ 0, & s < s_n - nt_0. \end{cases}$$

Each $s \in I_n$ can be written as $s = s_n + (k + \sigma - n)t_0$ for $k \in \{0, 1, \dots, 2n\}$ and $\sigma \in [0, 1)$. Using the exponential boundedness of $\big(U(t, s)\big)_{t\ge s}$ and (9.3), we estimate

$$\begin{aligned} \|Gg_n(s)\| &\le \frac{2}{nt_0}\, M\mathrm{e}^{|\omega|t_0}\, \big\|U\big(s_n + (k-n)t_0, s_n - nt_0\big)\, x_n\big\| \\ &= \frac{2M}{nt_0}\,\mathrm{e}^{|\omega|t_0}\big\|U\big(s_n + (k-n)t_0, s_n \\ &\qquad\qquad + (k-n)t_0 - kt_0\big) f_n\big(s_n + (k-n)t_0 - kt_0\big)\big\| \\ &\le \frac{4M}{nt_0}\,\mathrm{e}^{|\omega|t_0} \end{aligned}$$

for $s \in I_n$. Consequently, 0 is an approximate eigenvalue of $G$. □

In order to relate the spectra of $T(t)$ and $G$ to the hyperbolicity of the evolution family $\big(U(t, s)\big)_{t\ge s}$, we need some preliminary results and use the notation introduced in Definition 9.8.

**9.16 Lemma.** *Let $\big(T(t)\big)_{t\ge 0}$ be a hyperbolic evolution semigroup on $E$ with corresponding projection $\mathcal{P}$. Then $\varphi\,\mathcal{P}f = \mathcal{P}(\varphi f)$ for $\varphi \in \mathrm{C}_\mathrm{b}(\mathbb{R})$ and $f \in E$.*

Proof. Since $\big(T(t)\big)_{t\geq 0}$ is hyperbolic, there are constants $N, \delta > 0$ such that

$$N^{-1}\mathrm{e}^{\delta t}\,\|\mathcal{Q}f\|_\infty \leq \|T(t)\mathcal{Q}f\|_\infty \leq \|T(t)f\|_\infty + N\mathrm{e}^{-\delta t}\,\|\mathcal{P}f\|_\infty$$

for $f \in E$ and $t \geq 0$. This implies $\mathcal{P}E = \{f \in E : T(t)f \to 0 \text{ as } t \to \infty\}$. Hence, $\varphi\,\mathcal{P}f \in \mathcal{P}E$ for $\varphi \in \mathrm{C_b}(\mathbb{R})$. Further,

$$\begin{aligned}\|\mathcal{P}(\varphi\mathcal{Q}f)\|_\infty &= \big\|\mathcal{P}\big(\varphi T(t)T_Q^{-1}(t)\big)\mathcal{Q}f\big\|_\infty\\ &= \big\|T(t)\mathcal{P}\big(T_r(-t)\varphi\big)T_Q^{-1}(t)\mathcal{Q}f\big\|_\infty\\ &\leq N^2\mathrm{e}^{-2\delta t}\,\|\varphi\|_\infty\cdot\|f\|_\infty\end{aligned}$$

for $t \geq 0$, that is, $\mathcal{P}(\varphi\mathcal{Q}f) = 0$. As a result, we obtain $\mathcal{P}(\varphi f) = \mathcal{P}(\varphi\mathcal{P}f) + \mathcal{P}(\varphi\mathcal{Q}f) = \varphi\,\mathcal{P}f$. ☐

**9.17 Lemma.** *Let $\big(U(t,s)\big)_{t\geq s}$ be a hyperbolic evolution family on $X$ with projections $P(t)$ and constants $N, \delta > 0$. Then the following assertions hold.*

(i) *$U_Q(t,s)Q(s) = U_Q(t,r)U_Q(r,s)Q(s)$ for $t, r, s \in \mathbb{R}$.*

(ii) *The mapping $\mathbb{R}^2 \ni (t,s) \mapsto U_Q(t,s)Q(s) \in \mathcal{L}(X)$ is strongly continuous.*

Proof. Assertion (i) can easily be verified. For (ii) let $x \in X$ and $(t', s') \to (t,s)$. We may assume $t \leq s$ and $t' \leq s'$ and write

$$\begin{aligned}&U_Q(t',s')Q(s')x - U_Q(t,s)Q(s)x\\ &\qquad= U_Q(t',s')Q(s')\,\big[Q(s') - Q(s)\big]x\\ &\qquad\quad+ U_Q(t',s')Q(s')\,\big[U(s,t) - U(s',t')\big]\,U_Q(t,s)Q(s)x\\ &\qquad\quad+ \big[Q(t') - Q(t)\big]\,U_Q(t,s)Q(s)x.\end{aligned}$$

Since $\|U_Q(t',s')Q(s')\| \leq N$ and $U(\cdot,\cdot)$ and $Q(\cdot)$ are strongly continuous, (ii) follows. ☐

Given a hyperbolic evolution family, we define its so-called *Green's function* by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\\ -U_Q(t,s)Q(s), & t < s.\end{cases}$$

**9.18 Theorem.** *For an exponentially bounded evolution family $\big(U(t,s)\big)_{t\geq s}$ on a Banach space $X$ and the induced evolution semigroup $\big(T(t)\big)_{t\geq 0}$ on $E := \mathrm{C}_0(\mathbb{R}, X)$, the following assertions are equivalent.*

(a) *$\big(U(t,s)\big)_{t\geq s}$ is hyperbolic.*

(b) *$\big(T(t)\big)_{t\geq 0}$ is hyperbolic.*

(c) *$\rho\big(T(t)\big) \cap \Gamma \neq \emptyset$ for one/all $t > 0$.*

(d) *The generator $G$ of $\big(T(t)\big)_{t\geq 0}$ satisfies $\rho(G) \cap i\mathbb{R} \neq \emptyset$.*

*In this case, $G$ is invertible and $(G^{-1}f)(t) = -\int_{\mathbb{R}} \Gamma(t,s)f(s)\,ds$ for all $f \in E$ and $t \in \mathbb{R}$.*

Proof. The implications (b) $\iff$ (c) $\iff$ (d) are consequences of Proposition V.1.15 and Theorem 9.15.

(a) $\Rightarrow$ (b). By $(\mathcal{P}f)(s) := P(s)f(s)$ we define a bounded projection $\mathcal{P}$ on $E$ that commutes with $T(t)$ for $t \geq 0$ due to Definition 9.8.(i). Using Lemma 9.17, we see that the operator $T_Q(t) : \mathcal{Q}E \to \mathcal{Q}E$ has the inverse

$$T_Q^{-1}(t)f(s) = U_Q(s, s+t)f(s+t). \tag{9.4}$$

Finally, Definition 9.8.(iii) implies $\|T(t)\mathcal{P}\|, \|T_Q^{-1}(t)\mathcal{Q}\| \leq N\mathrm{e}^{-\delta t}$ for $t \geq 0$.

(b) $\Rightarrow$ (a). By Lemma 9.16 and Proposition 9.13, the projection $\mathcal{P}$ corresponding to $\big(T(t)\big)_{t\geq 0}$ is given by $P(\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$. Property (i) in Definition 9.8 then follows from $\mathcal{P}T(t) = T(t)\mathcal{P}$. Because of $T(t)\mathcal{Q}E = \mathcal{Q}E$, we obtain

$$Q(s)X = \{(\mathcal{Q}f)(s) : f \in E\} = \big\{\big(T(t)\mathcal{Q}f\big)(s) : f \in E\big\} = U(s, s-t)Q(s-t)X$$

for $s \in \mathbb{R}$ and $t \geq 0$. Let $s \in \mathbb{R}$, $t \geq 0$, $x \in Q(s-t)X$, and $\varepsilon > 0$. Choose $f \in \mathcal{Q}E$ with $f(s-t) = x$ and $\|T(t)f\|_\infty \leq \|T(t)f(s)\| + \varepsilon$. Since $\big(T(t)\big)_{t\geq 0}$ is hyperbolic, there are constants $N, \delta > 0$ such that

$$N^{-1}\mathrm{e}^{\delta t}\,\|x\| \leq N^{-1}\mathrm{e}^{\delta t}\,\|f\|_\infty \leq \|T(t)\mathcal{Q}f\|_\infty \leq \|U(s, s-t)x\| + \varepsilon.$$

So we have verified Definition 9.8.(ii) and the second estimate in (c). The other estimate is proved in the same way.

If (a)–(d) hold, then $G$ is invertible by Theorem 9.15. By Paragraph II.2.3, the restricted semigroups $\big(T_P(t)\big)_{t\geq 0}$ and $\big(T_Q^{-1}(t)\big)_{t\geq 0}$ on $\mathcal{P}E$ and $\mathcal{Q}E$ are generated by $G_P := G_{|\mathcal{P}E}$ and $-G_Q := -G_{|\mathcal{Q}E}$, respectively. Since both semigroups are uniformly exponentially stable, Theorem II.1.10.(i) and (9.4) imply

$$\begin{aligned} -G^{-1}f(t) &= \big(R(0, G_P)\mathcal{P}f\big)(t) - \big(R(0, -G_Q)\mathcal{Q}f\big)(t) \\ &= \int_0^\infty U(t, t-\tau)P(t-\tau)f(t-\tau)\,d\tau \\ &\quad - \int_0^\infty U_Q(t, t+\tau)Q(t+\tau)f(t+\tau)\,d\tau \\ &= \int_{\mathbb{R}} \Gamma(t,s)f(s)\,ds \end{aligned}$$

for $f \in E$ and $t \in \mathbb{R}$, where we have used that point evaluation is a continuous mapping from $E$ to $X$. ☐

Latushkin and Montgomery-Smith proved Theorem 9.15 in [LMS95]. We have presented a somewhat simpler proof taken from [RS96]. The equivalence (a) $\iff$ (b) in Theorem 9.18 is essentially due to Rau [Rau94]; see also [RS94]. The representation of $G^{-1}$ was found by Latushkin and Randolph [LR95]. By completely different methods, (9.18) (a) $\iff$ b was also

proved in [LMS95] and [LMSR96]. Both theorems still hold when $E$ is replaced by $\mathrm{L}^p(\mathbb{R}, X)$, $1 \leq p < \infty$; see the above references. For an alternative approach using mild solutions of the inhomogeneous Cauchy problem, we refer to [Cop78, §3], [DK74, §IV.3], [LRS98], [NRS98]; and the references therein.

## c. Perturbation Theory

In this section we derive perturbation results for evolution families from the perturbation theory for semigroups.

For operators $B(t)$, $t \in \mathbb{R}$, on a Banach space $X$, define the multiplication operator $\mathcal{B} := B(\cdot)$ on $E = \mathrm{C}_0(\mathbb{R}, X)$. Notice that $\mathcal{B}$ is closed in $E$ if all operators $B(t)$ are closed in $X$. A more general version of the following result can be found in [RRSV99] (e.g., one can allow for $\omega_0(U) = \infty$), see also [RRS96]. For the special case of bounded operators $B(t)$ a similar approach was already used in [Lum85a], [Lum85b], and [NR95]. We also refer to [Eva76] and [How74] for related applications to scattering theory.

**9.19 Theorem.** *Let $\big(U(t,s)\big)_{t\geq s}$ be an exponentially bounded evolution family on a Banach space $X$. Let $B(t)$, $t \in \mathbb{R}$, be closed operators on $X$ such that $U(t,s)X \subseteq D\big(B(t)\big)$, $t \mapsto B(t)U(t,s)$ is strongly continuous and $\|B(t)U(t,s)\| \leq k(t-s)$ for $t > s$ and some locally integrable function $k : \mathbb{R}_+ \to \mathbb{R}_+$. Then there is a unique exponentially bounded evolution family $\big(U_B(t,s)\big)_{t\geq s}$ on $X$ such that*

$$(9.5) \qquad U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x \, d\tau$$

*for all $t \geq s$ and $x \in X$. Moreover, for $x \in X$ and $s \in \mathbb{R}$, we have $U_B(t,s)x \in D\big(B(t)\big)$ for almost all $t > s$, the function $B(\cdot)U_B(\cdot,s)x$ is locally integrable on $[s,\infty)$, and*

$$(9.6) \qquad U_B(t,s)x = U(t,s)x + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)x \, d\tau$$

*for all $t \geq s$ and $x \in X$. The evolution semigroup $\big(T_B(t)\big)_{t\geq 0}$ on $E = \mathrm{C}_0(\mathbb{R}, X)$ induced by $\big(U_B(t,s)\big)_{t\geq s}$ is generated by $G_B = G + \mathcal{B}$ with domain $D(G_B) = D(G) \subseteq D(\mathcal{B})$.*

Proof. Let $\mathcal{T} := \big(T(t)\big)_{t\geq 0}$ be the evolution semigroup associated with $\big(U(t,s)\big)_{t\geq s}$ on $E$, and let $G$ be its generator. It is easy to see that $T(t)E \subseteq D(\mathcal{B})$ and $\|\mathcal{B}T(t)\| \leq k(t)$ for $t > 0$. Therefore, $\mathcal{B}T(\cdot)f$ is continuous on $(0,\infty)$ and

$$\int_0^{t_0} \|\mathcal{B}T(t)f\|_\infty \, dt \leq \int_0^{t_0} k(t) \, dt \, \|f\|_\infty \leq q \, \|f\|_\infty$$

for $f \in E$, some $0 \le q < 1$, and sufficiently small $t_0 > 0$. Also,

$$\int_0^\infty \mathrm{e}^{-\lambda t} \|\mathcal{B}T(t)f\|_\infty \, dt \le \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda k} \int_0^1 \|\mathcal{B}T(t)T(k)f\|_\infty \, dt \le c\,\|f\|_\infty$$

for $\lambda > \max\{\omega_0(\mathcal{T}), 0\}$ and a suitable constant $c$. So the closedness of $\mathcal{B}$ and (1.14) in Chapter II yield $\mathcal{B} \in \mathcal{L}(E_1, E)$. By the Miyadera–Voigt perturbation theorem, see Corollary III.3.16, the operator $G_B := G + \mathcal{B}$ with $D(G_B) := D(G)$ generates a strongly continuous semigroup $\big(T_B(t)\big)_{t\ge 0}$ on $E$. For $f \in D(G_B)$ and $\varphi \in \mathrm{C}_c^1(\mathbb{R})$, Theorem 9.14 implies that $\varphi f \in D(G) = D(G_B)$ and

$$G_B(\varphi f) = -\varphi' f + \varphi\, G f + \varphi\, \mathcal{B} f = -\varphi' f + \varphi\, G_B f,$$

and hence $\big(T_B(t)\big)_{t\ge0}$ is an evolution semigroup induced by an exponentially bounded evolution family $\big(U_B(t,s)\big)_{t\ge s}$ on $X$.

Fix $x \in X$, $s \in \mathbb{R}$, $t \ge 0$, and $f \in E$ with $f(s-t) = x$. Since the mapping $f \mapsto f(s)$ is continuous from $E$ to $X$, Corollary III.3.16 implies

$$\begin{aligned} U_B(s, s-t)x &= U(s,s-t)x + \int_{s-t}^{s} U_B(s,\tau)B(\tau)U(\tau, s-t)x \, d\tau \\ &= U(s,s-t)x + \int_{s-t}^{s} U(s,\tau)B(\tau)U_B(\tau, s-t)x \, d\tau, \end{aligned}$$

and the remaining assertions except for the uniqueness.

Let $\big(V(t,s)\big)_{t\ge s}$ be another exponentially bounded evolution family on $X$ satisfying (9.5). Then the associated evolution semigroup $\big(S(t)\big)_{t\ge0}$ satisfies Corollary III.3.15.(i) for $f \in E$. Thus, by Corollary III.3.15, $T_B(t) = S(t)$, and so $U_B(t,s) = V(t,s)$ for $t \ge s$. □

**9.20 Corollary.** *Let $\big(U(t,s)\big)_{t\ge s}$ be an exponentially bounded evolution family on $X$, and let $B(\cdot) \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$. Then the conclusions of Theorem* 9.19 *hold.*

To interpret the above results, assume that $\big(U(t,s)\big)_{t\ge s}$ solves (nACP). Then, by virtue of (9.6), the function $U_B(\cdot, s)x$ can be considered as a *mild solution* of the nonautonomous Cauchy problem

$$\text{(9.7)} \qquad \begin{cases} \frac{d}{dt} u(t) = \big(A(t) + B(t)\big)\, u(t) & \text{for } t \ge s,\ t, s \in \mathbb{R}, \\ u(s) = x; \end{cases}$$

cf. [Paz83, p. 129]. However, even if $B(\cdot) \in \mathrm{C_b}\big(\mathbb{R}, \mathcal{L}(X)\big)$, well-posedness of (nACP) (in the sense of Definition 9.2) does not imply well-posedness of (9.7). This follows from the next example, which is a version of [Phi53, Expl. 6.4]; see also [Lun95, Expl. 4.1.7].

**9.21 Example.** On $X := \mathrm{C}_0[0,1)$, the operator $A\varphi := \varphi'$ with domain $D(A) := \mathrm{C}_0^1[0,1)$ generates the left translation semigroup $\big(T_l(t)\big)_{t\geq 0}$; cf. Paragraph I.4.17. Define

$$\big(B(t)\varphi\big)(\xi) := \begin{cases} \varphi(\xi) & \text{if } 2(\xi+t)\geq 1, \\ 2(\xi+t)\varphi(\xi) & \text{if } 2(\xi+t)\leq 1, \end{cases}$$

for $\varphi \in X$, $0 \leq \xi < 1$, and $t \geq 0$. Also, set $B(t) := B(0)$ for $t < 0$. Then $B(\cdot) \in \mathrm{C_b}\big(\mathbb{R}, \mathcal{L}(X)\big)$. Each classical solution $u$ of (9.7) for $s = 0$ and $A(t) := A$ satisfies

$$u(t) = T_l(t)\varphi + \int_0^t T_l(t-\tau)B(\tau)u(\tau)\,d\tau \qquad \text{for } t \geq 0;$$

see Exercise 7.10.(1). On the other hand, for all $\varphi \in X$, the unique continuous solution of this integral equation is given by

$$u(t)(\xi) = \begin{cases} 0 & \text{if } \xi + t \geq 1, \\ \mathrm{e}^t \varphi(\xi+t) & \text{if } 1 \leq 2(\xi+t) \leq 2, \\ \mathrm{e}^{2(\xi+t)t}\varphi(\xi+t) & \text{if } 0 \leq 2(\xi+t) \leq 1, \end{cases}$$

for $t \geq 0$ and $0 \leq \xi < 1$. This function $u$ solves (9.7) if and only if $\varphi \in D(A)$ and $\varphi(1/2) = 0$.

The next corollary is an easy consequence of (9.1) and the fact that the evolution family $\big(\mathrm{e}^{c(t-s)}U(t,s)\big)_{t\geq s}$ solves the "rescaled" Cauchy problem given by $A(t)+c$. Results on differentiable solutions of a perturbed parabolic problem can be found in, e.g., [Hen81, §7.1] and [RRSV99, §4]. In the latter reference, the differentiability properties are deduced from a perturbation theorem of Dore–Venni type [MP97], which is applied to the evolution semigroup.

**9.22 Corollary.** *Let $A(\cdot) + c$ satisfy the parabolic condition from Paragraph* 9.6 *for some $c \in \mathbb{R}$. Let $Z_s$ be the domain of $(-A(s) - c)^\alpha$ for some $0 \leq \alpha < 1$ endowed with the norm $\|x\|_{\alpha,s} := \|(-A(s) - c)^\alpha x\|$. Assume that $B(s) \in \mathcal{L}(Z_s, X)$ is closable in $X$ for $s \in \mathbb{R}$ and $B(\cdot)(-A(\cdot) - c)^{-\alpha} \in \mathrm{C}_b(\mathbb{R}, \mathcal{L}_s(X))$. Then the conclusions of Theorem* 9.19 *hold.*

Here is an application of this result to partial differential equations.

**9.23 Example.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a compact boundary $\partial\Omega$ of class $\mathrm{C}^2$. Let $a_{kl} = a_{lk} \in \mathrm{C_b}\big(\mathbb{R}, \mathrm{C}^1(\overline{\Omega})\big) \cap \mathrm{C}^\mu\big(\mathbb{R}, \mathrm{C}(\overline{\Omega})\big)$ for $1/2 < \mu \leq 1$ and $b \in \mathrm{C_b}\big(\mathbb{R}, \mathrm{L}^q(\Omega)\big)$ for $\max\{1, n/2\} < q \leq \infty$. Assume further that $a_{kl}$ is real-valued and *uniformly elliptic*, i.e.,

$$\sum_{k,l=1}^n a_{kl}(t,\xi)\eta_k\eta_l \geq \varepsilon\,|\eta|^2$$

for $\eta \in \mathbb{R}^n$, $\xi \in \overline{\Omega}$, $t \in \mathbb{R}$, and a constant $\varepsilon > 0$. Set $X := \mathrm{L}^p(\Omega)$ for $p \in (1, q)$. We define

$$A(t)\varphi(\xi) := \sum_{k,l=1}^{n} \tfrac{\partial}{\partial \xi_k} \Big( a_{kl}(t,\xi)\, \tfrac{\partial}{\partial \xi_l}\, \varphi(\xi) \Big),$$

$$D\big(A(t)\big) := \Big\{ \varphi \in \mathrm{W}^{2,p}(\Omega) : \sum_{k,l=1}^{n} a_{kl}(t,\xi)\, n_k(\xi)\, \tfrac{\partial}{\partial \xi_l}\, \varphi(\xi) = 0 \text{ for } \xi \in \partial\Omega \Big\}$$

on $X$, where $n(\xi)$ is the outer unit normal vector at $\xi \in \partial\Omega$. By (the proof of) [Yag90, Thm. 4.1] and [Lun95, Thm. 3.1.3] there is a constant $c$ such that $A(t)+c$ satisfies the parabolic condition from Paragraph 9.6 for $\mu > 1/2$ and each $\nu \in (0, 1/2)$.

Fix $\widetilde{q} \in (n/2, q)$ with $\widetilde{q} > p$ and $\alpha \in (n/2\widetilde{q}, 1)$. Let $1/r := 1/p - 1/\widetilde{q}$ and $1/\widetilde{r} := 1/p - 1/q$. Then $p < \widetilde{r} < r$. Due to [Hen81, Thm. 1.6.1], the space $Z_t$ is continuously embedded in $\mathrm{L}^r(\Omega)$, and it can be seen that the norm of the embedding is uniformly bounded in $t \in \mathbb{R}$; cf. [RRSV99, §5]. Thus, $Z_t$ is contained in the (maximal) domain $D\big(B(t)\big)$ of the multiplication operator $B(t)\varphi = b(t,\cdot)\varphi(\cdot)$, and $B(t) : Z_t \to X$ is uniformly bounded. Clearly, $\big(B(t), D\big(B(t)\big)\big)$ is closed in $X$. Further, using Hölder's inequality and [HS75, Thm. 13.19], we estimate

$$\begin{aligned}
&\big\|B(t)(-A(t)-c)^{-\alpha}\varphi - B(s)(-A(s)-c)^{-\alpha}\varphi\big\|_p \\
&\qquad \le \|b(t)-b(s)\|_{\widetilde{q}} \cdot \big\|(-A(t)-c)^{-\alpha}\varphi\big\|_r \\
&\qquad\quad + \|b(s)\|_q \cdot \big\|(-A(t)-c)^{-\alpha}\varphi - (-A(s)-c)^{-\alpha}\varphi\big\|_{\widetilde{r}} \\
&\qquad \le c_1 \, \|b(t)-b(s)\|_q \cdot \|\varphi\|_p \\
&\qquad\quad + c_2 \, \|b(s)\|_q \cdot \big\|(-A(t)-c)^{-\alpha}\varphi - (-A(s)-c)^{-\alpha}\varphi\big\|_p^{1-\vartheta} \\
&\qquad\qquad \cdot \big\|(-A(t)-c)^{-\alpha}\varphi - (-A(s)-c)^{-\alpha}\varphi\big\|_r^{\vartheta} \\
&\qquad \le c_3 \Big( \|b(t)-b(s)\|_q \cdot \|\varphi\|_p \\
&\qquad\quad + \big\|(-A(t)-c)^{-\alpha}\varphi - (-A(s)-c)^{-\alpha}\varphi\big\|_p^{1-\vartheta} \cdot \|\varphi\|_p^{\vartheta} \Big)
\end{aligned}$$

for $t, s \in \mathbb{R}$, $\varphi \in X$, constants $c_k$, and $\vartheta := \frac{\widetilde{r}-p}{r-p}\frac{r}{\widetilde{r}} \in (0,1)$. Using the definition of fractional powers (Definition II.5.25) one can now derive the strong continuity of $B(\cdot)(-A(\cdot)-c)^{-\alpha}$, and Corollary 9.22 can be applied.

As an immediate application of Theorem 9.18, we obtain that the hyperbolicity of $\big(U(t,s)\big)_{t \ge s}$ is preserved under "small" perturbations $B(\cdot)$. This approach was already used in [LMSR96] and [LR95] for bounded perturbations. The following results are special cases of results in [Sch99].

**9.24 Theorem.** *Let $\big(U(t,s)\big)_{t\geq s}$ be a hyperbolic evolution family on a Banach space $X$. Let $\big(U(t,s)\big)_{t\geq s}$ and $B(\cdot)$ satisfy the hypotheses of Theorem 9.19. Assume that $\|B(t)\Gamma(t,s)\| \leq \beta(t-s)$ for $t,s\in\mathbb{R}$, $\beta\in \mathrm{L}^1(\mathbb{R})$ with $\int_{\mathbb{R}}\beta(t)dt =: q < 1$ and Green's function $\Gamma(\cdot,\cdot)$. Then the perturbed evolution family $\big(U_B(t,s)\big)_{t\geq s}$ is hyperbolic.*

Proof. First note that $U_Q(t,s) = U(t,r)U_Q(r,s)$ for $t > r$ and $t < s$, where $U_Q(t,s)$ is as in Definition 9.8. Thus, $\Gamma(t,s)X \subseteq D\big(B(t)\big)$ for $t\neq s$. From Theorem 9.18 and the closedness of $B(t)$ we derive

$$(\mathcal{B}G^{-1}f)(t) = -\int_{\mathbb{R}} B(t)\Gamma(t,s)f(s)\,ds$$

for $t\in\mathbb{R}$ and $f\in E$. Hence, $\|\mathcal{B}G^{-1}\|\leq q$, and by [Kat80, IV.1.16], $G_B$ is invertible. So the result follows from Theorem 9.18. □

**9.25 Corollary.** *In the situation of Corollary 9.20, let $\big(U(t,s)\big)_{t\geq s}$ be hyperbolic with constants $N,\delta>0$. If $\sup_{t\in\mathbb{R}}\|B(t)\| < \delta/2N$, then $\big(U_B(t,s)\big)_{t\geq s}$ is hyperbolic. If $\omega_0(U)<0$ and $\sup_{t\in\mathbb{R}}\|B(t)\| < \delta/N$, then $\omega_0(U_B)<0$.*

**9.26 Corollary.** *In the situation of Corollary 9.22, let $\big(U(t,s)\big)_{t\geq s}$ be hyperbolic with constants $N,\delta>0$ and projections $P(t)$. By (9.1) there is a constant $C$ such that $\|(-A(t)-c)^\alpha U(t,s)\|\leq C(t-s)^{-\alpha}$ for $0<t-s\leq 1$. If*

$$b := \sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(Z_t,X)} < \big(NC\,(\tfrac{1}{1-\alpha}+\tfrac{1+\mathrm{e}^{-\delta}}{\delta})\big)^{-1},$$

*then $\big(U_B(t,s)\big)_{t\geq s}$ is hyperbolic. If $\omega_0(U)<0$ and*

$$\sup_{t\in\mathbb{R}}\|B(t)\|_{\mathcal{L}(Z_t,X)} < \big(NC\,(\tfrac{1}{1-\alpha}+\tfrac{1}{\delta})\big)^{-1},$$

*then $\omega_0(U_B)<0$.*

Proof. We have only to observe that $\|P(t)\|\leq N$ and

$$\big\|B(t)\Gamma(t,s)\big\| \leq \begin{cases} b\,CN(t-s)^{-\alpha}, & t-s\in(0,1],\\ b\,C\,\|U(t-1,s)P(s)\|\leq b\,CN\mathrm{e}^{-\delta(t-1-s)}, & t-s>1,\\ b\,C\,\|U_Q(t-1,s)Q(s)\|\leq b\,CN\mathrm{e}^{-\delta(s-t+1)}, & t-s<0.\end{cases}$$

□

Among the many papers on robustness of exponential dichotomy under bounded perturbations, we mention only [Cop78, §4] and [DK74, §IV.5]. Certain classes of unbounded perturbations were considered in [CL96], [Hen81], and [Lin92].

## *d. Hyperbolic Evolution Families in the Parabolic Case*

If an evolution family $\big(U(t,s)\big)_{t\geq s}$ solves (nACP) on $X$, it is desirable to derive the hyperbolicity of $\big(U(t,s)\big)_{t\geq s}$ from properties of the operators $A(t)$. As a first guess, one could assume that each $A(t)$ generates a hyperbolic semigroup $\big(\mathrm{e}^{\tau A(t)}\big)_{\tau\geq 0}$ on $X$ with uniform constants $N,\delta > 0$. However, Example 9.9 already shows that one needs an additional hypothesis. To that purpose, it is natural to require $A(\cdot)$ to be Hölder continuous (in a suitable sense) with a sufficiently small Hölder constant. In fact, such results are known for bounded operators $A(t)$, [Bas94], [Cop78, §6], and for delay equations, [Liz92]. In the sequel, we adopt ideas due to Baskakov [Bas94] to our situation and apply Theorem 9.18. We make the following assumptions.

(P) *Let $\big(A(t)+c, D\big(A(t)\big)\big)$ for each $t\in\mathbb{R}$ and some fixed $c\in\mathbb{R}$ be generators of bounded analytic semigroups $\big(\mathrm{e}^{\tau A(t)}\big)_{\tau\geq 0}$ on $X$ of the same type $(M,\delta)$. Suppose that $D\big(A(t)\big)\equiv D\big(A(0)\big)$, $A(t)$ is invertible for all $t\in\mathbb{R}$, $\sup_{t,s\in\mathbb{R}}\|A(t)A(s)^{-1}\| < \infty$, and $\|A(t)A(s)^{-1}-I\| \leq L|t-s|^{\alpha}$ for $t,s\in\mathbb{R}$ and constants $L\geq 0$ and $0<\alpha\leq 1$.*

(ED) *Assume that $(\mathrm{e}^{\tau A(t)})_{\tau\geq 0}$ is hyperbolic with projection $P_t$ and constants $N,\delta>0$ for each $t\in\mathbb{R}$. Moreover, let $\|A(t)\mathrm{e}^{\tau A(t)}P_t\|\leq\psi(\tau)$ and $\|A(t)\mathrm{e}^{-\tau A_Q(t)}Q_t\|\leq\psi(-\tau)$ for $\tau>0$ and a function $\psi$ such that $\mathbb{R}\ni s\mapsto\varphi(s):=|s|^{\alpha}\psi(s)$ is integrable.*

Here, we have set $Q_t := I-P_t$, and $\mathrm{e}^{\tau A_Q(t)}$ is the restriction of $\mathrm{e}^{\tau A(t)}$ to $Q_tX$. Observe that (P) implies the parabolic condition from Paragraph 9.6 for $A(t)+c$. Thus, there is an exponentially bounded evolution family $\big(U(t,s)\big)_{t\geq s}$ solving (nACP), and we have a corresponding evolution semigroup $\big(T(t)\big)_{t\geq 0}$ on $E=\mathrm{C}_0(\mathbb{R},X)$ with generator $G$.

We start with some preliminary facts.

**9.27 Remark.** The second sentence in (ED) is a consequence of (P) and the first part of (ED). In fact, one can choose $\psi(\tau):={}^{CN}\!/_{\tau}$ for $0<\tau\leq 1$, $\psi(\tau):=CN\mathrm{e}^{-\delta(\tau-1)}$ for $\tau>1$, and $\psi(\tau):=CN\mathrm{e}^{\delta(\tau-1)}$ for $\tau<0$, where $\|\tau A(t)\mathrm{e}^{\tau A(t)}\|\leq C$ for $0\leq\tau\leq 1$ and $t\in\mathbb{R}$. For $\alpha=1$, this gives $\|\varphi\|_1 = CN\big(1+\delta^{-2}(1+\delta+\mathrm{e}^{-\delta})\big)$.

Proof. The existence of the constant $C$ follows from (4.9) in Chapter II. Moreover, $\|P_t\|\leq N$ and

$$A(t)\mathrm{e}^{-\tau A_Q(t)}Q_t = A(t)\mathrm{e}^{A(t)}\,\mathrm{e}^{-(\tau+1)A_Q(t)}Q_t \tag{9.8}$$

for $t\in\mathbb{R}$ and $\tau\geq 0$. This implies the asserted estimates. □

In view of this remark, condition (ED) reduces to the assumption that $[-\delta,\delta]+i\mathbb{R}\subseteq\rho\big(A(t)\big)$ and that the constant $N$ does not depend on $t$, see Theorem V.1.17 and Theorem IV.3.10.

For the next result, recall the definition of the operators $G_0$ and $\mathcal{A} = A(\cdot)$ following Definition 9.11.

**9.28 Lemma.** *Assume that* (P) *holds. Then* $\big(G, D(G)\big)$ *is the closure of* $\big(G_0 + \mathcal{A}, D(G_0) \cap D(\mathcal{A})\big)$. *Further, if* $f \in D(G) \cap D(\mathcal{A})$, *then* $f \in D(G_0)$.

PROOF. Using rescaling, we may assume that $c = 0$ in (P). Due to Remark 9.12.(iii) (or [LMSR96, Prop. 2.9]), the first assertion follows from $G_0 + \mathcal{A} \subseteq G$. In fact, we have

$$\begin{aligned} \big\|\tfrac{1}{t}\big(T(t)f - f\big) &- (-f' + \mathcal{A}f)\big\|_\infty \\ &\le \sup_{s\in\mathbb{R}} \Big( \big\|\tfrac{1}{t}\big(U(s+t,s)f(s) - f(s)\big) - A(s)f(s)\big\| \\ &\qquad\qquad + \|A(s)f(s) - A(s+t)f(s+t)\| \\ &\qquad\qquad + \big\|\tfrac{1}{t}\big(f(s) - f(s+t)\big) + f'(s)\big\| + \|f'(s+t) - f'(s)\|\Big) \end{aligned}$$

for $f \in D(G_0) \cap D(\mathcal{A})$ and $t > 0$. Clearly, the second, third, and fourth term on the right-hand side tend to 0 as $t \downarrow 0$. Further, one has

$$\begin{aligned} \big\|\tfrac{1}{t}\big(U(s+t,s)f(s) - f(s)\big) - A(s)f(s)\big\| &= \Big\|\tfrac{1}{t}\int_s^{s+t} \big(A(\tau)U(\tau,s) - A(s)\big)f(s)\,d\tau\Big\| \\ &\le \sup_{s\in\mathbb{R}, s\le\tau\le s+t} \big\|\big(A(\tau)U(\tau,s) - A(s)\big)f(s)\big\|. \end{aligned}$$

Since $(\tau, s) \mapsto A(\tau)U(\tau,s)A(s)^{-1}$ is strongly continuous and uniformly bounded for $s \le \tau \le s+1$ by [Ama95, Thm. II.4.4.1], the first summand also converges to 0 as $t \downarrow 0$.

Second, let $f \in D(G) \cap D(\mathcal{A})$. Then,

$$\tfrac{1}{t}\big(f(s-t) - f(s)\big) = \tfrac{1}{t}\big(U(s,s-t)f(s-t) - f(s)\big) + \tfrac{1}{t}\big(1 - U(s,s-t)\big)f(s-t).$$

The first term on the right-hand side converges in $E$ as $t \downarrow 0$. The convergence of the second one follows as above if one observes that

$$\begin{aligned} \big\|\tfrac{1}{t}\big(1 - U(\cdot,\cdot - t)\big)f(\cdot - t) + \mathcal{A}f\big\|_\infty \le{}& \big\|\tfrac{1}{t}\big(1 - U(\cdot + t,\cdot)\big)f(\cdot) + \mathcal{A}f\big\|_\infty \\ &+ \|\mathcal{A}f(\cdot + t) - \mathcal{A}f\|_\infty. \end{aligned}$$

□

By virtue of (ED), we can define

$$\Gamma_s(\tau) := \begin{cases} \mathrm{e}^{\tau A(s)}P_s, & \tau \ge 0,\ s \in \mathbb{R}, \\ -\mathrm{e}^{\tau A_Q(s)}Q_s, & \tau < 0,\ s \in \mathbb{R}. \end{cases}$$

Recall that the projections $P_t$ in (ED) are given by the spectral projections

$$P_t = \frac{1}{2\pi i}\int_\Gamma R(\lambda, \mathrm{e}^{A(t)})\,d\lambda$$

with respect to the spectral set $\big\{\lambda \in \sigma(\mathrm{e}^{A(t)}) : |\lambda| < 1\big\}$; cf. Section V.1.c.

**9.29 Lemma.** *Let* (P) *and* (ED) *hold. Then the mappings*

$$(s,\tau)\mapsto \Gamma_s(\tau)\in\mathcal{L}(X) \quad \text{and} \quad (t,s,\tau)\mapsto A(t)\Gamma_s(\tau)\in\mathcal{L}(X)$$

*are continuous for* $\tau\neq 0$ *and* $t,s\in\mathbb{R}$.

Proof. By (4.2) in Section II.4.a, the mapping $\tau\mapsto \mathrm{e}^{\tau A(s)}\in\mathcal{L}(X)$ is continuous for $\tau>0$ uniformly in $s\in\mathbb{R}$. It is straightforward to show that

$$\|R(\lambda,A(t)+c)-R(\lambda,A(s)+c)\|\le LM(M+1)\,|t-s|^{\alpha}\cdot|\lambda|^{-1}$$

for $t,s\in\mathbb{R}$ and $\lambda\in\Sigma_{\pi/2+\delta}$. So the representation (4.2) in Chapter II of $\mathrm{e}^{\tau A(t)}$ yields the Hölder continuity of $t\mapsto \mathrm{e}^{\tau A(t)}\in\mathcal{L}(X)$ uniformly for $0\le\tau\le d$. Moreover, due to (ED), the resolvents

$$R(\lambda,\mathrm{e}^{A(t)})=\sum_{n=0}^{\infty}\lambda^{-(n+1)}\,\mathrm{e}^{nA(t)}P_t-\sum_{n=1}^{\infty}\lambda^{n-1}\,\mathrm{e}^{-nA_Q(t)}Q_t$$

are uniformly bounded for $t\in\mathbb{R}$ and $|\lambda|=1$. Thus, $R(\lambda,\mathrm{e}^{A(t)})$ and hence the spectral projections $P_t$ are Hölder continuous with respect to $t$. Further, one has

$$\begin{aligned}\mathrm{e}^{-\tau A_Q(t)}Q_t-\mathrm{e}^{-\sigma A_Q(s)}Q_s&=\mathrm{e}^{-\tau A_Q(t)}Q_t\,(Q_t-Q_s)\\&\quad+\mathrm{e}^{-\tau A_Q(t)}Q_t\left(\mathrm{e}^{\sigma A(s)}-\mathrm{e}^{\tau A(t)}\right)\mathrm{e}^{-\sigma A_Q(s)}Q_s\\&\quad+(Q_t-Q_s)\,\mathrm{e}^{-\sigma A_Q(s)}Q_s\end{aligned}$$

for $t,s\in\mathbb{R}$ and $\tau,\sigma>0$. Therefore, $(s,\tau)\mapsto\Gamma_s(\tau)\in\mathcal{L}(X)$ is continuous for $\tau\neq0$. From [Paz83, Lem. 5.6.2] and (9.8) it now follows that $(t,s,\tau)\mapsto A(t)\Gamma_s(\tau)\in\mathcal{L}(X)$ is continuous for $\tau\neq0$. ☐

We now give a sufficient condition for the hyperbolicity of $\big(U(t,s)\big)_{t\ge s}$ taken from [Sch99]. We remark that [Cop78, Prop. 6.2] shows that, roughly speaking, (ED) is a necessary condition if $A(\cdot)$ has a small Lipschitz constant.

**9.30 Theorem.** *Assume that* (P) *and* (ED) *hold. Let* $q:=L\|\varphi\|_1<1$. *Then* $\big(U(t,s)\big)_{t\ge s}$ *is hyperbolic with an exponent* $0<\delta'<{}^{\delta(1-q)}/_{2N}=:\eta$.

Proof. (1) Using (ED), we define for $f\in E=\mathrm{C}_0(\mathbb{R},X)$ and $t\in\mathbb{R}$ the operators

$$\begin{aligned}(Rf)(t)&:=\int_{-\infty}^{\infty}\Gamma_s(t-s)f(s)\,ds\\&=\int_{-\infty}^{t}\mathrm{e}^{(t-s)A(s)}P_s\,f(s)\,ds-\int_t^{\infty}\mathrm{e}^{(t-s)A_Q(s)}Q_s\,f(s)\,ds,\\(Lf)(t)&:=\int_{-\infty}^{\infty}\Gamma_t(t-s)f(s)\,ds\\&=\int_{-\infty}^{t}\mathrm{e}^{(t-s)A(t)}P_t\,f(s)\,ds-\int_t^{\infty}\mathrm{e}^{(t-s)A_Q(t)}Q_t\,f(s)\,ds.\end{aligned}$$

By means of Lemma 9.29, it is straightforward to verify that $R, L \in \mathcal{L}(E)$ and $\|R\|, \|L\| \leq {}^{2N}/_{\delta}$. Let $G$ be the generator of the evolution semigroup $\big(T(t)\big)_{t\geq 0}$ on $E = \mathrm{C}_0(\mathbb{R}, X)$ induced by $\big(U(t,s)\big)_{t\geq s}$. We show in step (2) and (3) that $G$ is bijective.

(2) For $f \in D(\mathcal{A})$, we have

$$\int_{-\infty}^{\infty} A(t)\Gamma_s(t-s)f(s)\,ds = \int_{-\infty}^{\infty} A(t)A(s)^{-1}\,\Gamma_s(t-s)A(s)f(s)\,ds.$$

Due to (ED) and Lemma 9.29, $Rf(t) \in D\big(A(t)\big)$ and $\mathcal{A}Rf \in E$. Moreover,

$$\left(\tfrac{d}{dt}Rf\right)(t) = P_t f(t) + Q_t f(t) + \int_{-\infty}^{\infty} \Gamma_s(t-s)A(s)f(s)\,ds,$$

so that ${}^{d}/_{dt}Rf \in E$. Now, Lemma 9.28 yields $RD(\mathcal{A}) \subseteq D(\mathcal{A}) \cap D(G_0) \subseteq D(G)$ and $(GR+I)f = Sf$ for $f \in D(\mathcal{A})$, where

$$(Sf)(t) := \int_{-\infty}^{\infty} \big(A(t) - A(s)\big)\,\Gamma_s(t-s)f(s)\,ds.$$

Using (P), (ED), and Lemma 9.29, we obtain for $f \in E$ and $t \in \mathbb{R}$ that

$$\begin{aligned} \|Sf(t)\| &\leq \int_{-\infty}^{\infty} \big\|\big(A(t)-A(s)\big)A(s)^{-1}\big\|\;\|A(s)\Gamma_s(t-s)f(s)\|\,ds \\ &\leq L\,\|f\|_\infty \int_{-\infty}^{\infty} \varphi(t-s)\,ds = q\,\|f\|_\infty \end{aligned}$$

and $Sf \in E$. This means that $S$ is a strict contraction on $E$. Thus, $S - I$ is invertible.

Fix $g \in E$. Let $f := (S-I)^{-1}g$ and choose $D(\mathcal{A}) \ni f_n \to f$. Then $Rf_n \to Rf$ and $GRf_n = (S-I)f_n \to g$ in $E$. Since $G$ is closed, $Rf \in D(G)$ and $GRf = g$. So $G$ is surjective and has the right inverse $R(S-I)^{-1}$.

(3) Let $f \in D(\mathcal{A}) \cap D(G_0) \subseteq D(G)$. Integrating by parts, we compute

$$\begin{aligned} (LGf)(t) &= \int_{-\infty}^{\infty} \Gamma_t(t-s)A(s)f(s)\,ds - \int_{-\infty}^{\infty} \Gamma_t(t-s)f'(s)\,ds \\ &= \int_{-\infty}^{\infty} \Gamma_t(t-s)A(s)f(s)\,ds - (P_t+Q_t)f(t) \\ &\quad - \int_{-\infty}^{\infty} \Gamma_t(t-s)A(t)f(s)\,ds \\ &= -\,f(t) + (Vf)(t) \end{aligned} \tag{9.9}$$

for $t \in \mathbb{R}$, where

$$(Vf)(t) := \int_{-\infty}^{\infty} \Gamma_t(t-s)\big(A(s)-A(t)\big)f(s)\,ds.$$

Again, for $f \in D(\mathcal{A})$, conditions (P) and (ED) and Lemma 9.29 yield

$$\begin{aligned}
&\|A(t)\,(Vf)(t)\| \\
&\qquad \le \int_{-\infty}^{\infty} \|A(t)\Gamma_t(t-s)\| \cdot \big\|\big(A(t)-A(s)\big)A(s)^{-1}\big\| \cdot \|A(s)f(s)\|\,ds \\
&\qquad \le L\,\|\mathcal{A}f\|_\infty \int_{-\infty}^{\infty} \varphi(t-s)\,ds = q\,\|\mathcal{A}f\|_\infty
\end{aligned}$$

and $Vf \in D(\mathcal{A})$. Therefore, $V$ is a strict contraction on $D(\mathcal{A})$ endowed with the norm $\|f\|_{\mathcal{A}} := \|\mathcal{A}f\|_\infty$ (notice that $\mathcal{A}^{-1} = A(\cdot)^{-1} \in \mathcal{L}(E)$ by (P)).

Assume $Gf = 0$ for some $f \in D(G)$. Then it follows that $f(s) = \big(T(1)f\big)(s) = U(s,s-1)f(s-1)$ for $s \in \mathbb{R}$. Since the function $s \mapsto A(s)U(s,s-1) \in \mathcal{L}(X)$ is strongly continuous and uniformly bounded by (9.1), we have $f \in D(\mathcal{A}) \cap D(G)$. Thus, $f \in D(\mathcal{A}) \cap D(G_0)$ due to Lemma 9.28. Now, the identity (9.9) implies $Vf = f + LGf = f$, and hence $f = 0$.

(4) Steps (2) and (3) yield $0 \in \rho(G)$ and $G^{-1} = R(S-I)^{-1}$. Consequently, $\|G^{-1}\| \le \eta^{-1}$. Due to the symmetry of $\sigma(G)$, see Theorem 9.15, this implies $(-\eta,\eta) + i\mathbb{R} \subseteq \rho(G)$. Using rescaling and Theorem 9.18, one sees that the evolution families $\big(\mathrm{e}^{\pm\delta'(t-s)}U(t,s)\big)_{t\ge s}$ are hyperbolic with exponent $0 \le \delta' < \eta$ and the same projections $P(s)$. □

## *Notes and Further Reading to Section* 9

Well-posedness of abstract and concrete nonautonomous Cauchy problems is treated in the books [Ama95], [Fat83], [Gol85], [Kre71], [Lun95], [Paz83], [Tan79], and [Tan97]. The asymptotic behavior of solutions is investigated in, for instance, [Cop78], [DK74], and [Hen81] and, in the time parabolic case, [DKM92], [Lun95]. The approach via evolution semigroups is developed systematically in the monograph [CL99], where one finds plenty of further references.

# Chapter VII

# A Brief History of the Exponential Function

**(by Tanja Hahn and Carla Perazzoli*)**

## 1. A Bird's-Eye View

From a philosophical perspective, the exponential function may be viewed as a link between the seemingly contradictory positions of HERACLITUS on the one side and PARMENIDES on the other, as quoted in the Epilogue (p. 549). While the time-dependent function $t \mapsto T(t)$—the *semigroup*—reflects the aspect of permanent change in a deterministic autonomous system, its *generator* $A$ stands for the eternal, timeless principle behind the system. The exponential function ties both aspects together through the formula

$$T(t) = \exp(tA).$$

From the beginning, the scalar exponential function $t \mapsto \exp(ta)$ drew much of its significance from two very peculiar properties it enjoys. For one thing, it satisfies the functional equation

$$\text{(FE)} \qquad f(t+s) = f(t) \cdot f(s).$$

On the other hand, it satisfies the differential equation

$$\text{(DE)} \qquad \frac{du(t)}{dt} = au(t).$$

(FE) expresses the idea behind the slide rule and was systematically exploited for the first time by JOHN NAPIER (1550–1617). (DE), on the other

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hand, grew out of an apparently quite different circle of ideas: The growth rate of an amount of money under the influence of continuously calculated compound interest is proportional, at any time, to the amount attained at that time. It was LEONHARD EULER (1707–1783) who put earlier results into a coherent context and showed that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n = \mathrm{e}^t,$$

with $\mathrm{e} = \lim_{n\to\infty}(1 + {}^1/_n)^n$ (see [Eul48], see also Section 3).

In general terms, we could say that if we identify $\mathcal{L}(\mathbb{C})$ with $\mathbb{C}$, (FE) is the law of linear, autonomous, and deterministic evolution (see Epilogue, Section 1), while (DE) describes the same phenomenon in the language of calculus. Based on this interpretation, one might think that only very special autonomous systems can possibly be described by (DE), which presupposes differentiability. Our understanding and appreciation of the exponential function unfolds with the realization that (DE) is very close to (FE) far beyond the realm of classical calculus and that the scalar exponential function serves us well as a guide into unknown territory. The following is a striking example of visionary reasoning, under the guidance of the scalar situation, by one of the great masters of calculus. In 1772, JOSEPH LOUIS LAGRANGE (1736–1830) ventured to write Taylor's formula

$$g(s+t) = \sum_{n=0}^{\infty} \frac{t^n g^{(n)}(s)}{n!}$$

as

$$g(s+t) = \exp\left(t\,\tfrac{d}{dt}\right) g(s);$$

see [Lag72]. On might be tempted to disqualify this as purely formal thinking that need not even make sense for $C^\infty$-functions. A more adequate comment, using the modern concept of exponential function, would be that it took almost two hundred years and the invention of functional analysis and modern semigroup theory to realize that LAGRANGE'S formula is correct if we only suppose $f$ to be integrable: The translation semigroup is generated by the first derivative on the space of integrable functions on the line.

LAGRANGE did not care much about the justification of this formula. Rather, he used it with great skill, and it is quite interesting that this sort of symbolic operational calculus anticipated in many details vital parts of the modern treatment of evolution equations. LAGRANGE himself noted that his formula led him to find many new theorems that otherwise would have been difficult to discover. Other authors who used, in different disguises, formulas like LAGRANGE'S include JEAN-BAPTISTE JOSEPH FOURIER (1768–1830) (see, e.g., [Fou22]), GEORGE BOOLE (1815–1864), and OLIVER HEAVISIDE (1850–1925).

Boole and Heaviside were especially outspoken concerning the benefits of the symbolic calculus. Here is an excerpt from Boole's book [Boo59, pp. 388–389]:

*There exist forms of the functional symbol $f$, for which we can attach a meaning to the expression $f(m)$, but cannot directly attach a meaning to the symbol $f\,({}^{d}\!/_{dx})$. And the question arises: Does this difference restrict our freedom in the use of that principle which permits us to treat expressions of the form $f\,({}^{d}\!/_{dx})$ as if ${}^{d}\!/_{dx}$ were a symbol of quantity? For instance, we can attach no direct meaning to the expression $\mathrm{e}^{h\,{}^{d}\!/_{dx}} f(x)$, but if we develop the exponential as if ${}^{d}\!/_{dx}$ were quantitative, we have*

$$\mathrm{e}^{h\,{}^{d}\!/_{dx}} f(x) = \left(1 + h\frac{d}{dx} + \frac{1}{1\cdot 2}h^2\frac{d^2}{dx^2} + \&c.\right) f(x) = f(x+h)$$

*by Taylor's theorem. Are we then permitted, on the above principle, to make use of symbolic language; always supposing that we can, by the continued application of the same principle, obtain a final result of interpretable form?*

He answered his question in the affirmative:

*Now all special instances point to the conclusion that this is permissible, and seem to indicate, as a general principle, that the mere processes of symbolical reasoning are independent of the conditions of their interpretation.*

At the end of the nineteenth century Heaviside commented in [Hea93] on a similar (rhetorical) question:

*Shall I refuse my dinner because I do not fully understand the process of digestion? No, not if I am satisfied with the result.*

Of course, a rigorous foundation for these ideas required the basic concepts of functional analysis. Still, also in this respect a few pioneering contributions stand out and stun us by their visionary quality. Thus the work of Giuseppe Peano (1858–1932) in 1887 on systems of ordinary linear differential equations with constant coefficients, where the exponential of a complex matrix is the decisive tool, is perfectly prepared for an interpretation in infinite-dimensional Banach spaces. And Marshal Harvey Stone's (1903–1989) representation, in 1930, of a group of unitary operators as the exponential of a skew-adjoint unbounded operator paved the way for the Hille–Yosida theorem of 1947. A synopsis of the state of affairs achieved by that time can be found in the book of Carl Einar Hille (1894–1980) on one-parameter semigroups, which appeared in 1948 and comprises a good portion of modern functional analysis. In its revised edition of 1957, coauthored and strongly influenced by Ralph S. Phillips (†1998), this book is

a focal point in the history of operator semigroups insofar as it represents an encyclopedic achievement, firmly anchoring semigroup methods in the field of evolution equations. Here we find a careful analysis of the influence of continuity properties on the way an operator semigroup may be understood as the exponential of its infinitesimal generator, and ample evidence for the philosophy that the exponential function governs deterministic and autonomous evolution. Eventually, it became customary to reserve the notation $T(t) = \exp(tA)$ mainly for the strongly continuous situation. More general cases do occur but seem to be of lesser importance, since techniques have been developed to produce a strongly continuous situation through modifications of the underlying Banach space (see, e.g., Proposition II.6.6).

We have concentrated in this short survey on aspects of the exponential function that particularly reflect the spirit of this book. There are other aspects, equally interesting and important and also intimately related to evolution equations, such as the Fourier and the Laplace transforms, and the theory of Lie groups. We do not attempt to place these aspects into our picture. Rather, we give up our general point of view now and proceed with a more elaborate account of specific details.

## 2. The Functional Equation

In connection with the use and the computation of logarithms, the equation (FE) had implicitly attracted much attention since the time of Napier. However, it was Augustin Louis Cauchy (1789–1857) who approached this equation for the first time in a systematic way. In 1821 he published his *Cours d'Analyse* [Cau21] and considered on pages 98–113 and 220–229 in detail the real and complex functional equation

$$\varphi(s+t) = \varphi(s) \cdot \varphi(t) \tag{FE}$$

as well as the equations

$$\begin{aligned} \varphi(s+t) &= \varphi(s) + \varphi(t), \\ \varphi(s \cdot t) &= \varphi(s) + \varphi(t), \\ \varphi(s \cdot t) &= \varphi(s) \cdot \varphi(t) \end{aligned}$$

(cf. the quotation in Section I.1). Trying to determine all continuous solutions, he showed that the general continuous solution of (FE) is given by the exponential function. Although Cauchy is generally considered the father of rigorous calculus, he was not aware of the notions of uniform continuity and of uniform convergence, so some of his arguments are not quite convincing. He may have thought of the following (wrong) principle, which appears on p. 234 of his book [Cau21] and which would have filled the gaps:

> *Lorsque les différents termes de la série $u_0 + u_1 + u_2 + \cdots$ sont des fonctions d'une même variable $x$, continues par rapport à cette variable dans le voisinage d'une valeur particulière pour laquelle la série est convergente, la somme $s$ de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de $x$.*[1]

Niels Henrik Abel (1802–1829) gave a counterexample[2] in [Abe26] and clarified some details, so partial credit for the following theorem goes to Abel.

**2.1 Theorem.** (Cauchy 1821, Abel 1826). *The nontrivial continuous complex-valued solutions of* (FE) *on the real line are exactly the functions* $\mathbb{R} \ni t \mapsto \varphi_a(t) := \exp(ta)$ *with* $a \in \mathbb{C}$.

It is remarkable that, as a result, a continuous solution of (FE) must necessarily be smooth. Today it would be natural to immediately comment this result by asking whether weaker conditions on $\varphi$ also force $\varphi$ to be an exponential. Actually, it was more than 70 years later that David Hilbert (1862–1943) asked this question at the International Congress of Mathematicians in Paris (cf. Paragraph I.1.6).

In fact, Stefan Banach (1892–1943) and Wacław Sierpiński (1892–1969) then proved in [Ban20] and [Sie20] that it is enough to assume measurability of $\varphi$.

**2.2 Theorem.** (Banach, Sierpiński 1920). *Let $\varphi$ be a nontrivial measurable solution of* (FE) *on the real line. Then there exists a unique $a \in \mathbb{C}$ such that $\varphi(t) = \exp(ta)$ for all $t \in \mathbb{R}$.*

The question whether there exist other solutions of (FE) at all had already been solved by Georg Hamel (1897–1954) in 1905. In his famous paper [Ham05] on the functional equation

$$f(x+y) = f(x) + f(y) \tag{2.1}$$

he considered the real numbers $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and obtained solutions $f$ of (2.1) that are not continuous. Applying the exponential function on both sides of (2.1) then yields a solution $\varphi := \exp \circ f$ of (FE) that is not continuous. By Banach's and Sierpiński's result such a solution cannot be measurable, and one can summarize the result as follows (see Exercise I.1.7.(1)).

---

[1] If the different terms of the series $u_0 + u_1 + u_2 + \cdots$ are functions of the same variable $x$, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, then also the sum $s$ of the series is, in the neighborhood of this particular value, a continuous function with respect to $x$.

[2] A beautiful picture illustrating this episode can be found in [HW97, p. 171].

**2.3 Theorem.** (HAMEL 1905). *There exist solutions of* (FE) *that are not measurable. Furthermore, all solutions of* (FE) *can be written in the form* $\varphi = \exp \circ f$, *where* $f$ *solves the functional equation* (2.1).

## 3. The Differential Equation

Motivated by RENÉ DESCARTES'S (1596–1650) treatise on geometry [Des37], F. DEBEAUNE (1601–1652) posed in 1638 the problem of finding a curve $y = f(x)$ such that for any point $P$ on the curve the distance between the abscissa belonging to $P$ and the point where the tangent in $P$ cuts the $x$-axis has a constant, preassigned value $c$ (see [HW97, p. 25] for details). The problem turned out to be quite hard. GOTTFRIED WILHELM LEIBNIZ (1646–1716) proposed an approximate step-by-step construction of $f(x_0 + kb)$ for a small increment $b$, which led to the formula

$$f(x_0 + kb) = \left(1 + \frac{b}{c}\right)^k f(x_0).$$

The connection to the compound interest formula is obvious. LEIBNIZ himself was not fully satisfied, and it took the ingenuity of EULER to bring the idea to a conclusion: If we put $x_0 = 0$, $f(x_0) = c = 1$ and aim at a fixed point $t > 0$, approximating it in $k$ steps of length $t/k$, then Leibniz's formula yields the approximation

$$f_k(t) = \left(1 + \frac{t}{k}\right)^k.$$

Isolating the case $t = 1$, EULER proved[3] through a bold application of the binomial formula the convergence of these terms to

$$\mathrm{e} := \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

He then put $t/k := 1/n$ and obtained

$$f(t) = \lim_{k\to\infty} \left(1 + \frac{t}{k}\right)^k = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{nt} = \mathrm{e}^t = \exp(t).$$

Since $(1 + t/k)^k\,(1 - t/k)^k$ tends to 1 as $k \to \infty$, we also have

$$\exp(t) = \lim_{k\to\infty} \left(1 - \frac{t}{k}\right)^{-k}.$$

[3] Euler did not specifically refer to Debeaune's problem, but rather to several other problems in connection with the compound interest formula.

Now, Debeaune's curve with constant $c$ solves

$$\frac{df(t)}{dt} = \frac{1}{c} \cdot f(t);$$

hence the solution of (DE) with $u(0) = u_0$ is

$$u(t) = \exp(at) \cdot u_0.$$

EULER'S treatment of the exponential function was ingenious, but depended largely on his superior intuition. His delicate convergence arguments were put on a sound basis by the work of CAUCHY (and others), so that in 1887 PEANO was able to leave the one-dimensional context and to deal, in a rigorous way, with systems of ordinary linear differential equations with constant coefficients. Using the matrix and vector notation he had introduced, he wrote the system

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \alpha_{11}x_1(t) + \cdots + \alpha_{1n}x_n(t), \\ \vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\ \frac{dx_n(t)}{dt} &= \alpha_{n1}x_1(t) + \cdots + \alpha_{nn}x_n(t) \end{aligned} \tag{3.1}$$

as

$$\frac{d\mathrm{x}(t)}{dt} = \alpha\mathrm{x}(t),$$

where $\alpha$ stands for the matrix of coefficients

$$\alpha = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}.$$

In complete analogy to the one-dimensional case he found the solution as

$$\mathrm{x}(t) := \mathrm{e}^{t\alpha}\mathrm{x}(0),$$

where he defined $\mathrm{e}^{t\alpha} := \sum_{n=0}^{\infty} {t^n\alpha^n}/{n!}$ and proved convergence of this series (see the quotation from [Pea87] in Section I.2). We can state his result as follows.

**3.1 Theorem.** (PEANO 1887). *Let $A$ be a complex $m \times m$ matrix. Then the series*

$$U(t) := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

*converges for all $t \in \mathbb{R}$, and $t \mapsto \mathrm{x}(t) := U(t)\mathrm{x}_0$ is the unique solution of the system* (3.1) *with the initial condition $\mathrm{x}(0) = \mathrm{x}_0$.*

Subsequently, it was one of G. Peano's students, Maria Gramegna (†1915), who again took up this idea in 1910 and generalized Peano's result in [Gra10] to infinite systems of differential equations and to integral equations. A few years before, in 1894, Henri Poincaré (1854–1912) had already dealt successfully with infinite systems. But he considered only special cases (see, for example, [Poi94]). By contrast, Gramegna used very general methods for solving these problems. This gives her work a modern flair. She defined convergence with respect to the sup-norm $\|\cdot\|_\infty$ and considered linear operators and the operator norm on $\ell^\infty$. With these tools she introduced the exponential function for bounded operators and proved convergence of the exponential series. She wrote:

> *Allora abbiasi un sistema di infinite equazioni differenziali lineari con infinite incognite:*
>
> $$\frac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \cdots$$
> $$\frac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \cdots$$
> $$\cdots\cdots\cdots\cdots\cdots\cdots$$
>
> *dove le $u$ sono costanti rispetto al tempo. Indichiamo con $a$ la sostituzione rappresentata dalla matrice delle $u$, (...). Chiamo $x$ il complesso $(x_1, x_2, \ldots)$ et sia $x_0$ il suo valore iniziale. Le equazioni differenziali date si potranno scrivere: $Dx = ax$. E l'integrale è: $x_1 = \mathrm{e}^{ta}x_0$ ossia i diversi valori di $x$ corrispondenti ai diversi valori di $t$ si hanno applicando al complesso $x_0$ la sostituzione $\mathrm{e}^{at}$ cioè la sostituzione:*
>
> $$1 + ta + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \cdots.$$ [4]

After that a similar treatment of integral equations followed. Using modern terminology, she proved the following.

[4] We now consider an infinite system of differential linear equations with an infinite number of unknowns:

$$\frac{dx_1}{dt} = u_{11}x_1 + u_{12}x_2 + \cdots$$
$$\frac{dx_2}{dt} = u_{21}x_1 + u_{22}x_2 + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where the $u$ are constant with respect to time. Let us denote by $a$ the substitution represented by the matrix of the $u$'s, (...). Let $x$ be the vector $(x_1, x_2, \ldots)$ and $x_0$ its initial value. One can then write the given differential equation as: $Dx = ax$. And the integral is given by $x_1 = \mathrm{e}^{ta}x_0$ or also with values of $x$ corresponding to values of $t$; taking the totality $x_0$ under consideration, one can confer that $\mathrm{e}^{at}$ has the representation:

$$1 + ta + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \cdots.$$

**3.2 Theorem.** (GRAMEGNA 1910).

(i) *If* $A = (a_{ij}) \in \mathcal{L}(\ell^\infty)$, *then the series* $U(t) := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ *converges, and the function* $t \mapsto \big(x_k(t)\big)_{k\in\mathbb{N}} := U(t)\big((x_k)_{k\in\mathbb{N}}\big)$ *is the unique solution of the infinite system of differential equations*

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + \cdots, & x_1(0) &= x_1,\\
&\vdots & &\vdots\\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + \cdots, & x_n(0) &= x_n,\\
&\vdots & &\vdots
\end{aligned}$$

(ii) *Let* $Af := \int_0^1 k(\cdot,y)f(y)dy$ *for* $k \in \mathrm{C}([0,1]^2)$ *and* $f \in \mathrm{C}[0,1]$. *Then* $U(t) := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ *converges, and* $t \mapsto u(t) := U(t)f$ *is the unique solution of the integro-differential equation*

$$\frac{\partial u(t,x)}{\partial t} = \int_0^1 k(x,y)\,u(t,y)\,dy, \quad u(0) = f.$$

With respect to GRAMEGNA'S article,[5] H.C. Kennedy wrote in his biography of PEANO [Ken80, p. 132].

*Many of those who wrote theses under* PEANO'S *direction were women and, according to Terracini, they were not always well prepared, but the 1910 graduate,* MARIA GRAMEGNA, *was one of the most promising. In fact,* PEANO *had presented a long article by her on differential and integral equations to the Academy of Sciences on 13 March, 1910. In it she anticipated the modern application of matrix theory to the study of systems of differential equations; the idea for this probably came from* PEANO. *Her abilities were not to be realized, however, for she went to teach at the Normal School in Avezzano (L'Aquila) and died on 13 January, 1915, a victim of the earthquake that destroyed that town and killed 96% of its inhabitants.*

[5] When I was looking for the volume of the "Atti della Reale Accademia delle Science di Torino" containing GRAMEGNA'S article in the library of the Department of Mathematics at the University of Rome "La Sapienza" in 1994, I was utterly surprised to find the respective pages of this volume still connected so that I had to cut them open in order to read the text. I must say that I could not refrain from being touched by the fact that this important article had gone unnoticed until the very day I held it in my hands (C.P.).

## 4. The Birth of Semigroup Theory

With the results of HAMEL, BANACH, and SIERPIŃSKI the functional equation (FE) in its classical meaning was completely solved by 1920. However, in the meantime a more general perspective with respect to this equation had gradually emerged. It was JACQUES HADAMARD (1865–1963) who pointed out in [Had24] that what is usually called HUYGENS' principle in the theory of propagation of waves has as a consequence an abstract principle, applicable to autonomous Cauchy problems: If such a problem admits unique solutions for all times, then these solutions are the orbits of the initial values under a (semi) group of transformations (see Epilogue, Section 1, and p. 152). Thus an operator-valued version of (FE), which is the (semi) group law

$$T(t+s) = T(t)\,T(s), \tag{FE$^*$}$$

became increasingly important.

On the other hand, such Cauchy problems are, in modern language, a Banach-space-valued version of (DE). Taking into account the classical results concerning (FE) and (DE) and the success of PEANO'S strategy, (FE$^*$) and the abstract Cauchy problem from Definition II.6.1 appear like the two sides of the same medal. Semigroup theory in the sense of this book is the fusion of these two aspects into one coherent theory. However, the extent to which the exponential function is the connecting link is truly surprising and was not immediately realized. In the Hilbert space context, M.H. STONE developed the operational calculus for general (unbounded) self-adjoint transformations and characterized strongly continuous unitary groups in [Sto30] (see also Theorem II.3.24).[6]

**4.1 Theorem.** (STONE 1930).

(i) *Let $A$ be a self-adjoint operator in a Hilbert space $H$. Then $U(t) = \mathrm{e}^{itA}$, $-\infty < t < \infty$, is a family of unitary transformations with the group property*

$$U(s+t) = U(s) \cdot U(t) \tag{4.1}$$

*and the continuity property*

$$U(s) \to U(t) \quad \text{in the strong sense, when } s \to t. \tag{4.2}$$

(ii) *If $U(t)$, $-\infty < t < \infty$, is a family of unitary transformations on a Hilbert space for which the properties* (4.1) *and* (4.2) *hold, then there exists a unique self-adjoint mapping $A$ with $U(t) = \mathrm{e}^{itA}$ for all $t \in \mathbb{R}$.*

[6] The genesis of this theorem is an interesting story by itself. We refer to [Sto32b] and the foreword in [Sto32a].

Outside the Hilbert space scenario, the advance was quite cautious, and it appears as if, for some time, nobody dared to hope that, *as a rule*, differential operators could be plugged into the exponential function. We mention publications of D.S. NATHAN [Nat35], M. NAGUMO [Nag36], and KOSAKU YOSIDA (1909–1990) [Yos36] that seem to be particularly noteworthy and contained the following result.

**4.2 Theorem.** (NATHAN 1935, NAGUMO 1936, YOSIDA 1936).

(i) *Let $A$ be a bounded linear operator on a Banach space $X$. Then $A$ generates a norm-continuous solution $U$ of* (FE*) *via the formula*

$$U(t) = \exp(tA) := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}. \tag{4.3}$$

(ii) *Let $U$ be a norm-continuous solution of* (FE*) *in a Banach space. Then there exists a unique bounded operator $A$ such that $U$ is given by* (4.3).

Of course, the situation where the semigroup is not uniformly continuous is the real challenge. Then the exponential series does not seem to be of much value, cf. Exercise II.3.12.(2.ii). Still, the following result, due to IZRAIL MOISEEVITCH GELFAND (*1913) [Gel39], indicates that the exponential function retains its value. We point out that weak continuity is the same as strong continuity in this context (cf. Theorem I.5.8), a fact GELFAND was aware of, and that the group property is essential in order to get the result.

**4.3 Theorem.** (GELFAND 1939). *Let $U$ be a solution of* (FE*), $-\infty < t, s < \infty$, *that is norm bounded and continuous for the weak operator topology on $\mathfrak{L}(X)$. Then there exists a linear operator $A$ on $X$ such that*

$$U(t)x = \exp(tA)x := \sum_{n=0}^{\infty} \frac{t^n A^n x}{n!}$$

*for all $x$ in a dense subset of $X$.*

In 1944, HILLE and NELSON DUNFORD (1906–1986) characterized the situation as follows (see [HD47]).

> *The problem of representing a one-parameter group of operators on a Banach space reduces according to several well-known methods of attack to establishing differentiability of the function $T_\xi$ at $\xi = 0$. The derivative*
>
> $$Ax = \lim_{\xi \to 0} \xi^{-1}(T_\xi - I)x$$
>
> *exists as a closed operator with domain $D(A)$ dense, provided that $T_\xi$ is continuous in the strong operator topology. It is then possible to assign*

*a meaning to* $\exp(\xi A)$ *in a natural way and so that* $T_\xi = \exp(\xi A)$, $-\infty < \xi < \infty$. *The operator* $A$ *is bounded if and only if* $T_\xi$ *is continuous in* $\xi$ *in the uniform topology, in which case* $A = \lim_{\xi \to 0} \xi^{-1}(T_\xi - I)$ *exists in the uniform topology. This implies that* $T_\xi$ *is an entire function of* $\xi$*; conversely, if* $T_\xi$ *is analytic anywhere, then* $A$ *is bounded. These considerations extend to the semi-group case, in which* $T_{\xi+\zeta} = T_\xi T_\zeta$ *is known to hold only for positive values of the parameter, although the number of distinct cases is much larger, and in particular, analyticity does not imply that* $A$ *is bounded.*

A few years later HILLE and YOSIDA simultaneously succeeded in characterizing generators of strongly continuous semigroup of contractions.

**4.4 Theorem.** (HILLE, YOSIDA 1948). *Let* $\bigl(A, D(A)\bigr)$ *be a linear operator on a Banach space. Then the following are equivalent:*

(a) *$A$ is the generator of a strongly continuous solution* $U : \mathbb{R}_+ \to \mathcal{L}(X)$ *of* (FE*) *that is bounded in norm by* 1.

(b) *$A$ is closed and densely defined,* $(0, \infty) \subset \rho(A)$, *and* $\|(\lambda - A)^{-1}\| \le 1/\lambda$ *for all* $\lambda > 0$.

*In this case* $u(t) := U(t)x$ *for* $x \in D(A)$ *is the unique solution of* (DE).

YOSIDA'S famous article [Yos48] (cf. also [Kat92]) and HILLE'S celebrated book [Hil48] were published in the same year. Of course, part of the theorem was known before, as can be seen from the passage of [HD47] cited above. It is interesting that HILLE discovered the missing part while correcting the galley proofs of his book, which sheds some light on the significance of the theorem with respect to the making of the text. However, the result marks the breakthrough of semigroup theory and has remained its focus ever since.

As mentioned in Paragraph II.3.3, there are remarkable differences in the way the semigroup was constructed from a given operator $A$ satisfying condition (b):

While YOSIDA plugged the bounded operator

$$A_\lambda := \lambda^2 R(\lambda, A) - \lambda I$$

into the exponential series and let $\lambda \to \infty$, HILLE considered

$$(I - {}^{tA}/_{n})^{-n} = \left[{}^{n}/_{t} R(A, {}^{n}/_{t})\right]^n ,$$

again a bounded operator, and proved convergence for $n \to \infty$.

The condition $\|U(t)\| \le 1$ was later removed independently by WILLIAM FELLER (1906–1970), ISAO MIYADERA, and R.S. PHILLIPS in 1952; cf. Generation Theorem II.3.8.

At this point, our story comes full circle and we see the genius of EULER and the boldness of LAGRANGE shine through this marvelous piece of modern analysis.

# Appendix A

# A Reminder of Some Functional Analysis

Our book is written in a functional-analytic spirit. Its main objects are operators on Banach spaces, and we use many, sometimes quite sophisticated, results and techniques from functional analysis and operator theory. As a rule, we refer to textbooks like [Con85], [DS58], [Lan93], [RS72], [Rud73], [TL80], or [Yos65]. However, for the convenience of the reader we add this appendix, where we

- introduce our notation,
- list some basic results, and
- prove a few of them.

To start with, we introduce the following classical sequence and function spaces. Here, $J$ is a real interval; $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$; and $\Omega$, depending on the context, is a domain in $\mathbb{R}^n$, a locally compact metric space, or a measure space. The symbol $X$ always stands for a Banach space.

$$\ell^\infty(X) = \ell^\infty(\mathbb{N}, X) := \Big\{(x_n)_{n\in\mathbb{N}} \subset X : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\Big\},$$

$$\|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} \|x_n\|,$$

$$c(X) := c(\mathbb{N}, X) := \Big\{(x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} x_n \text{ exists}\Big\} \subset \ell^\infty(X),$$

$$c_0(X) = c_0(\mathbb{N}, X) := \Big\{(x_n)_{n\in\mathbb{N}} \subset X : \lim_{n\to\infty} x_n = 0\Big\} \subset c(X),$$

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \ell^\infty(\mathbb{N}, \mathbb{C}), \quad c := c(\mathbb{N}) := c(\mathbb{N}, \mathbb{C}), \quad c_0 := c_0(\mathbb{N}) := c_0(\mathbb{N}, \mathbb{C}),$$

$$\ell^p := \ell^p(\mathbb{N}) := \ell^p(\mathbb{N},\mathbb{C}) := \Big\{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\Big\}, \quad p \in [1,\infty),$$

$$\|(x_n)_{n\in\mathbb{N}}\| := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p},$$

$\mathrm{C}(\Omega) := \{f : \Omega \to \mathbb{K} \mid f \text{ is continuous}\},$

$$\|f\|_\infty := \sup_{s\in\Omega} |f(s)| \quad (\text{if } \Omega \text{ is compact}),$$

$\mathrm{C}_0(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ vanishes at infinity}\};$ cf. p. 25,

$\mathrm{C}_\mathrm{b}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ is bounded}\},$

$\mathrm{C}_\mathrm{c}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ has compact support}\};$ cf. p. 25,

$\mathrm{C}_\mathrm{ub}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ is bounded and uniformly continuous}\},$

$\mathrm{AC}(J) := \{f : J \to \mathbb{K} \mid f \text{ is absolutely continuous}\};$ cf. p. 64,

$\mathrm{C}^k(J) := \{f \in \mathrm{C}(J) : f \text{ is } k\text{-times continuously differentiable}\},$

$\mathrm{C}^\alpha(J) := \{f \in \mathrm{C}(J) : f \text{ is Hölder continuous of order } \alpha\};$ cf. p. 136,

$\mathrm{C}^\infty(J) := \{f \in \mathrm{C}(J) : f \text{ is infinitely many times differentiable}\},$

$\mathrm{Lip}_\mathrm{u}(J) := \{f \in \mathrm{C}_\mathrm{ub}(\Omega) : f \text{ is Lipschitz continuous}\},$

$$\|f\|_{\mathrm{Lip}} := |f(0)| + \sup_{r\neq s}\left|\frac{f(r)-f(s)}{r-s}\right|,$$

$\mathrm{L}^p(\Omega,\mu) := \{f : \Omega \to \mathbb{K} \mid f \text{ is } p\text{-integrable on } \Omega\},$

$$\|f\|_p := \Big(\int_\Omega |f|^p(s)\, d\mu(s)\Big)^{1/p},$$

$\mathrm{L}^\infty(\Omega,\mu) := \{f : \Omega \to \mathbb{K} \mid f \text{ is measurable and } \mu\text{-essentially bounded}\},$

$\|f\|_\infty := \operatorname{ess\,sup} |f|;$ cf. p. 32 and p. 524,

$\mathrm{L}^\infty(J,X) := \{f : J \to X \mid f \text{ is measurable and essentially bounded}\};$
cf. p. 524,

$\mathrm{L}^p(J,X) := \{f : J \to X \mid f \text{ is } p\text{-Bochner integrable on } J\};$ cf. p. 524,

$\mathrm{M}_\mathrm{b}(\mathbb{R}) := \{\mu : \mu \text{ is a regular (signed or complex) Borel measure}\},$

$$\|\mu\| := \sup\Big\{\sum_{k=1}^\infty |\mu(\Omega_k)| : (\Omega_k)_{k\in\mathbb{N}} \text{ is a partition of } \Omega\Big\};$$

cf. [Rud86, Chap. 6],

$$\mathrm{W}^{k,p}(\Omega) := \left\{ f \in \mathrm{L}^p(\Omega) : \begin{array}{c} f \text{ is } k\text{-times distributionally differentiable,} \\ \text{with } f^{(k)} \in \mathrm{L}^p(\Omega) \end{array} \right\},$$

$\mathrm{W}^{k,p}(J,X) :=$ Sobolev space of Bochner integrable functions; cf. p. 525,

$\mathrm{h}^\alpha(J) :=$ little Hölder space of order $\alpha$; cf. p. 137,

$\mathrm{H}^k(\Omega) := \mathrm{W}^{k,2}(\Omega);$ cf. p. 407,

$\mathrm{H}^k_0(J) := \{f \in \mathrm{H}^k(J) : f(s) = 0 \text{ for } s \in \partial J\},$

$\mathscr{S}(\mathbb{R}^n) :=$ Schwartz space of rapidly decreasing functions; cf. p. 405,

$\mathrm{UBV}(\mathbb{R}) := \{f \in \mathrm{L}^1(\mathbb{R}) : f$ is of uniformly bounded variation$\}$,

$$\|f\| := \sup\Big\{\sum_{k=1}^{n} |f(s_k) - f(s_{k-1})| : \begin{array}{l} -b = s_0 < s_1 < \cdots < s_n = b, \\ \text{for } b > 0,\ n \in \mathbb{N} \end{array}\Big\}.$$

Clearly, we may combine the various sub- and superscripts for the spaces of continuous functions and obtain, e.g., $\mathrm{C}^1_{\mathrm{c}}(J) = \mathrm{C}^1(J) \cap \mathrm{C}_c(J)$.

Moreover, we will use the following notations. If $X_n$ is a Hilbert space, then

$$\text{(A.1)} \qquad X := \bigoplus_{n\in\mathbb{N}}^{2} X_n := \Big\{(x_n)_{n\in\mathbb{N}} : x_n \in X_n \text{ and } \sum_{n\in\mathbb{N}} \|x_n\|^2 < \infty\Big\},$$

equipped with the inner product

$$\big((x_n) \,|\, (y_n)\big) := \sum_{n\in\mathbb{N}} (x_n \,|\, y_n),$$

again is a Hilbert space, called the *Hilbert direct sum* of the spaces $X_n$.

For an abstract complex Banach space $X$ we denote its dual by $X'$ and the *canonical bilinear form* by

$$\langle x, x' \rangle \quad \text{for } x \in X, \quad x' \in X'.$$

As usual, we also write $x'(x)$ for $\langle x, x' \rangle$ and denote by $\sigma(X, X')$ the *weak topology* on $X$ and by $\sigma(X', X)$ the weak* topology on $X'$. Then the following properties hold.

**A.1 Proposition.**

(i) *For convex sets in $X$ (in particular, for subspaces) the weak and norm closure coincide.*

(ii) *The closed, convex hull $\overline{\mathrm{co}}\, K$ of a weakly compact set $K$ in $X$ is weakly compact (Kreĭn's theorem).*

(iii) *The dual unit ball $U^0 := \{x' \in X' : \|x'\| \le 1\}$ is weak* compact (Banach–Alaoglu's theorem).*

The space of all bounded, linear operators on $X$ will be denoted[1] by $\mathcal{L}(X)$and becomes a Banach space for the norm

$$\|T\| := \sup\{\|Tx\| : \|x\| \le 1\}, \quad T \in \mathcal{L}(X).$$

[1] For the space of all bounded, linear operators between two normed spaces $X$ and $Y$ we use the notation $\mathcal{L}(X, Y)$.

The operators $T \in \mathcal{L}(X)$ satisfying

$$\|Tx\| \le \|x\| \qquad \text{for all } x \in X$$

are called *contractions*, while *isometries* are defined by

$$\|Tx\| = \|x\| \qquad \text{for all } x \in X.$$

Besides the *uniform operator topology* on $\mathcal{L}(X)$, which is the one induced by the above operator norm, we frequently consider two more topologies on $\mathcal{L}(X)$.

We write $\mathcal{L}_s(X)$ if we endow $\mathcal{L}(X)$ with the *strong operator topology*, which is the topology of pointwise convergence on $(X, \|\cdot\|)$.

Finally, $\mathcal{L}_\sigma(X)$ denotes $\mathcal{L}(X)$ with the *weak operator topology*, which is the topology of pointwise convergence on $\big(X, \sigma(X, X')\big)$.

A net $(T_\alpha)_{\alpha\in\mathcal{A}} \subset \mathcal{L}(X)$ converges to $T \in \mathcal{L}(X)$ if and only if

(A.2) $\|T_\alpha - T\| \to 0$ (uniform operator topology),

(A.3) $\|T_\alpha x - Tx\| \to 0 \ \forall\, x \in X$ (strong operator topology),

(A.4) $|\langle T_\alpha x - Tx, x'\rangle| \to 0 \ \forall\, x \in X,\ x' \in X'$ (weak operator topology).

With these notions, the *principle of uniform boundedness* can be stated as follows.

**A.2 Proposition.** *For a subset $K \subset \mathcal{L}(X)$ the following properties are equivalent.*

(a) *$K$ is bounded for the weak operator topology.*

(b) *$K$ is bounded for the strong operator topology.*

(c) *$K$ is uniformly bounded, i.e., $\|T\| \le c$ for all $T \in K$.*

Continuity with respect to the strong operator topology will be shown frequently by using the following property (b) (see [Sch80, Sec. III.4.5]).

**A.3 Proposition.** *On bounded subsets of $\mathcal{L}(X)$, the following topologies coincide.*

(a) *The strong operator topology.*

(b) *The topology of pointwise convergence on a dense subset of $X$.*

(c) *The topology of uniform convergence on relatively compact subsets of $X$.*

The advantage of using the strong or weak operator topology instead of the norm topology on $\mathcal{L}(X)$ is that the former yield more continuity and more compactness. This becomes evident already from the definition of a strongly continuous semigroup in Section I.5. Another example is provided by our discussion of asymptotic properties of semigroups in Section V.2. There, we use compactness in $\mathcal{L}_s(X)$ and $\mathcal{L}_\sigma(X)$, which can be characterized by the following two results.

**A.4 Proposition.** *Let $\mathcal{K} \subset \mathcal{L}(X)$ be a bounded set of operators and consider*

$$\mathcal{K}x := \{Tx : T \in \mathcal{K}\} \qquad \text{for } x \in X.$$

*Then the subspaces*

$$X_s := \big\{x \in X : \mathcal{K}x \text{ is relatively (norm) compact}\big\}$$

*and*

$$X_\sigma := \big\{x \in X : \mathcal{K}x \text{ is relatively weakly compact}\big\}$$

*are closed in $X$.*

Proof. The assertion for $X_s$ follows by a standard diagonal procedure, while the argument for $X_\sigma$ is more complicated.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $X_\sigma$ converging to $x \in X$. By Eberlein's theorem ([Sch80, Sec. IV.11.2]) it suffices to show that every sequence $(T_k x)_{k\in\mathbb{N}}$ with $T_k \in \mathcal{K}$ has a weakly converging subsequence. Since $x_1 \in X_\sigma$, there is a subsequence $(T_{k(i_1)}x_1)$ converging weakly to some $y_1 \in X$. Similarly, for $x_2 \in X_\sigma$ there exists a subsequence $(T_{k(i_2)})$ of $(T_{k(i_1)})$ such that $(T_{k(i_2)}x_2)$ converges weakly to $y_2 \in X$, and so on. Applying a diagonal procedure we find a subsequence $(T_{k(i)})_{i\in\mathbb{N}}$ of $(T_k)_{k\in\mathbb{N}}$ such that

$$T_{k(i)}x_n \overset{\text{weakly}}{\longrightarrow} y_n \qquad \text{for every } n \in \mathbb{N}.$$

From

$$\begin{aligned}
\|y_n - y_m\| &= \sup\big\{\langle y_n - y_m, x'\rangle : \|x'\| \le 1\big\}\\
&= \sup\Big\{\lim_{i\to\infty} \big|\big\langle T_{k(i)}x_n - T_{k(i)}x_m, x'\big\rangle\big| : \|x'\| \le 1\Big\}\\
&\le \big\|T_{k(i)}\big\| \cdot \|x_n - x_m\|,
\end{aligned}$$

it follows that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with limit $y \in X$. A standard $3\varepsilon$-argument shows

$$y = \sigma(X, X')\text{-}\lim_{i\to\infty} T_{k(i)}x.$$

□

An important consequence is the following characterization of (relatively) compact sets of operators.

**A.5 Corollary.** *For a bounded subset $\mathcal{K} \subset \mathcal{L}(X)$, the following assertions are equivalent.*

(a) *$\mathcal{K}$ is relatively compact for the strong (weak) operator topology.*

(b) *$\mathcal{K}x$ is relatively strongly (weakly) compact for every $x \in X$.*

(c) *$\mathcal{K}x$ is relatively strongly (weakly) compact for every $x$ in a dense subset of $X$.*

PROOF. The implication (a) $\Rightarrow$ (b) follows by the continuity of the map $T \mapsto Tx$, while the converse is, in some sense, a consequence of Tychonoff's theorem on products of compact spaces (see [Dug66, Chap. XI, Thm. 1.4.(4)]). The equivalence (b) $\iff$ (c) follows from Proposition A.4. $\square$

The vector space $\mathcal{L}(X)$ is an algebra for the operator multiplication. We state the continuity properties of this multiplication with respect to the three operator topologies.

**A.6 Proposition.** *The multiplication*

$$(S,T) \mapsto S \cdot T$$

*on $\mathcal{L}(X)$ is*

(i) *jointly continuous for the norm topology,*
(ii) *separately continuous for the strong and for the weak operator topologies, and*
(iii) *jointly continuous on bounded sets for the strong operator topology.*

As an example for the functional-analytic constructions made throughout the text, we consider the following setting.

Let $\mathcal{X}_{t_0} := \mathrm{C}\big([0,t_0], \mathcal{L}_s(X)\big)$ be the space of all functions on $[0,t_0]$ into $\mathcal{L}(X)$ that are continuous for the strong operator topology. For each $F \in \mathcal{X}_{t_0}$ and $x \in X$, the functions $t \mapsto F(t)x$ are continuous, hence bounded, on $[0,t_0]$. The uniform boundedness principle then implies

$$\|F\|_\infty := \sup_{s\in[0,t_0]} \|F(s)\| < \infty.$$

Clearly, this defines a norm making $\mathcal{X}_{t_0}$ a complete space.

**A.7 Proposition.** *The space*

$$\mathcal{X}_{t_0} := \Big(\mathrm{C}\big([0,t_0], \mathcal{L}_s(X)\big), \|\cdot\|_\infty\Big)$$

*is a Banach space.*

PROOF. Let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{X}_{t_0}$. Then, by the definition of the norm in $\mathcal{X}_{t_0}$, $\big((F_n(\cdot)x\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}([0,t_0],X)$ for all $x \in X$. Since $\mathrm{C}([0,t_0],X)$ is complete, the limit $\lim_{n\to\infty} F_n(\cdot)x =: F(\cdot)x \in \mathrm{C}([0,t_0],X)$ exists, and we obtain $\lim_{n\to\infty} F_n = F$ in $\mathcal{X}_{t_0}$. $\square$

# Appendix B

# A Reminder of Some Operator Theory

Familiarity with linear operators, in particular unbounded operators, is essential for an understanding of our semigroups and their generators. The best introduction is still Kato's monograph [Kat80] (see also [DS58], [GGK90], [Gol66], [TL80], [Wei80]), but we briefly restate some of the basic definitions and properties.[1]

**B.1 Definition.** *A linear operator $A$ with domain $D(A)$ in a Banach space $X$, i.e., $D(A) \subset X \to X$, is closed if it satisfies one of the following equivalent properties.*

(a) *If for the sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ the limits $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Ax_n = y \in X$ exist, then $x \in D(A)$ and $Ax = y$.*

(b) *The graph $\mathcal{G}(A) := \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$.*

(c) *$X_1 := \big(D(A), \|\cdot\|_A\big)$ is a Banach space[2] for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

(d) *$A$ is weakly closed, i.e., property* (a) *(or property* (b)*) holds for the $\sigma(X, X')$-topology on $X$.*

---

[1] Most of the following concepts also make sense for operators acting between different Banach spaces. However, for simplicity we state them for a single Banach space only and leave the straightforward generalization to the reader.

[2] This definition of $X_1$ also makes sense if $A$ has empty resolvent set. Since if $\rho(A) \neq \emptyset$, the graph norm and the norms $\|\cdot\|_{1,\lambda}$ from Exercise II.5.9.(1) are all equivalent, this definition of $X_1$ will not conflict with Definition II.5.1 for $n = 1$.

*If $\lambda - A$ is injective for some $\lambda \in \mathbb{C}$, then the above properties are also equivalent to*

(e) $(\lambda - A)^{-1}$ *is closed.*

While the additive perturbation of $A$ by a bounded operator $B \in \mathcal{L}(X)$ yields again a closed operator, the situation is slightly more complicated for multiplicative perturbations.

**B.2 Proposition.** *Let $\big(A, D(A)\big)$ be a closed operator and take $B \in \mathcal{L}(X)$. Then the following holds.*

(i) *$AB$ with domain $D(AB) := \{x \in X : Bx \in D(A)\}$ is closed.*

(ii) *$BA$ with domain $D(BA) := D(A)$ is closed if $B^{-1} \in \mathcal{L}(X)$.*

PROOF. (i) is easy to check and implies (ii) after the similarity transformation $BA = B(AB)B^{-1}$. □

It will be important to find closed extensions of not necessarily closed operators. Here are the relevant notions.

**B.3 Definition.** *An operator $\big(B, D(B)\big)$ is an extension of $\big(A, D(A)\big)$, in symbols $A \subset B$, if $D(A) \subset D(B)$ and $Bx = Ax$ for $x \in D(A)$. The smallest closed extension of $A$, if it exists, is called the closure of $A$ and is denoted by $\overline{A}$. Operators having a closure are called closable.*

**B.4 Proposition.** *An operator $\big(A, D(A)\big)$ is closable if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ with $x_n \to 0$ and $Ax_n \to z$ one has $z = 0$. In that case, the graph of the closure is given by*

$$\mathcal{G}(\overline{A}) = \overline{\mathcal{G}(A)}.$$

A simple operator that is not closable is

$$Af := f'(0) \cdot \mathbb{1} \quad \text{with domain} \quad D(A) := \mathrm{C}^1[0,1]$$

in the Banach space $X := \mathrm{C}[0,1]$. This follows, e.g., from the following characterization of bounded linear forms and the fact that the kernel of a closed operator is always closed.[3]

**B.5 Proposition.** *Let $X$ be a normed vector space and take a linear functional $x' : X \to \mathbb{C}$. Then $x'$ is bounded if and only if its kernel $\ker x'$ is closed in $X$. Hence, $x'$ is unbounded if and only if $\ker x'$ is dense in $X$.*

[3] Here, for a linear map $\Phi : X \to Y$ between two vector spaces $X$ and $Y$ its kernel is defined by $\ker \Phi := \{x \in X : \Phi x = 0\}$.

Proof. If $x'$ is bounded, then clearly $\ker(x')$ is closed. On the other hand, if $\ker x'$ is closed, then the quotient ${}^{X}\!/_{\ker x'}$ is a normed vector space of dimension 1. Moreover, we can decompose $x' = i\,\widehat{x'}$ by the canonical maps $i : {}^{X}\!/_{\ker x'} \to \mathbb{C}$ and $\widehat{x'} : X \to {}^{X}\!/_{\ker x'}$. Since $\|\widehat{x'}\| \leq 1$, this proves that $x'$ is bounded. The remaining assertions follow from the fact that for each linear form $x' \neq 0$ the codimension of $\ker x'$ in $X$ is 1. □

A subspace $D$ of $D(A)$ that is dense in $D(A)$ for the graph norm is called a *core* for $A$. If $\big(A, D(A)\big)$ is closed, one can recover $A$ from its restriction to a core $D$, i.e., the closure of $(A, D)$ becomes $\big(A, D(A)\big)$; see Exercise II.1.15.(2).

The closed graph theorem states that everywhere defined closed operators are already bounded. It can be phrased as follows.

**B.6 Theorem.** *For a closed operator $A : D(A) \subset X \to X$ the following properties are equivalent.*

(a) *$\big(A, D(A)\big)$ is a bounded operator, i.e., there exists $c \geq 0$ such that*

$$\|Ax\| \leq c\,\|x\| \qquad \text{for all } x \in D(A).$$

(b) *$D(A)$ is a closed subspace of $X$.*

By the closed graph theorem, one obtains the following surprising result.

**B.7 Corollary.** *Let $A : D(A) \subset X \to X$ be closed and assume that a Banach space $Y$ is continuously embedded in $X$ such that the range $\operatorname{rg} A := A\big(D(A)\big)$ is contained in $Y$. Then $A$ is bounded from $(D(A), \|\cdot\|_A)$ into $Y$.*

If an operator $A$ has dense domain $D(A)$ in $X$, we can define its adjoint operator on the dual space $X'$.[4]

**B.8 Definition.** *For a densely defined operator $\big(A, D(A)\big)$ on $X$, we define the adjoint operator $\big(A', D(A')\big)$ on $X'$ by*

$$\begin{aligned} D(A') &:= \big\{x' \in X' : \exists\, y' \in X' \text{ such that } \langle Ax, x'\rangle = \langle x, y'\rangle \ \forall\, x \in D(A)\big\}, \\ A'x' &:= y' \text{ for } x \in D(A). \end{aligned}$$

**B.9 Example.** Take $A_p := {}^{d}\!/_{ds}$ on $X_p := \mathrm{L}^p(\mathbb{R})$, $1 \leq p < \infty$, with domain $D(A_p) := \mathrm{W}^{1,p}(\mathbb{R}) := \{f \in X_p : f \text{ absolutely continuous}, f' \in X_p\}$. Then $A_p{}' = -A_q$ on $X_q$, where ${}^{1}\!/_{p} + {}^{1}\!/_{q} = 1$. For a proof and many more examples we refer to [Gol66, Sec. II.2 & Chap. VI] and [Kat80, Sec. III.5]. Compare also Exercise II.4.12.(12).

While the adjoint operator is always closed, it may happen that $D(A') = \{0\}$ (e.g., take the nonclosable operator following Proposition B.4).

[4] Similarly, one can define the Hilbert space adjoint $A^*$ by replacing the canonical bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle$ by the *inner product* $(\,\cdot\,|\,\cdot\,)$.

On reflexive Banach spaces there is a nice duality between densely defined and closable operators.

**B.10 Proposition.** *Let $\big(A, D(A)\big)$ be a densely defined operator on a reflexive Banach space $X$. Then the adjoint $A'$ is densely defined if and only if $A$ is closable. In that case, one has*

$$(A')' = \overline{A}.$$

We now prove a close relationship between inverses and adjoints.

**B.11 Proposition.** *Let $\big(A, D(A)\big)$ be a densely defined closed operator on $X$. Then the inverse $A^{-1} \in \mathcal{L}(X)$ exists if and only if the inverse $(A')^{-1} \in \mathcal{L}(X')$ exists. In that case, one has*

$$(A')^{-1} = (A^{-1})'.$$

Proof. Assume $A^{-1} \in \mathcal{L}(X)$. Since $(A^{-1})' \in \mathcal{L}(X')$, one has

$$\big\langle x, (A^{-1})'A'x' \big\rangle = \big\langle A^{-1}x, A'x' \big\rangle = \big\langle AA^{-1}x, x' \big\rangle = \langle x, x' \rangle$$

for all $x \in X$, $x' \in D(A')$, i.e., $A'$ has a left inverse. Similarly,

$$\big\langle Ax, (A^{-1})'x' \big\rangle = \big\langle A^{-1}Ax, x' \big\rangle = \langle x, x' \rangle$$

holds for all $x \in D(A)$, $x' \in X'$, i.e., $(A^{-1})'x' \in D(A')$ and $A'(A^{-1})'x' = x'$.

On the other hand, assume $(A')^{-1} \in \mathcal{L}(X')$. Then

$$\big\langle Ax, (A')^{-1}x' \big\rangle = \big\langle x, A'(A')^{-1}x' \big\rangle = \langle x, x' \rangle$$

for all $x \in D(A)$ and $x' \in X'$. For every $x \in D(A)$, choose $x' \in X'$ such that $\|x'\| = 1$ and $|\langle x, x' \rangle| = \|x\|$ and obtain

$$\|x\| = \big|\big\langle Ax, (A')^{-1}x' \big\rangle\big| \le \|Ax\| \cdot \big\|(A')^{-1}\big\|.$$

This shows that $A$ is injective and its inverse satisfies

$$\big\|A^{-1}\big\| \le \big\|(A')^{-1}\big\|,$$

hence is bounded. By Theorem B.6, $D(A^{-1}) = \operatorname{rg} A$ must be closed. A simple Hahn–Banach argument shows that $\operatorname{rg} A = X$, hence $A^{-1} \in \mathcal{L}(X)$. ☐

**B.12 Corollary.** *For a densely defined closed operator $\big(A, D(A)\big)$ the spectra of $A$ and of $A'$ coincide, i.e.,*

$$\sigma(A) = \sigma(A')$$

*and $R(\lambda, A)' = R(\lambda, A')$ for all $\lambda \in \rho(A)$.*

The next result shows that adjoints are also very useful in relating the ranges of two operators. We will use this fact in Section VI.8 to characterize various controllability concepts. For a proof we refer to [CP78, Chap. 3, Cor. 3.5 and Thm. 3.6] or [Zab92, Part IV, Sec. 2.1].

**B.13 Lemma.** *Let $V$, $W$, $Z$ be Banach spaces and let $S \in \mathcal{L}(V, Z)$ and $T \in \mathcal{L}(W, Z)$.*

(i) *The following conditions are equivalent.*
   (a) $\overline{\operatorname{rg} S} \subset \overline{\operatorname{rg} T}$.
   (b) $\ker S' \supset \ker T'$.

(ii) *If, in addition, the spaces $V$, $W$, and $Z$ are reflexive, then the following conditions are equivalent.*
   (a) $\operatorname{rg} S \subset \operatorname{rg} T$.
   (b) *There exists $\gamma > 0$ such that $\|S'z'\| \le \gamma \|T'z'\|$ for all $z' \in Z'$.*

Now we turn again to the unbounded situation and define iterates of unbounded operators.

**B.14 Definition.** *The $n$th power $A^n$ of an operator $A : D(A) \subset X \to X$ is defined successively as*

$$\begin{aligned} A^n x &:= A(A^{n-1}x), \\ D(A^n) &:= \big\{x \in D(A) : A^{n-1}x \in D(A)\big\}. \end{aligned}$$

In general, it may happen that $D(A^2) = \{0\}$ even if $A$ is densely defined and closed. However, if $A^{-1} \in \mathcal{L}(X)$ exists (or if $\rho(A) \neq \emptyset$), the infinite intersection

$$D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n)$$

is still dense. This is proved in Proposition II.1.8 for semigroup generators and in [Len94] or [AEMK94, Prop. 6.2] for the general case.

Next, we give some results concerning the continuity and differentiability of products of operator-valued functions.

**B.15 Lemma.** *Let $J$ be some real interval and $P$, $Q : I \to \mathcal{L}(X)$ be two strongly continuous operator-valued functions defined on $J$. Then the product $(PQ)(\cdot) : J \to \mathcal{L}(X)$, defined by $(PQ)(t) := P(t)Q(t)$, is strongly continuous as well.*

Proof. We fix $x \in X$ and $t \in J$ and take a sequence $(t_n)_{n\in\mathbb{N}} \subset J$ with $\lim_{n\to\infty} t_n = t$. Then, by the uniform boundedness principle, the set $\{P(t_n) : n \in \mathbb{N}\} \subset \mathcal{L}(X)$ is bounded, and therefore

$$\begin{aligned} \|P(t_n)Q(t_n)x - P(t)Q(t)x\| \le \|P(t_n)\| \cdot \|Q(t_n)x - Q(t)x\| \\ + \big\|\big(P(t_n) - P(t)\big)Q(t)x\big\|, \end{aligned}$$

where the right-hand side converges to zero as $n \to \infty$. □

**B.16 Lemma.** *Let $J$ be some real interval and $P$, $Q : J \to \mathcal{L}(X)$ be two strongly continuous operator-valued functions defined on $J$. Moreover, assume that $P(\cdot)x : J \to X$ and $Q(\cdot)x : J \to X$ are differentiable for all $x \in D$ for some subspace $D$ of $X$, which is invariant under $Q$. Then $(PQ)(\cdot)x : J \to X$, defined by $(PQ)(t)x := P(t)Q(t)x$, is differentiable for every $x \in D$ and*

$$\tfrac{d}{dt}\Big(P(\cdot)Q(\cdot)x\Big)(t_0) = \tfrac{d}{dt}\Big(P(\cdot)Q(t_0)x\Big)(t_0) + P(t_0)\Big(\tfrac{d}{dt}Q(\cdot)x\Big)(t_0).$$

Proof. Let $t_0 \in J$ and $(h_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} h_n = 0$ and $t_0 + h_n \in J$ for all $n \in \mathbb{N}$. Then, for $x \in D$, we have

$$\begin{aligned}
&\frac{P(t_0+h_n)Q(t_0+h_n)x - P(t_0)Q(t_0)x}{h_n} \\
&\qquad = P(t_0+h_n)\,\frac{Q(t_0+h_n)x - Q(t_0)x}{h_n} + \frac{P(t_0+h_n) - P(t_0)}{h_n}\,Q(t_0)x \\
&\qquad =: L_1(n,x) + L_2(n,x).
\end{aligned}$$

Clearly, the sequence $\big(L_2(n,x)\big)_{n\in\mathbb{N}}$ converges for all $x \in D$ and its limit is $\lim_{n\to\infty} L_2(n,x) = P'(t_0)Q(t_0)x$. In order to show that $\big(L_1(n,x)\big)_{n\in\mathbb{N}}$ converges for $x \in D$, note that

$$\left\{\frac{Q(t_0+h_n)x - Q(t_0)x}{h_n} : n \in \mathbb{N}\right\}$$

is relatively compact in $X$ and that $\{P(t_0+h_n) : n \in \mathbb{N}\}$ is bounded. Since by Proposition A.3 the topologies of pointwise convergence and of uniform convergence on relatively compact sets coincide, we conclude that $\big(L_1(n,x)\big)_{n\in\mathbb{N}}$ converges for $x \in D$ and

$$\lim_{n\to\infty} L_1(n,x) = P(t_0)Q'(t_0)x.$$

This completes the proof. □

In the context of operators on spaces of vector-valued functions it is convenient to use the following *tensor product* notation.

Assume that $X$, $Y$ are Banach spaces, $\mathrm{F}(J,Y)$ is a Banach space of $Y$-valued functions defined on an interval $J \subseteq \mathbb{R}$, $T \in \mathcal{L}(X,Y)$ is a bounded linear operator, and $f : J \to \mathbb{C}$ is a complex-valued function. If the map $f \otimes y : J \ni s \mapsto f(s)\cdot y \in Y$ belongs to $\mathrm{F}(J,Y)$ for all $y \in Y$, then we define the linear operator $f \otimes T : X \to \mathrm{F}(J,Y)$ by

$$\big((f\otimes T)x\big)(s) := (f \otimes Tx)(s) = f(s)\cdot Tx$$

for all $x \in X$, $s \in J$.

Independently, for a Banach space $X$ and elements $x \in X$, $x' \in X'$, we frequently use the tensor product notation $x \otimes x'$ for the rank-one operator on $X$ defined by

$$(x \otimes x')\, v := x'(v) \cdot x, \qquad v \in X.$$

We close this appendix with a classical theorem to be used in the proof of Theorem V.2.21.

**B.17 Gelfand's $T{=}I$ Theorem.** *Let $T \in \mathcal{L}(X)$ satisfy $\sigma(T) = \{1\}$. If $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$, then $T = I$.*

Proof (see [AR89]). Since $z \mapsto \log z$ is analytic in a neighborhood of $z_0 = 1$, we can, by the usual functional calculus, define $S := -\mathrm{i}\log T$. This operator satisfies $T = \mathrm{e}^{\mathrm{i}S}$, and by the spectral mapping theorem, $\sigma(mS) = \{0\}$ for all $m \in \mathbb{N}$. Now take the operators $\sin(mS) := {}^1\!/\!{}_{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}mS} - \mathrm{e}^{-\mathrm{i}mS}) = {}^1\!/\!{}_{2\mathrm{i}}(T^m - T^{-m})$ and observe that

$$\sigma\big(\sin(mS)\big) = \sin\big(\sigma(mS)\big) = \{0\}$$

and

$$\big\|\big[\sin(mS)\big]^n\big\| = \left\| \left(\frac{T^m - T^{-m}}{2\mathrm{i}}\right)^n \right\| \le \sup_{n\in\mathbb{Z}} \|T^n\|.$$

The principal branch of arcsin admits a Taylor series $\sum_{n=0}^{\infty} c_n z^n$ at $z_0 = 0$ such that $c_n \ge 0$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} c_n = \arcsin(1) = {}^{\pi}\!/\!{}_2$. This implies the estimate

$$\begin{aligned} \|mS\| &= \big\|\arcsin\big(\sin(mS)\big)\big\| \\ &\le \sum_{n=0}^{\infty} c_n \big\|[\sin(mS)]^n\big\| \\ &\le \frac{\pi}{2} \sup_{n\in\mathbb{N}} \|T^n\|. \end{aligned}$$

Since this holds for all $m \in \mathbb{N}$, we obtain $S = 0$ and $T = \mathrm{e}^{\mathrm{i}S} = I$. $\square$

# Appendix C

# Vector-Valued Integration

## *a. The Bochner Integral*

In the first part of this appendix we give a brief introduction to the so-called Bochner integration of vector-valued functions. For a detailed treatment we refer to [DS58, Chap. III], [DU77] and [HP57, Sec. III.1].

To start with, we take a Banach space $X$ and consider a function

$$f : J \to X$$

defined on some interval $J \subset \mathbb{R}$. If $f$ is continuous, we can, as in the scalar case, define the integral $\int_J f(s)\,ds$ as the limit of Riemann sums. However, in many situations (e.g. in Section III.3, Section IV.3.c, Section VI.7, or Section VI.8) this is too restrictive, and one has to extend Lebesgue's integration theory to vector-valued functions. To this end we first introduce the following notation.

**C.1 Definition.** *Let $f : J \subset \mathbb{R} \to X$ be a vector-valued function.*

(a) *The function $f$ is called simple if it can be represented as*[1]

$$f = \sum_{k=1}^{n} x_k \cdot \mathbb{1}_{J_k} \qquad \text{(C.1)}$$

*for elements $x_k \in X$ and measurable subsets $J_k \subset J$.*

[1] Here, as usual, $\mathbb{1}_{J_k}$ denotes the characteristic function of the set $J_k$.

*For a simple function $f$ we define its integral by*

$$\int_J f(s)\,ds := \sum_{k=1}^{n} x_k \lambda(J_k),$$

*which is independent of the special representation in* (C.1) *and where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$.*

(b) *If $f$ can be approximated pointwise by simple functions, i.e., if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions on $J$ such that*

$$\lim_{n\to\infty} \|f(s) - f_n(s)\| = 0 \qquad \text{a.e.,} \tag{C.2}$$

*then we call $f$ (strongly) measurable.*

(c) *If $f$ is measurable and there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions on $J$ such that*

$$\lim_{n\to\infty} \int_J \|f(s) - f_n(s)\|\,ds = 0, \tag{C.3}$$

*then we call $f$ (Bochner) integrable. For an integrable function $f$ we define its integral by*

$$\int_J f(s)\,ds := \lim_{n\to\infty} \int_J f_n(s)\,ds,$$

*which is independent of the special choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$.*

It is now easy to verify that the set of measurable or integrable functions $f : J \to X$, respectively, form a vector space. We list some elementary properties of measurable functions, cf. [HP57, Cor. 1 after Thm. 3.5.3] and [DU77, Chap. II].

**C.2 Proposition.** *Let $f : J \subset \mathbb{R} \to X$ be a vector-valued function.*

(i) *If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions on $J$ converging to $f$ in the sense of* (C.2)*, then $f$ is measurable as well.*

(ii) *If $f$ is measurable and $F : J \to \mathcal{L}(X)$ is strongly continuous, then the composition $F \circ f : J \to X$ is measurable as well.*

Integrable functions can be characterized in the following way cf. [HP57, Thm. 3.7.4] and [DU77, Chap. II, Thm. 2].

**C.3 Proposition.** *For a measurable function $f : J \subset \mathbb{R} \to X$ the following conditions are equivalent.*

(a) *$f$ is integrable.*

(b) *$\int_J \|f(s)\|\,ds < \infty$.*

As an immediate consequence of Proposition C.2.(ii) and Proposition C.3 we remark that the composition $F \circ f : J \to X$ of an integrable function $f : J \to X$ and a strongly continuous function $F : J \to \mathcal{L}(X)$ is again integrable. Moreover, we note that for the Bochner integral the

- *triangle inequality* $\left\|\int_J f(s)\,ds\right\| \le \int_J \|f(s)\|\,ds$ (cf. [HP57, Thm. 3.7.6] and [DU77, Chap. II, Thm. 4]),
- *Fubini's theorem* (cf. [HP57, 3.7.13]), and
- *Lebesgue's dominated convergence theorem* (cf. [HP57, Thm. 3.7.9] and [DU77, Chap. II, Thm. 3])

prevail. Furthermore, we have the following result on interchanging integration with the application of closed operators, cf. [HP57, Thm. 3.7.12] and [DU77, Chap. II, Thm. 6].

**C.4 Proposition.** *Let $A : D(A) \subseteq X \to Y$ be a closed operator acting between two Banach spaces $X$ and $Y$. If $f : J \to X$ is an integrable function with $f(s) \in D(A)$ for almost all $s \in J$ and if $Af : J \to Y$ given by $(Af)(s) := A\,f(s)$ is integrable, then $\int_J f(s)\,ds \in D(A)$ and*

$$A\Big(\int_J f(s)\,ds\Big) = \int_J Af(s)\,ds.$$

Next, we introduce $\mathrm{L}^p$-spaces of vector-valued functions.

**C.5 Definition.** *If we identify functions $f : J \to X$ that differ only on sets with Lebesgue measure zero, then the spaces $\big(\mathrm{L}^p(J,X), \|\cdot\|_p\big)$ defined by*

$$\mathrm{L}^p(J,X) := \left\{ f : J \to X : \begin{array}{l} f \text{ is measurable and} \\ \int_J \|f(s)\|^p\,ds =: \|f\|_p^p < \infty \end{array} \right\} \quad \text{if } 1 \le p < \infty,$$

$$\mathrm{L}^\infty(J,X) := \left\{ f : J \to X : \begin{array}{l} f \text{ is measurable and} \\ \operatorname{ess\,sup} \|f\| =: \|f\|_\infty < \infty \end{array} \right\}$$

*are Banach spaces (see* [DS58, Thm. III.6.6]*).*

Here the *essential supremum* of a function $q : J \to \mathbb{R}$ is

$$\operatorname{ess\,sup} q := \sup q_{\mathrm{ess}}(J),$$

where

$$q_{\mathrm{ess}}(J) := \Big\{ r \in \mathbb{R} : \lambda\big(\{s \in J : |q(s) - r| < \varepsilon\}\big) \ne 0 \text{ for all } \varepsilon > 0 \Big\}$$

is its *essential range* (cf. I.4.9). If $\operatorname{ess\,sup} q < \infty$, then we call $q$ *essentially bounded*.

One can show that the subspace of simple functions and even the subspace of step functions is dense in $\mathrm{L}^p(J,X)$ for all $1 \le p < \infty$. Moreover, if $J$ is bounded, it follows that $\mathrm{L}^{p_2}(J,X) \subset \mathrm{L}^{p_1}(J,X)$ for all $1 \le p_1 < p_2 \le \infty$.

Using the above $\mathrm{L}^p$-spaces, we introduce vector-valued Sobolev spaces.

**C.6 Definition.** *For $1 \le p \le \infty$ we define the Sobolev space $\mathrm{W}^{1,p}(J,X)$ by*

$$\text{(C.4)} \quad f \in \mathrm{W}^{1,p}(J,X) \quad :\Longleftrightarrow \quad \begin{cases} f \in \mathrm{L}^p(J,X) \quad \text{and} \\ f(s) = f(s_0) + \int_{s_0}^{s} g(s)\,ds \\ \text{for some } s_0 \in J \text{ and } g \in \mathrm{L}^p(J,X). \end{cases}$$

Again, the spaces $\mathrm{W}^{1,p}(J,X)$ endowed with the norms

$$\|f\|_{\mathrm{W}^{1,p}(J,X)} := \|f\|_p + \|g\|_p,$$

where $g$ satisfies (C.4), become Banach spaces for all $1 \le p \le \infty$ and are contained in the space $\mathrm{C}(J,X)$ of all continuous functions from $J$ to $X$, cf. [Ama95, Sec. III.1.1].

We close this subsection with Voigt's *convex compactness property* for the strong operator topology (see [Voi92]). It is used in Lemma III.1.13, Proposition IV.2.12, and Lemma IV.4.6.

**C.7 Theorem.** *Let $K : [a,b] \to \mathcal{L}(X,Y)$ be a strongly continuous function. If $K(t)$ is a compact operator for each $t \in (a,b)$, then $\int_a^b K(t)\,dt$ is compact as well.*

Proof. Since $\mathcal{K}(X,Y)$ is norm closed in $\mathcal{L}(X,Y)$ and $\int_\alpha^\beta K(s)\,ds$ converges for $\alpha \downarrow a$ and $\beta \uparrow b$ in norm to $\int_a^b K(s)\,ds$, we can assume without loss of generality that $K(s)$ is compact for all $s \in [a,b]$.

Now, an operator is compact if and only if any restriction to a separable subspace is compact. Hence, we can without loss of generality assume that $X$ is separable. Moreover, we may also assume that $Y$ is separable, otherwise we replace $Y$ by the separable space

$$\overline{\mathrm{lin}}\{K(t)x : x \in X,\, t \in [a,b]\}.$$

Since by [Ban32, Chap. XI, §8, Thm. 9] every separable Banach space can be embedded isomorphically into $\mathrm{C}[0,1]$, we can further assume that $Y = \mathrm{C}[0,1]$. In particular, this implies that there exists a sequence $(P_n)_{n\in\mathbb{N}} \subset \mathcal{K}(Y)$ of compact operators converging strongly to the identity operator $I$ (e.g., the Bernstein operators from Paragraph III.5.7).

For this sequence $(P_n)_{n\in\mathbb{N}}$, an operator $K \in \mathcal{L}(X,Y)$ is compact if and only if $\lim_{n\to\infty} \|(I-P_n)K\| = 0$.

In fact, if $K$ is compact, then $KU$ (with $U$ the unit ball in $X$) is relatively compact. Hence, by Proposition A.3, the sequence $\big((I-P_n)K\big)_{n\in\mathbb{N}}$ converges uniformly on $U$ to 0, i.e., $\lim_{n\to\infty} \|(I-P_n)K\| = 0$.

Conversely, if $\lim_{n\to\infty} \|(I - P_n)K\| = 0$, then the sequence $(P_n K)_{n\in\mathbb{N}} \subset \mathcal{K}(X,Y)$ converges in norm to $K$; hence $K$ is compact as the norm limit of compact operators.

Using this criterion we now conclude the proof, observing that

$$\left\|(I - P_n)\int_a^b K(t)\,dt\right\| = \left\|\int_a^b (I - P_n)K(t)\,dt\right\| \le \int_a^b \|(I - P_n)K(t)\|\,dt.$$

By the uniform boundedness principle the integrand is bounded and converges pointwise to 0. Lebesgue's dominated convergence theorem then implies that $\int_a^b \|(I - P_n)K(t)\|\,dt$ converges to zero, and the assertion follows. □

## b. The Fourier Transform

In this book we encounter the Fourier transform $\mathcal{F}$ in basically two different situations. In the context of partial differential operators in Section VI.5 we consider it as a map acting on functions from $\mathbb{R}^N$ to $\mathbb{R}$. On the other hand, we use it in the context of harmonic analysis (e.g., in the proofs of Theorem II.4.20, Theorem V.1.11, or in Section IV.3.c) as a map defined on $\mathrm{L}^1(\mathbb{R})$ or, more generally, on $\mathrm{L}^1(\mathbb{R}, X)$ for a Banach space $X$. While the results for the first case are collected in Section VI.5.a, we now give a brief account for the second one. Moreover, we consider the analogous notion on the convolution algebra $\big(\ell^1(\mathbb{Z}), *\big)$.

To this end, we assume that $X$ is a Banach space. For a function $f \in \mathrm{L}^1(\mathbb{R}, X)$ we define its *Fourier transform* $\mathcal{F}f := \widehat{f}$ by

$$\widehat{f}(s) := \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-ist}\,dt$$

for all $s \in \mathbb{R}$.

The definition of the Fourier transform is not uniform in the literature. In fact, when considering it on $\mathrm{L}^2$-spaces it is more common to use the factor $(2\pi)^{-1/2}$ (or $(2\pi)^{-N/2}$ in the $N$-dimensional case; cf. Section VI.5.a) in front of the integral, making it an isometry on $\mathrm{L}^2$ by Plancherel's theorem. On the other hand, considered as a map on $\mathrm{L}^1$-spaces and in the context of harmonic analysis, the Fourier transform (defined as above) is just the Gelfand transformation. In particular, it maps convolutions into products (see Lemma C.12.(i) below), a property that gets lost otherwise.

As in Lemma VI.5.5, the Fourier transforms of vector-valued $\mathrm{L}^1$-functions vanish at infinity.

**C.8 Riemann–Lebesgue Lemma.** *If* $f \in \mathrm{L}^1(\mathbb{R}, X)$, *then* $\widehat{f} \in \mathrm{C}_0(\mathbb{R}, X)$, *i.e., we have* $\lim_{s\to\pm\infty} \widehat{f}(s) = 0$.

For the proof it suffices to consider step functions, for which, as in the scalar case, the assertion follows by integration by parts.

The next results are needed in the proof of the weak spectral mapping theorem in Section IV.3.c.

**C.9 Inversion Theorem.** *If the Fourier transform of $f \in \mathrm{L}^1(\mathbb{R})$ satisfies $\widehat{f} \in \mathrm{L}^1(\mathbb{R})$, then*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(s)\, \mathrm{e}^{\mathrm{i}st}\, ds$$

*for almost all $t \in \mathbb{R}$.*

We now consider the analogous concepts on the algebra $\big(\ell^1(\mathbb{Z}), *\big)$. To this end we define

$$\widehat{f}(z) := \sum_{n\in\mathbb{Z}} a_n z^n \qquad \text{for } z \in \Gamma \text{ and } f := (a_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}).$$

It is then clear that

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \widehat{f}(z) z^{-(n+1)}\, dz, \qquad n \in \mathbb{Z}.$$

Moreover, we have the following result.

**C.10 Uniqueness Theorem.** *If $h \in \mathrm{C}(\Gamma)$ such that $\int_\Gamma h(z) z^{-(n+1)}\, dz = 0$ for all $n \in \mathbb{Z}$, then $h = 0$.*

Next, we introduce the convolution product on $\mathrm{L}^1(\mathbb{R})$ and $\ell^1(\mathbb{Z})$.

**C.11 Definition.** *For $f, g \in \mathrm{L}^1(\mathbb{R})$ we define the convolution by*

$$(f * g)(t) := \int_{-\infty}^{\infty} f(s) g(s-t)\, ds, \quad t \in \mathbb{R}.$$

*For $f := (a_n)_{n\in\mathbb{Z}}$, $g := (b_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ its convolution $f * g = (c_n)_{n\in\mathbb{Z}}$ is given by*

$$c_n := \sum_{m\in\mathbb{Z}} a_m b_{n-m}, \quad n \in \mathbb{Z}.$$

We refer to [SW71] for a complete treatment of these notions. Here we only discuss some properties of the algebras $\big(\mathrm{L}^1(\mathbb{R}), *\big)$ and $\big(\ell^1(\mathbb{Z}), *\big)$ that are needed to prove the weak spectral mapping theorem for bounded groups in Section IV.3.c.

**C.12 Lemma.** *The space* $\big(\mathrm{L}^1(\mathbb{R}), *\big)$ *is a commutative Banach algebra having the following properties.*

(i) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ *for all* $f, g \in \mathrm{L}^1(\mathbb{R})$.

(ii) *Let* $K \subset \mathbb{R}$ *be closed, and* $s_0 \in \mathbb{R} \setminus K$. *Then there exists a function* $f \in \mathrm{L}^1(\mathbb{R})$ *such that* $\widehat{f}$ *has compact support and* $\widehat{f}(s_0) \neq 0$ *and* $\widehat{f} \equiv 0$ *in a neighborhood of* $K$.

(iii) *The subspace*

$$\mathcal{K} := \Big\{ f \in \mathrm{L}^1(\mathbb{R}) : \widehat{f} \text{ has compact support} \Big\} \tag{C.5}$$

*is norm dense in* $\mathrm{L}^1(\mathbb{R})$.

Proof. The fact that $\big(\mathrm{L}^1(\mathbb{R}), *\big)$ is a commutative Banach algebra can be found in [Rud62, Thm. 1.1.7]. For (i) we refer to [Rud62, Thm. 1.2.4].

To prove (ii), let $h$ be a $\mathrm{C}^2$-function on $\mathbb{R}$ such that $h(-s_0) \neq 0$ and $h_{|-V} = 0$ for some neighborhood $V$ of the set $K$ having bounded complement $\mathbb{R} \setminus V$. Then for $f := \widehat{h}$ and $0 \neq s \in \mathbb{R}$ we obtain from integration by parts that

$$\begin{aligned} f(s) = \widehat{h}(s) &= \int_{-\infty}^{\infty} h(t)\, \mathrm{e}^{-\mathrm{i}st}\, dt \\ &= \frac{1}{\mathrm{i}s} \int_{-\infty}^{\infty} h'(t)\, \mathrm{e}^{-\mathrm{i}st}\, dt \\ &= -\frac{1}{s^2} \int_{-\infty}^{\infty} h''(t)\, \mathrm{e}^{-\mathrm{i}st}\, dt. \end{aligned}$$

It follows that $f \in \mathrm{L}^1(\mathbb{R})$ and thus, by the inversion formula,

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, \mathrm{e}^{\mathrm{i}st}\, ds = \frac{1}{2\pi} \widehat{f}(-t),$$

where, since $h$ and $\widehat{f}$ are both continuous, this equality holds for all $t \in \mathbb{R}$. Hence $\widehat{f}(s_0) = h(-s_0) \neq 0$ and $\widehat{f} \equiv 0$ in $V$, proving (ii).

To prove (iii), we note that $\mathcal{K}$ is invariant under translations and multiplication by functions $t \mapsto \mathrm{e}^{\mathrm{i}\lambda t}$ for all $\lambda \in \mathbb{R}$. Hence, if $g \in \mathrm{L}^1(\mathbb{R})' = \mathrm{L}^\infty(\mathbb{R})$ vanishes on $\mathcal{K}$, then

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t} f(s+t) g(t)\, dt = 0$$

for all $f \in \mathcal{K}$, $s, \lambda \in \mathbb{R}$. Now fix $f \in \mathcal{K}$ and $s \in \mathbb{R}$. Then the function $f(s + \cdot) g(\cdot)$ is in $\mathrm{L}^1(\mathbb{R})$, and its Fourier transform is identical to zero. It follows from the uniqueness theorem that $f(s+t)g(t) = 0$ for all $f \in \mathcal{K}$, $s \in \mathbb{R}$, and almost all $t \in \mathbb{R}$. However, by (ii) we know that $\mathcal{K} \neq \{0\}$ and therefore $g = 0$. Hence, $\mathcal{K}$ is norm dense in $\mathrm{L}^1(\mathbb{R})$ by the Hahn–Banach theorem. □

Next, we consider the analogous statements for $\ell^1(\mathbb{Z})$.

**C.13 Lemma.** *The space* $\big(\ell^1(\mathbb{Z}), *\big)$ *is a commutative Banach algebra with the following properties.*

(i) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ *for all* $f, g \in \ell^1(\mathbb{Z})$.

(ii) *Let* $K \subset \Gamma$ *be closed and* $z_0 \in \Gamma \setminus K$. *Then there exists a sequence* $f \in \ell^1(\mathbb{Z})$ *such that* $\widehat{f}(z_0) \neq 0$ *and* $\widehat{f} \equiv 0$ *in a neighborhood of* $K$.

Proof. For the fact that $\big(\ell^1(\mathbb{Z}), *\big)$ forms a commutative Banach algebra we again refer to [Rud62], while assertion (i) is clear by the Cauchy product formula for series.

To prove (ii), let $h$ be a $\mathrm{C}^2$-function on $\Gamma$ such that $h(z_0) \neq 0$ and $h \equiv 0$ in a neighborhood of $K$. Consider $f := (a_n)_{n\in\mathbb{Z}}$, where

$$a_n := \frac{1}{2\pi \mathrm{i}} \int_\Gamma h(z) z^{-(n+1)}\, dz.$$

Then, using integration by parts, we obtain

$$a_n = \frac{1}{n(n-1)} \cdot \frac{1}{2\pi \mathrm{i}} \int_\Gamma h''(z) z^{-n+1}\, dz$$

and thus

$$|a_n| \leq \frac{1}{|n(n-1)|} \cdot \sup_{|z|=1} |h''(z)| \quad \text{for } n \neq 0, 1.$$

Therefore, $f \in \ell^1(\mathbb{Z})$, and since by the uniqueness theorem $\widehat{f} = h$, the sequence $f$ has all desired properties. □

We close this subsection by the following Hilbert-space-valued analogue of Plancherel's equation. Here, as in Section VI.5.a, we have to extend $\mathcal{F}$ to $\mathrm{L}^2$ and then obtain a map with the following property.

**C.14 Plancherel's Theorem.** *If* $f \in \mathrm{L}^2(\mathbb{R}, H)$ *for a Hilbert space* $H$, *then*

$$\|\mathcal{F}f\|_2 = \sqrt{2\pi}\, \|f\|_2,$$

*i.e.,* $1/\sqrt{2\pi}\,\mathcal{F}$ *is an isometry.*

For a proof we refer to [GN83, Lem. 2]. The fact that this result holds only for Hilbert-space-valued functions is the reason for many fundamental differences between the semigroup theory on Hilbert spaces and that on Banach spaces (see, e.g., Theorem II.4.20 and Theorem V.1.11).

## *c. The Laplace Transform*

As can be seen from identity (1.13) in Chapter II, the resolvent of a generator can be interpreted as the Laplace transform of the corresponding strongly continuous semigroup. While this connection is the leitmotif of the monograph [ABHN99], we use this fact only occasionally. For this reason we state only the following results and refer to [Are87b], [ABHN99], [deL94], [Doe74], [HP57, Sec. 6.2] and [Wid46] for a more detailed study.

**C.15 Definition.** *Let $X$ be a Banach space and assume that $f : \mathbb{R}_+ \to X$ is a measurable, exponentially bounded function of exponent $w \in \mathbb{R}$, i.e., $\|f(t)\| \le M\mathrm{e}^{wt}$ for all $t \ge 0$ and some constant $M > 0$. Then, we define its Laplace transform $\mathcal{L} f : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > w\} \to X$ by*

$$(\mathcal{L} f)(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t} f(t)\, dt.$$

For the Laplace transform the following analogue to Theorem C.10 holds.

**C.16 Uniqueness Theorem.** *Assume that $f, g \in \mathrm{C}(\mathbb{R}_+, X)$ are exponentially bounded. If $(\mathcal{L} f)(\lambda) = (\mathcal{L} g)(\lambda)$ for $\operatorname{Re}\lambda$ sufficiently large, then $f = g$.*

By extending two operator-valued functions $F, G \in \mathrm{L}^1(\mathbb{R}_+, \mathcal{L}(X))$ by zero onto the negative real line, we can, as in Definition C.11, define their *convolution* $F * G$ by

$$(F * G)(t) := \int_0^\infty F(s)G(t-s)\, ds, \qquad t \ge 0.$$

Then the analogue to Lemma C.12.(i) holds.

**C.17 Convolution Theorem.** *Let $F, G : \mathbb{R}_+ \to \mathcal{L}(X)$ be strongly continuous and exponentially bounded functions of exponent $w \in \mathbb{R}$. Then their convolution $F * G$ is exponentially bounded of exponent $w$ and*

$$\big[\mathcal{L}(F * G)\big](\lambda) = (\mathcal{L} F)(\lambda) \cdot (\mathcal{L} G)(\lambda)$$

*for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re}\lambda > w$.*

# Epilogue

## Determinism: Scenes from the Interplay Between Metaphysics and Mathematics

**(by Gregor Nickel*)**

*The subject of this book is evolution equations, that is, a mathematical treatment of motion in time. In this epilogue we will thus review some scenes from the history of the attempts to describe motion by mathematical means. However, we will begin with the more general question concerning a relation of "mathematics" and "reality," which we will try to exemplify by discussing the problem of motion and its determinism. In the first section we recall the mathematical framework of almost all contemporary scientific theory concerned with temporally changing systems. Section* 2 *shows that to a large extent, modern science fits into this framework. In Section* 3 *we will draw a historical line through the discussion on determinism extending from* Isaac Newton *to* Albert Einstein*. In Section* 4 *this discussion will be continued on philosophical grounds. Finally, in Section* 5 *there will be an attempt to illuminate a more encompassing dimension of the problem.*

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*As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

ALBERT EINSTEIN [11, p. 233]

This well-known, ironically resigned remark by ALBERT EINSTEIN (1879–1955) completely separates the spheres of "mathematical certainty" and "reference to reality." Thereby it not only prevents from the outset a potential conflict between them, but also defeats any possible constructive contact between mathematics and reality.

In its content the remark is indeed beyond reproach, which explains why it, in one form or another, constitutes the last line of defense for (natural) scientists confronted with the discomforting question, "What does your work tell us about reality?" At least unconsciously, however, the scientist often assumes that the world (or relevant properties thereof) in fact corresponds to its mathematical description (cf., e.g., [5], [13], and [33]). If such a naive realism is probed more deeply, then the scientist can do no better than offer the triumphs of technology as evidence of such a correspondence, while the theoretician and in particular the mathematician take cover in the complacent view that he is concerned "only with models," and the question of their relation and "application" to reality falls beyond the bounds of his accountability. Such a strict distinction makes the question of interpreting mathematics with respect to reality easy to dismiss.

However, an opposing standpoint need not directly question EINSTEIN'S remark but instead merely note—and this will be the guiding principle in what follows—that mathematicians and natural scientists have always held, and will continue to hold, that mathematical propositions refer to reality.[1] And it is almost impossible to understand, let alone stimulate, the ongoing of science without reference to that connection.

The interplay of mathematics and metaphysics will be examined here with regard to the problem of *determinism*, the conflict between freedom[2] and universal determination. The focus is on the philosophical question of whether and to what extent the course of natural events is predetermined. Inextricably bound up with this is the question of human freedom. While this is certainly true from a historical perspective, in so far as the various historical positions are to be understood in terms of their motives and inner justifications, it is also true from a systematic perspective, in so far as being human is to be understood as the ambivalent state of being simultaneously natural and rational. Determinism is of paradigmatic significance to us here not only because this book deals with one-parameter semigroups, which—as will be shown in Section 1—can be seen as the mathematical for-

[1] It is often not a matter of how the position is explicitly formulated; for example, even in EINSTEIN'S work the connection between mathematical-scientific theory and the world view he personally favored shows up clearly (cf. Section 3).

[2] The question of freedom as such, however, will not be discussed deeply; at this point it is sufficient to note that freedom is not to be confused with "indeterminism" or "chance."

mulation of autonomous, deterministic motion,[3] but also because it is one of the fundamental problems that have been repeatedly discussed and reformulated.[4] It lies at the core of the long-lasting debate on representing nature by mathematical terms.[5]

## 1. The Mathematical Structure

A large part of contemporary natural science is concerned with investigating the motion of systems. "Motion" will designate, here and in what follows, all forms of temporal change, and is thus a much more general term than mere change of location. The mathematical framework for this investigation can be outlined as follows:[6]

(1) The object of inquiry is the *motion* of a *system* in *time*.

(2) *Time* is represented by the additive group of real numbers $\mathbb{R}$ (or the additive semigroup $\mathbb{R}_+$). We thus use a one-dimensional, homogeneous continuum consisting of single points.[7]

(3) The *system* under consideration is characterized by a set $\mathcal{Z}$—the *state space*—of distinct states $z \in \mathcal{Z}$ whose temporal change is to

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[3] An attempt to look at determinism from the point of view of mathematics can be found in the monograph [10, Sec. XV.13] by N. Dunford and J.T. Schwartz.

[4] So Gottfried Wilhelm Leibniz remarks in the preface to his well-known theodicy: *There are two famous labyrinths where our reason very often goes astray: one concerns the great question of the Free and the Necessary (...) the other consists in the discussion of continuity and of the indivisibles which appear to be the elements thereof, and where the consideration of the infinite must enter in. The first perplexes almost all the human race, the other exercises philosophers only* [26, p. 53].

[5] Today, however, the position is often held that determinism is a concept now out of style. At this point it suffices to remark that most of today's science deals with deterministic motion in our sense (see Section 1 below). Though there may exist many objections against a strict determinism, modern science has not (yet?) overcome this concept. A concise discussion and critique of scientific determinism may be found in [34]. The two building blocks of science, experiment and mathematical theory, are shown to be grounded on the one hand on the *freedom* of the scientist to choose his or her experiment or description, yet on the other hand on strict determinism of nature and its accordance with the logical laws of thought. Thus, science is a paradoxical enterprise based on extracting the human observer from the natural world.

[6] It should be emphasized that none of the following assumptions are evident; every one could be criticized; see, e.g., the next footnote. Of course, also in mathematics there are various concepts for modeling motion. We concentrate here on the case of (reversible) motion with continuous time and global existence and assume a certain time regularity. However, more complicated behavior also can be discussed in a similar setting. For instance, stochastic time evolutions in some original state space $\mathcal{Z}$ may fit into this scheme by taking the space of probability densities $\mathrm{L}^1(\mathcal{Z})$ as a new state space.

[7] This identification is not so innocent as it might appear. While many criticisms of the definition could be cited here, one from David Hume will suffice:

*An infinite number of real parts of time, passing in succession, and exhausted one after the other, appears so evident a contradiction, that no man, one should think, whose judgement is not corrupted, instead of being improved, by the sciences, would ever be able to admit of it* [18, p. 424].

be determined. The set of all possible states of the system is thus fixed from the outset. For example, the state space of a "planetary system" may be taken as the positions and velocities (or momenta) of all planets, or the state space of an "ecosystem" can be chosen as the number of individuals belonging to each relevant species.

(4) The *motion* of the system is represented by the temporal change of states or, in mathematical terms, by a function $\mathbb{R} \ni t \mapsto z(t) \in \mathcal{Z}$ that maps each instant $t \in \mathbb{R}$ to a unique state $z(t) \in \mathcal{Z}$.

Up to now we have described a single motion by means of a function $z(\cdot)$.[8] In this sense, the motion can already be called deterministic. This perspective develops its full force if all possible motions are taken into account.

(5) For every instant $t_0 \in \mathbb{R}$ and every initial state $z_0 \in \mathcal{Z}$ there exists a unique motion $z_{t_0,z_0} : \mathbb{R} \to \mathcal{Z}$ satisfying $z_{t_0,z_0}(t_0) = z_0$.

We point out that now the role of the observer has changed essentially. Rather than just "describing" what happens, the observer requires that the system can be restarted with any prescribed initial data. In this way, the observer becomes an experimenter, which is constitutive for modern natural science. Due to assumption 5, we can vary the initial value $z_0 \in \mathcal{Z}$ at time $t_0$ and obtain a uniquely determined state $z_{t_0,z_0}(t_1)$ of the system at a target time $t_1$. This defines a mapping

$$\Phi_{t_1,t_0} : \mathcal{Z} \to \mathcal{Z}, \qquad \Phi_{t_1,t_0}(z_0) := z_{t_0,z_0}(t_1).$$

In the next step, we take $z_1 = \Phi_{t_1,t_0}(z_0)$ as a new initial state and $t_1$ as a new initial time. Again by assumption 5, we obtain at time $t$ a unique state given by $\Phi_{t,t_1}(z_1) = \Phi_{t,t_1}\big(\Phi_{t_1,t_0}(z_0)\big)$. This state must coincide with the state achieved at time $t$ by the original motion $z_{t_0,z_0}(\cdot)$, which also passes through $z_1$ at time $t_1$, that is,

$$\Phi_{t,t_1}\big(\Phi_{t_1,t_0}(z_0)\big) = \Phi_{t,t_0}(z_0)$$

holds for all $z_0 \in \mathcal{Z}$ and $t, t_1, t_0 \in \mathbb{R}$.[9] For the family of mappings $\{\Phi_{t,s} : t, s \in \mathbb{R}\}$ this means

$$\Phi_{t,s} \circ \Phi_{s,r} = \Phi_{t,r} \tag{1.1}$$

[8] The observer outside the system can (at least theoretically) consider this motion as a whole by regarding the function $z(\cdot)$. The system itself has at no time another option to "choose" than that prescribed.

[9] Here we adopt the—apparently self-evident—view that we can evaluate the motion at any intermediate time. However, the discussion of the implications of dissecting a motion to individual steps goes back at least to Aristotle (384–322). In his lecture on nature, he sharply distinguishes between an *actual* interruption of movement (that of a *"mobile,"* i.e., a moving object, along a line) and its mere *possibility:*

> *(...) whereas any point between the extremities may be made to function dually in the sense explained [as beginning and as end, G.N.], it does not actually function unless the mobile actually divides the line by stopping and beginning to move again. Else there were one movement, not two, for it is just this that erects the "point between" into a beginning and an end (...)* [2, p. 373].

and, trivially,

$$\Phi_{t,t} = I \tag{1.2}$$

for all $t, s, r \in \mathbb{R}$ and the identity $I : z \mapsto z$. We call such a system consisting of the state space $\mathcal{Z}$, the time space $\mathbb{R}$ (or $\mathbb{R}_+$), and a family of mappings $\Phi_{t,s} : \mathcal{Z} \to \mathcal{Z}$ satisfying (1.1) and (1.2) a *deterministic system*.

In many situations, e.g., if the governing physical laws do not change in the course of time and no external force acts on the system, it is reasonable to make the following assumption.

(6) The state $z_1 = \Phi_{t,s}(z_0)$ at time $t$ depends only on the initial state $z_0$ at time $s$ and the time difference $\tau = t - s$.

Such systems will be called *autonomous*.[10] For the mappings $\Phi_{t,s}$ this means $\Phi_{t,s} = \Phi_{r,u}$ whenever $t - s = r - u$. We can now write

$$T(t) := \Phi_{r,r-t}$$

and obtain a one-parameter family $\{T(t) : t \in \mathbb{R}\}$ of mappings on the state space $\mathcal{Z}$ satisfying

$$T(s)\,T(t) = T(t+s) \quad \text{and} \quad T(0) = I \tag{1.3}$$

for all $t, s \in \mathbb{R}$. A family of mappings fulfilling equation (1.3) is called a *one-parameter group* (or a *one-parameter semigroup* if $t \in \mathbb{R}_+$). It is our mathematical model of autonomous, deterministic motion of a system in time.[11]

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If a continual motion occurs, then there is no justification for saying the object is in the middle position (during a given period of time):

> *But if anyone should say that it [the mobile A, G.N.] has "arrived" at every potential division in succession and "departed" from it, he will have to assert that as it moved it was continually coming to a stand. For it cannot "have arrived" at a point [B, G.N.] (which implies that it is there) and "have departed" from it (which implies that it is not there) at the same point in time. So there are two points of time concerned, with a period of time between them; and consequently A will be at rest at B (...)* [2, p. 375].

From this quite consistent perspective, the deduction of a relation as given in (1.1) certainly seems problematic. We will return to this question when considering quantum mechanics in Section 2.

[10] Every system whatsoever can be embedded in a larger autonomous system by integrating the changing environment into the system until external change is eliminated. For a corresponding mathematical procedure for associating an autonomous system, see [31] or Section VI.9.

[11] The explicit use of the semigroup equation (1.3) seems to have occurred rather late in the literature of mathematical physics. Carl Einar Hille (1894–1980), one of the founders of modern semigroup theory, writes:

> *Like Monsieur Jourdain in Le Bourgeois Gentilhomme, who found to his great surprise that he had spoken prose all his life, mathematicians are becoming aware of the fact that they have used semi-groups extensively even if not always consciously. (...). The*

We add a final word on the choice of the state space $\mathcal{Z}$. The state space should describe those properties essential for the observer, and a given state at a specific time should determine any further motion. These (scientific, not objective!) requirements necessitate a careful balancing; the history of physics shows numerous examples of how a state space was chosen, but then later was altered—usually enlarged—with the goal of "saving" a deterministic motion, or the semigroup property. The "correct" state space is precisely that which, on the one hand, contains all relevant properties, and on the other hand guarantees deterministic motion.

This can be illustrated nicely by the examples mentioned in assumption 3. The future behavior of the planetary system is not determined solely by the positions of the planets at a given time but only if we take into account positions and momenta. In a second example, we may describe the size of a population at time $t$ by the real number $x(t)$. If we assume that the number of newborns depends on the size $x(s)$ of the population during the time interval $t - \tau \leq s \leq t$, then the correct state space is not $\mathbb{R}$ but a space of functions $f : [-\tau, 0] \to \mathbb{R}$. We refer to Section VI.6 for a systematic treatment of such situations and also to [19] for a look at historic discussions concerning state spaces.

## 2. Are Relativity, Quantum Mechanics, and Chaos Deterministic?

It is a frequently voiced opinion that twentieth century scientific theory has revolutionized philosophy's view on determinism. In particular, one refers to *relativity*, *quantum mechanics*, and *chaos theory* in this context. In this section we intend to show that these three more or less closed and mathematically codified theories can be integrated into the scheme sketched in Section 1.

First, EINSTEIN'S theory of *relativity* elaborates in its special relativistic part indeed a new structure of space-time compared with NEWTONIAN

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*concept was formulated as recently as 1904, and it is such a primitive notion that one may well be in doubt of its value and possible implications* [17].

One of the first scientists to have used the concept of semigroups to formulate a mathematical concept of determinism appears to be JACQUES HADAMARD (1865–1963) in his lectures on differential equations [Had23, p. 53]. With reference to CHRISTIAN HUYGENS' (1629–1695) treatment of light diffusion, HADAMARD discusses HUYGENS' "principle" in the form of a syllogism, whose major premise implicitly contains the semigroup law: *"(major premise). The action of phenomena produced at the instant* $t = 0$ *on the state of matter at the instant* $t = t_0$ *takes place by the mediation of every intermediate instant* $t = t'$*, i.e. (assuming* $0 < t' < t_0$*), in order to find out what takes place for* $t = t_0$*, we can deduce from the state at* $t = 0$ *the state at* $t = t'$ *and, from the latter, the required state at* $t = t_0$*".* The premise is designated as a *"law of thought"* or as a *"truism,"* which nevertheless has interesting consequences. For it corresponds *"to the fact that the integration of partial differential equations defines certain groups of functional operations; and this for instance leads to quite remarkable identities (...)."* See also [Hil48, Sec. 20.2] and Section II.6.

mechanics. However, the causal structure is neither globally nor locally very different. To obtain accord with the picture of Section 1 you have only to consider the *Eigenzeit* for every body or every observer. In cosmology, the standard model uses a global, absolute "cosmologic" time within which matter evolves deterministically.[12] Finally, some of the most famous researchers in the field of relativity and cosmology are very fond of the idea of strict determinism; compare, for example, [14] and, of course, EINSTEIN himself.

Second, *quantum mechanics* offers a more complex picture. A thorough examination of its various interpretations would lead far beyond the limits of this epilogue (for a deeper study see [20], [32], [36]). In particular, we concentrate on the closed theory of ca. 1930. With respect to the question of determinism one has to distinguish between, roughly speaking, two different time evolutions in quantum mechanics. There is first the *unitary time evolution* of the individual state (Ψ-function) following SCHRÖDINGER's equation

$$\dot{\Psi}(t) = \frac{-i}{\hbar} H \Psi(t)$$

with Hamiltonian $H$. Indeed, this is no better example of an abstract Cauchy problem solved by a group $(\mathrm{e}^{-\mathrm{i}/\hbar Ht})_{t\in\mathbb{R}}$. The state of a quantum system thus evolves deterministically in the sense of Section 1 as long as no measurement process takes place.

The measurement process represents the second form of time evolution in quantum mechanics, and there has been a long-lasting discussion about the "primary" probabilities occurring with the measurement process.[13] Measuring a quantum system—also called "reduction of the wave packet" or "projection"—is codified by *stochastic time evolution*, where the (absolute square of the) state $\Psi(t)$ can be interpreted as a probability density (BORN's interpretation). Whether these primary probabilities (thus a weak form of indeterminism) will give way to a more fundamental (crypto)-deterministic description is a problem of current research (see [36, p. 120]).

Thus, in quantum mechanics a new state space has been introduced that allows a deterministic theory. Indeed, the fundamental scientific paradigm shift in the wake of quantum mechanics does not concern determinism, but raises the question of the reducibility of the (material) world into single, trivially combined elements, as well as the related question of the observer's perspective. This question, only loosely connected to the problem of determinism, is given a negative answer within the framework of quantum

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[12] The space-time metric is taken to be separated with respect to space and time, and it is possible to define an absolute time coordinate given by the *Eigenzeit* in the inertial system of the "freely falling" cosmological substrate.

[13] It is remarkable that the *real dissection* of the motion due to measurement causes these troubles. As in ARISTOTLE's description (see footnote 9) it changes the course of time evolution and is to be discussed as distinct from a mere *potential dissection*. Interestingly enough, in contrast to KANT, the perspective of the observer might here be the reason for a certain indeterminism (see Section 4).

mechanics; *each* (elementary or nonelementary) quantum-mechanical system is typically a unit that is not to be regarded as consisting of building blocks. In the transition from classical to quantum mechanics, this new concept of the object is much more essential than any change in the notion of causality.[14]

Third, the so-called *chaos paradigm*, from which quite often far-reaching "philosophical conclusions" are derived, appears (with a few possible exceptions, e.g., [37]) as a disheartened return to classical physics, since an interpretation of or reconciliation with quantum mechanics has proved to be too difficult. This paradigm (apparently) once again refers to objects in nature as if they behaved like billiard balls. On the level of the objects there holds thus a strict determinism. The only new development is an improvement of the mathematical rigor in discussing the (epistemic) problem that knowledge of the initial state with finite accuracy gives rise to a (dramatically) reduced possibility of forecast. Yet this *sensitive dependence on the initial conditions*, today's completely "new" insight, can already be found in historical examples. LEIBNIZ, for instance, had already described the "butterfly effect" in his small essay "On Destiny":

> *And often, such small things can cause very important changes. I used to say a fly can change the whole state, in case it should buzz around a great king's head while he is weighing important counsels of state (...). And even this effect of small things causes those who do not consider things correctly to imagine some things happen accidently and are not determined by destiny, for this distinction arises not in the facts but in our understanding* [25, pp. 571–72].

## 3. Determinism in Mathematical Science from Newton to Einstein

The dominant role mathematics plays in the interpretation of nature is not obvious, but it is the result of a scientific revolution whose essence is well captured in GALILEO GALILEI'S (1564–1642) statement "the book of nature is written in mathematical language" and that culminates in

[14] This is, in summary version, an exemplification of the claim of ERNST CASSIRER (1874–1945) presented in his book *Determinism and Indeterminism in Modern Physics*, which he published in 1936. This work's systematic hypothesis states:

> *The answer that an epistemology of science gives to the problem of causality never stands alone but always depends on a certain assumption as to the nature of the object in science. These two are intimately connected and mutually determine each other* [9, p. 6].

That is, one cannot assume that the object to be investigated is completely given in all its possible facets, in order *then* to examine its causal interactions. Rather, the reverse is true: The inquiry into a particular kind of causality determines what kind of objects will be perceived by the observer.

Isaac Newton's (1643–1727) magnum opus *Philosophiae Naturalis Principia Mathematica*. With Newton's equations of motion, the behavior of a mechanical system could be calculated for the first time, and this made determinism, in the formal sense as described in Section 1, for the first time thinkable.

It is interesting to note that Newton himself explicitly ruled out the possibility of a complete determinism for all events in the universe. After establishing in the first book of his *Principia* the principles of motion, the structure of space and time, and then the relationship of Kepler's laws of planetary motion to his own law of gravity, Newton proceeds to investigate in the third book the well-known data concerning planets, moons and comets. He singles out for emphasis the regularity of the concentric motion of all planets and their moons in the same plane and in the same direction, and concludes that

> *it is not to be conceived that mere mechanical causes could give birth to so many regular motions (...). This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being (...) and lest the system of fixed stars should, by their gravity, fall onto each other, he hath placed those systems at immense distances from one another* [30, p. 544].

The regular harmony of the solar system and the stable order of the fixed stars, which Newton claimed could not have arisen by mechanical laws alone, suggest the hand of a Creator. This view is expressed more precisely in the following opposition between metaphysical necessity and the God of creation:

> *Blind metaphysical necessity, which is certainly the same always and everywhere, could produce no variety of things. All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing (...)* [30, p. 544].

According to Newton, God not only has to create an ordered planetary system, but he must also incessantly intercede. He gives two reasons for this: First, matter in motion continuously loses kinetic energy so that an external impulse is necessary to keep the system in motion; second, Newton was unable to demonstrate the entire system's stability. Here, God's intervention came to his aid.

Newton's contemporary Gottfried Wilhelm Leibniz (1646–1716) formulated an incisive critique of these views. For both metaphysical as well as scientific reasons, Leibniz found Newton's image of God unacceptable. The *mathematical* difference can be expressed in modern terminology in the following way: Newton's thought was dominated by the causal

structure of a differential equation, or more precisely, an initial value problem that repeatedly arises at different instants, whereas LEIBNIZ'S thought was structured by a conservation law and a (temporally) global variational principle of least action subject to a more teleological interpretation. In this case, real motion results as the minimum of an integral of action defined for all possible motions.[15] Only later did it become clear that both approaches, the principle of least action and the initial value problem, are mathematically equivalent (for a modern account, see [3, p. 55]).

LEIBNIZ opens the debate with NEWTON in a letter to the princess of Wales—which was of course intended to be aired in public—in which he writes:

> *Sir Isaac Newton, and his followers, have also a very odd opinion concerning the work of God. According to their doctrine, God Almighty wants to wind up his watch from time to time: otherwise it would cease to move. He had not, it seems, sufficient foresight to make it a perpetual motion (...). According to my opinion, the same force and vigor remains always in the world, and only passes from one part of matter to another, agreeably to the laws of nature, and the beautiful pre-established order* [27, pp. 11–12].

NEWTON did not personally reply to the charge; he had his student and associate SAMUEL CLARKE (1675–1729) respond. However, that the responses largely grew out of NEWTON'S direct intervention can be shown from scattered notes in his handwriting. So began a correspondence, consisting of five increasingly long letters from each, that was one of the most intriguing controversies of the age. CLARKE wrote in his first response:

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[15] A vehement controversy arose as early as the eighteenth century concerning the question of priority in the discovery of the principle of least action. One of the first versions stems from MAUPERTUIS (1698–1759), which was then given its precise formulation by LEONHARD EULER (1707–1783) (a description of this controversy is offered in [38]). LEIBNIZ can be regarded only as predecessor of this discovery, although in his philosophical writing it appears with all desirable clarity.

He describes this universal principle at the conclusion of his theodicy, and it is no accident that his account is cloaked in mythical guise. Moreover, it is remarkable that the description is closely connected with the problem of strict determinism. Theodorus, a priest of Jupiter, is witness to a terrible fate foretold by the god to a man seeking his advice, a fate the latter cannot avoid. In response to Theodorus's question whether God could not have created the world differently so as to prevent such a terrible fate, Pallas Athena appears to him in a dream and shows him the splendor of the divine (actually LEIBNIZ'S) universal order. They come to a place whose every chamber represents a possible course of events:

> *The chambers were ordered in the shape of a pyramid: the closer one came to the apex, the more beautiful they and the worlds they represented became. Finally one arrives at the highest chamber, which crowns the pyramid, and this was the most beautiful of all. For the pyramid had indeed a beginning, but its end could not be seen; it had a top, but no base: it continued on into infinity. This is because (...) in an infinity of possible worlds one of them is the best, for otherwise God would have had no reason to create even one of them. But there is no world under which a less perfect one could not be found: that is why the pyramid continually descends into infinity* [28, p. 408] (author's translation).

> *The notion of the world's being a great machine, going on without the interposition of God (...) is the notion of materialism and fate, and tends (under pretence of making God a supra-mundane intelligence), to exclude providence and God's government in reality out of the world. And by the same reason that a philosopher can represent all things going on from the beginning of the creation, without any government or interposition of providence; a skeptic will easily argue still farther backwards, and suppose that things have from eternity gone on (as they do now) without any true creation or original author at all, but only what such arguers call all-wise and eternal nature* [27, p. 14].

Clarke points out the loss of kinetic energy that results from inelastic impacts, which makes an intervention by God necessary to maintain the system. Leibniz answers this by referring to a principle of conservation that is consonant with the perfection of divine creation:

> *If active force should diminish in the universe, by the natural laws which God has established; so that there should be need of him to give a new impression in order to restore that force (...) the disorder would not only be with respect to us but to God himself. He might have prevented it (...) and therefore, indeed, he has actually done it* [27, p. 29].

Clarke then poses the question of human freedom in a universe governed by Leibniz's principle of conservation:

> *To suppose that in spontaneous animal-motion, the soul gives no new motion or impression to matter; but that all spontaneous animal-motion is performed by mechanical impulse of matter; is reducing all things to mere fate and necessity* [27, p. 51].

Leibniz's reply refers to his system of preestablished harmony, which he claims permits him simultaneously and without contradiction to pair human freedom with a determinism of the human body (see [27, pp. 95 and 99]).

The respective positions in this controversy can be briefly characterized in the simple antithesis that Leibniz accuses Newton of constructing a theory of the heavens founded on the capriciousness of God, whereas Newton (or in his stead, Clarke) accuses Leibniz of deism or fatalism. Newton's God does too much, Leibniz's too little.

The critique of Newton's concept of a God who indulgently intervenes time and again in the system of nature, which Leibniz developed from (essentially) metaphysical grounds, was taken up again nearly a century later by Pierre Simon de Laplace (1749–1827). In his case, however, it is supported by further developments in mathematical physics. He was able to analyze the equations for the motion of the planetary system more

precisely in order to answer (as he thought) the question of stability.[16] In addition, LAPLACE[17] attempted to derive from just these equations the laws ruling the origin of the planetary system, whose orderliness NEWTON found wonderful yet inexplicable.

In his treatise on the origin of the planetary system LAPLACE wrote:

> *I cannot leave off remarking how much Newton, in this matter, departed from the method he otherwise applied so skillfully (...). He would consider the point all the more substantiated, if he had known what we have proved, namely, that the conditions for the order of the planets and their satellites are precisely the same as those which guarantee their stability (...). But cannot this very order of the planets itself be a result of the laws of motion?* [23, p. 474].

His contributions to the development and application of classical mechanics to astronomical problems led therefore in a twofold way to important results: First, an "explanation" of the stability of the solar system and, second, a purely mechanical model for its coming into being. In the introduction to his essay on the theory of probability (1814), LAPLACE extended this paradigm constituted by classical mechanics to the motion of the entirety of the universe:

> *All events, even those which on account of their insignificance do not seem to follow the great laws of nature, are as a result of [them] just as necessary as the revolutions of the sun* [24, p. 3].

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[16] LAPLACE discussed a perturbation problem considering the elliptic orbit of some planet perturbed by gravitation due to the other planets in terms of a series expansion. He was able to show that the first few terms of this expansion are only of the form $(a \cos \omega t)$, thus "harmless" periodic deviations; no "dangerous" "secular" term of the form $(at)$ or "mixed term" of the form $(at \cos \omega t)$ appeared. He thus concluded that under the condition of elliptic motion in the same plane and direction the planetary system is stable. Without these conditions it was clear that no stability was to be expected, e.g., if the moon would circulate perpendicular to the ecliptic, it would fall onto the earth within four years due to the influence of the sun. However, even though one could show that *all* terms of this expansion are only of periodic form, the convergence of the whole series is not guaranteed. LAPLACE's stability is thus valid only on a sufficiently small time interval. For the mathematical content see [4, p. 69].

[17] Earlier even than LAPLACE, the young IMMANUEL KANT had already undertaken to explain the form of the planetary system on the basis of NEWTON'S mechanics. He describes the formation of material whirlpools that arise from an initially homogeneous distribution of matter due to the effects of gravitational forces and eventually form the milky way and the planetary systems. In this, KANT takes up LEIBNIZ'S argumentation. The perfect Creator designs the universe from the very beginning so that the currently visible order is necessarily generated according to deterministic laws:

> *Matter (...) is thus tied to certain laws, and will necessarily bring forth beautiful combinations if it freely follows their lead. It has not the freedom to stray from this plan of perfection (...) and there is a God precisely because nature, even in utter chaos, cannot proceed otherwise than orderly and in accordance with rules* [21, A XXVIII, pp. 234–35].

Only ignorance of the causes of a particular motion could lead to adopting contingency or final purposes as an explanation. Slowly but surely, all true causes unveil themselves to the inquiring mind. In this context appears the famous reference to the comprehensive determination and the potential calculability of the universe. Revealingly enough, and despite his previous reference to LEIBNIZ, he does not mention a metaphysical God as the unique example of omniscient, absolute spirit:

> *We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence* [24, p. 4].

LAPLACE'S daemon, the notion of an entity overseeing the entire universe in all its details, reappears in 1872 in the famous speech "On the Limits of Our Knowledge of Nature" delivered by the biologist, physiologist, and philosopher EMIL DU BOIS-REYMOND (1818–1896). Just before the upcoming crisis initiated by the rise of quantum mechanics, the classical physical paradigm found a particularly clear expression in DU BOIS-REYMOND. Within the framework of this scientific description of nature there emerges a suggestion of a universal formula that would guarantee complete transparency:

> *It is even conceivable that our scientific knowledge will reach a point which would allow the workings of the entire universe to be represented by One mathematical formula, by One immeasurable system of simultaneous differential equations, from which the position, the direction of motion, and the speed of every atom in the universe could be calculated at any time* [7, p. 443] (author's translation).

At the same time, however, DU BOIS-REYMOND sets up a strict limit to knowledge: First, the atomic matter presupposed by mechanics is nothing more than a useful fiction; a *"philosophical atom"* conceived as existing beyond this pragmatic construction is *"on closer examination an absurdity"* [7, p. 447]. Second, not only consciousness, but even the most simple qualitative sensations, are irremediably out of reach for the natural scientist. Even a complete and *"astronomically exact"* knowledge of all material systems including the human brain, which is in principle attainable, leaves the question of the nature of consciousness untouched, and natural scientists will always have to answer the question with the reply *"Ignorabimus."* This

claim is substantiated by the unbridgeable gulf separating the quality-less descriptions of mechanics and the qualities of perception as well as intentionality:

> *Astronomical knowledge of the brain (...) reveals it to be nothing but matter in motion (...). What conceivable connection is there between certain movements of certain atoms in my brain, on the one side, and on the other, the facts which are for me primary, undefinable, indisputable: "I feel pain, I feel pleasure; I taste something sweet, smell the scent of roses, hear the piping of the organ, see red" (...). It is quite incomprehensible, and shall remain so forevermore, that for a number of carbon, hydrogen, nitrogen, and oxygen atoms it is not a matter of complete indifference where they are and where they are going, where they were and where they went, where they will be and where they will be going (...)* [7, p. 457].

Du Bois-Reymond accords great importance to the question of the *"irreconcilable contradiction"* between the *"world-view established by mechanical physics"* and *"freedom of the will."* However, this question is held to be logically subordinate to the above problem of sensory qualities. Du Bois-Reymond's own position in this matter is peculiarly vague. After having curtly brushed aside as *"most dark and self-inflicted aberrations"* the various historical efforts[18] at grappling with the problem of free will, he formulates his *"monistic view"* as the result of a consequent application of the law of conservation of energy:

> *Conservation of energy means that force is created or destroyed just as little as matter (...). The molecules of the brain can only ever fall in a certain way, just as ineluctably as dice after leaving the tumbler (...). Now if, as monism conceives it, our thoughts and inclinations, and this includes our acts of volition, are incomprehensible yet necessary side effects of the stirrings and fluctuations of our brain molecules, then it makes sense to say there is no freedom of the will. For monism, the world is a single mechanism, and in a mechanism, there is no room for freedom of the will* [8, p. 82].

Yet in the end, du Bois-Reymond considerably qualifies his position in view of the exigencies of practical life. Even the *"most resolute monist"* could hardly maintain that each and every action is already predetermined by mechanical necessity.

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[18] He refers, albeit negatively, to the efforts of contemporary French mathematicians to make room for free will within the framework of a theory of solutions for differential equations. According to these attempts, free will could be integrated into the sphere of mechanical descriptions by means of the phenomenon of bifurcation, which refers to a breakdown of uniqueness of the solutions of differential equations (see [19]).

For du Bois-Reymond, there remained only the fundamental and not rationally decidable alternative of strictly denying free will or of asserting such freedom but at the cost of conceding a *"mystery"* unable to be solved. Thus in du Bois-Reymond's own account, the problem finds not a solution but a new formulation:

> *The writings of the metaphysicians offer a long series of attempts at reconciling freedom of the will and moral law with a mechanical order of the universe. If anyone, Kant for example, had achieved this squaring of the circle, then this series would reach its end. Only inconquerable problems are in the habit of being so immortal* [8, p. 87].

In view of the fact that he is the last great determinist of the twentieth century, this account cannot overlook Albert Einstein. His debate with Niels Bohr (1885–1962) on the interpretation of quantum mechanics—a discussion nearly as complex and far-reaching as the one between Leibniz and Clarke—cannot be treated here in full detail. With respect to the question of determinism it exhibits considerable similarities to the Leibniz–Clarke debate. It can more or less even be read as its repetition under altered historical conditions: Einstein (Leibniz) insists on causal determinacy; Bohr (Newton) refuses such a metaphysical commitment.[19] On the one hand there is the pragmatic position of Bohr, who claims that the results of quantum mechanics

> *(...) led us to recognize that the adequacy of our whole customary attitude, which is characterized by the demand for causality, depends solely upon the smallness of the quantum of action in comparison with the actions with which we are concerned in ordinary phenomena* [6, p. 116],

and for whom the question of determinism versus freedom of the will cannot be answered in terms of either-or. Rather, there is a

> *(...) parallelism between the renewed discussion of the validity of the principle of causality and the discussion of a free will which has persisted from earliest times. Just as the freedom of the will is an experiential category of our psychic life, causality may be considered as a mode of perception by which we reduce our sense impressions to order. At the same time, however, we are concerned in both cases with idealizations (...) which depend upon one another in the sense that the feeling of volition and the demand for causality are equally indispensable elements*

[19] However, there are also differences in style and content. For example, while Bohr and Einstein are discussing their subject in terms of a much more advanced theory (quantum mechanics and relativity) and they are talking about boxes with clocks, weights, and small machines, most of the debate between Leibniz and Clarke—almost completely omitting technical jargon—is concerned with the question of freedom, God, and metaphysical concepts of this type. For a detailed account of the Bohr–Einstein debate see [20].

> *in the relation between subject and object which forms the core of the problem of knowledge* [6, p. 116].

On the other hand there is Einstein's position of "metaphysical mathematicism:"

> *But the scientist is possessed by the sense of universal causation. The future, to him, is every whit as necessary and determined as the past (...). His religious feeling takes the form of a rapturous amazement at the harmony of natural law, which reveals an intelligence of such superiority that, compared with it, all systematic thinking and acting of human beings is an utterly insignificant reflection* [11, p. 40].

Such a position can consequently lead to a rejection of the belief in freedom:

> *I do not at all believe in human freedom in the philosophical sense. Everybody acts not only under external compulsion, but also in accordance with inner necessity* [11, p. 8].

Einstein's cosmological religion repeats almost literally the vision articulated by Leibniz; it reveals the same reverent wonder at a universe experienced in perfect harmony. And in the mathematical background there lies a correspondence between the field equations of general relativity and the universal principle of least action.

Despite not providing an ultimate answer to the question of determinism, the debate between Leibniz and Newton, the contributions of Laplace and du Bois-Reymond, and the controversy of Einstein and Bohr have shown that this question was an important motivating force for the development of natural science although answered in quite divergent ways (even at the same time). Metaphysics remains entangled with natural science: *"Not worrying about philosophical questions in science is too cheap a solution"* [36, p. 251].

## 4. Developments in the Concept of Object from Leibniz to Kant

We will try to give in this section a concise summary of the philosophical reflections on determinism that followed the rise of modern science, and which were best expressed in the works of Leibniz, Hume, and Kant. Especially the changes in the concepts of object and observer and their relationship will be emphasized here.

As early as in Leibniz's work there can be found an omniscient "world-spirit." He claims

*that everything proceeds mathematically, that is, infallibly in the whole wide universe, to the extent that if someone were to have sufficient insight into the inner workings of things, and had enough memory and intelligence to take on and account for all circumstances, he would be a prophet, and see the future in the present, as if in a mirror* [25, p. 571].

The basis of this "picture" is the *unique, universal perspective* of a metaphysical God detached from the world but omnisciently observing the whole.

Soon after his controversy with CLARKE, LEIBNIZ'S harmonically ordered universe, in which knowledge and being, concept and object, enjoy a preestablished agreement, was shaken to its foundations under the weight of DAVID HUME'S (1711–1776) radical skepticism. In his *An Enquiry Concerning Human Understanding,* HUME repeatedly warns against extending philosophical investigation to questions *"that lie entirely out of the sphere of experience"* [18, p. 358]. While treating the question of causal connection, HUME makes a preliminary decision: He rejects LEIBNIZ'S assumption of a preexisting world of simple substances persisting independently of perception. For HUME reality consists of *"perceptions"* and experience, which is the recollection of past perception. It is thus the *individual perspective of any observer* that provides the starting point for all further considerations. Under these conditions, there is no logical argument for any kind of causal connection between perceptions:

*That the sun will not rise tomorrow is no less intelligible a proposition, and implies no more contradiction than the affirmation, that it will rise* [18, p. 322].

Accordingly, one cannot differentiate between an expectation supported by natural causality and one stemming from human acts:

*A man who at noon leaves his purse full of gold on the pavement at Charing Cross, may as well expect that it will fly away like a feather, as that he will find it untouched an hour after* [18, p. 372].

At the end of this analysis remains the useful belief, generated by the power of custom, in the causal connection of events. Such belief cannot be rationally justified but is based on mere feeling. Consequently, mathematics is denied the power to found causal relations between facts. According to HUME, the certainty of mathematical deduction cannot be transferred to the sphere of perception:

*Nor is geometry, when taken into the assistance of natural philosophy, ever able to (...) lead us into the knowledge of ultimate causes (...). Every part of mixed mathematics proceeds upon the supposition that certain laws are established by nature in her operations (...) but still the discovery of the law itself is owing merely to experience, and all the*

> *abstract reasonings in the world could never lead us one step towards the knowledge of it* [18, p. 327].

With his skeptical analysis of philosophical reason, HUME had set a standard of reflection that forbade any further uncritical use of metaphysical ideas—even the apparently clear and fundamental concept of causality. But a rigorous application of his skeptical results made completely incomprehensible the far-reaching success of the mathematical description of physical processes given by NEWTON'S mechanical physics. Moreover, and characteristically enough, the phenomenon of human freedom was also left underdeveloped in HUME'S writings.

At this point KANT'S critical revolution sets in. On the one hand, KANT not only confirms HUME'S skeptical analysis in its full extent but he takes the argument still further. There is simply no possibility whatsoever for reason to make grounded claims concerning the causality of objects. On the other hand, if reason is not to fall prey to a blanket skepticism and if one wants to understand the successes of a mathematical approach to nature, then the focus of philosophical inquiry must be altered. Thus, *"the critical question is to be addressed not immediately to things, but rather to knowledge"* [9, p. 17]. The principle of causality now expresses nothing more than a mode of relation between objects and the perceiving subject. KANT'S *"critical determinism"* is a *"principle for the formation of empirical concepts, an assertion and a prescription as to how we should grasp and form our empirical concepts in order that they may discharge their task—the task of the 'reification' [Objektivierung] of phenomena"* [9, p. 19]. This, however, poses the question of freedom in a completely new way. Human action can now be regarded from two different perspectives rigorously founded in an analysis of knowledge. As KANT states, a rational being can regard him or herself from two standpoints: In the first sense it is like everything an empirical object and as such subject to universal determinism. In the second sense human beings may be accepted as things in themselves and therefore as free. Based on this picture ERNST CASSIRER comes to the following conclusion:

> *By virtue of this doctrine Kant can remain a strict empirical determinist and can nevertheless assert that precisely this empirical determination leaves the way open for another determination, different in principle, which he calls the determination through the moral law or the pure autonomy of the will. The two are not mutually exclusive in the Kantian system, but rather require and condition each other (...)* [9, p. 202].

The object investigated by natural science is thereby fundamentally altered, which KANT acknowledges in approaching the historical fact of natural science in terms of mathematics and experiment.

> *They [all students of nature] learned that reason has insight only into that which it produces after a plan of its own (...). Reason, holding in one hand its principles, according to which alone concordant appearances can be admitted as equivalent to laws, and in the other hand the experiment which it has devised in conformity with these principles, must approach nature in order to be taught by it. It must not, however, do so in the character of a pupil who listens to everything that the teacher chooses to say, but of an appointed judge who compels the witnesses to answer questions which he has himself formulated* [22, B XIIIf, p. 20].

The debate on determinism goes on, unfortunately often below the level of Leibniz and Clarke or Bohr and Einstein. One major point is often neglected, although it had already been clarified by Kant's analysis: The observer and his perspective have to be taken into account. Cassirer expresses it well by saying

> *that after the decisive advance attained through Hume and Kant in the analysis of the causal problem, it is no longer possible to regard the causal relation as a simple connection between things, or to prove or disprove it in this sense* [9, p. 20].

## 5. Back to Some Roots of Our Problem: Motion in History

It does not seem to be mere chance that determinism is such an perennial. Indeed, the problem of describing motion can be traced as far back as the beginning of ancient Greek philosophy.[20]

In this section we will look at some of the oldest formulations of this problem as well as two types of solution—Platonism and atomism—which are two of the main paradigms for all subsequent Western natural philosophy and science.

*Into the same river we step*
*and do not step;*
*we are it,*
*and we are not.*
Heraclitus of Ephesus
[1, p. 78]

*How could What Is be something of the future?*
*How could it come-to-be?*
*For if it were coming-to-be,*
*or if it were going to be in the future,*
*in either case there would be a time when it is not.*
*Thus coming-to-be is quenched,*
*and destruction is unthinkable.*
Parmenides of Elea [1, p. 98]

20 The subsequent analysis will follow the historical reconstruction offered by Georg Picht (see [34], but also [15, Chap. IV]).

The question concerning the nature of motion was deemed important as early as the fifth century B.C. in the reflections handed down from the pre-Socratic philosophers. Two fundamental experiences found expression in the opposing views of PARMENIDES OF ELEA (ca. 540–480) and HERACLITUS OF EPHESUS (ca. 550–480). According to HERACLITUS, all things are in a continual flux, nothing remains the same, and every moment has already slipped into the past. Just the opposite position is taken by PARMENIDES. He denies change of any kind. True being is unchanged and immutable for the simple argument that "being is and nonbeing is not." Hence the pre-Socratic experience of the world was caught in a contradiction: Either motion is universal or it is nothing at all. This dilemma seems to have been a motivating force for creative thought throughout Western history. Every concept of motion is compelled to come to terms with this fundamental contradiction, since motion means *change* of something that *is*.

As early as in PLATO'S (427–348) dialogue *Timaeus*, the main problem was portrayed in precisely these terms:

> *(...) in my judgement, we must make a distinction and ask, what is that which always is and has no becoming, and what is that which is always becoming and never is? That which is apprehended by intelligence and reason is always in the same state, but that which is conceived by opinion with the help of sensation and without reason is always in a process of becoming and perishing and never really is* [35, 27d-28a, p. 1161].

The (ontological) separation of *"that which is becoming"* from *"that which is"* corresponds to PLATO'S (epistemological) distinction between opinion and knowledge. Opinion or probability is the highest status attainable with regard to that which is becoming, whereas we can attain true knowledge only of that which is. This incipient value hierarchy is also apparent in SOCRATES' ironic remark that the ancients became so dizzy in their researches that they held all things to be in flux (cf. PLATO'S dialogue *Cratylus*, 411b-c). Despite such ironic asides, PLATO saw it as his task to mediate the realm of becoming perceived by the senses with the realm of the eternal being grasped in thought. One result of this effort is presented in PLATO'S myth of creation. The visible, mutable world is created by a mythical demiurge as a likeness of the unchangeable ideas that lie beyond the reach of sensible experience:

> *Everyone will see that he must have looked to the eternal, for the world is the fairest of creations (...). And having been created in this way, the world has been framed in the likeness of that which is apprehended by reason and mind and is unchangeable* [35, 28c–29a, pp. 1161–62].

Here, too, an implicit value hierarchy structures the argument: Because the world is perceived as a "fair" and well-ordered cosmos, it can only have been created in view of the eternal and not in view of the impermanent.

While reason can at least apprehend the original of the world, every representation of the original's image—the sensible world of change—will be but a probable approximation of the truth grasped by reason. This is why PLATO is merely being consistent in framing his account as myth and not as theory.

His narrative on the causes of the world of becoming distinguishes between *rational* and *necessary* causes:

> *(...) the creation of this world is the combined work of necessity and mind. Mind, the ruling power, persuaded necessity to bring the greater part of created things to perfection (...). But if a person will truly tell of the way in which the work was accomplished, he must include the variable cause as well* [35, 47e–48a, p. 1175].

Accordingly, PLATO furnishes two different causal accounts, the second of which, based on *blind necessity*, ushers in a theory of elementary particles. He analytically reduces the manifold of sensible phenomena to the (spatial) motion of elementary, symmetrical, and mathematically describable entities (elementary triangles) and then attempts to construct the same phenomena by resynthesizing them out of particle motion. PLATO'S first account, based on *reason*, utilizes the original-image theory to bridge the gap between being and becoming. The far-reaching consequences of PLATO'S theory cannot be overestimated in their importance for the entire European metaphysical tradition, including the notion of mathematical ideas providing the original template for the phenomenal world.

Yet another, so-to-speak naturalized, variant of the problem originated in pre-Socratic philosophy and has had an effect, as transmitted by the Roman author LUCRETIUS (ca. 96–55), on the development of modern physics. This version is known as atomism. It assumes indestructible, eternal material particles—atoms—whose continual motion is the reason for the variety of changes in the material world as perceived by the senses. Building on the ideas of DEMOCRITUS (ca. 470–380) and EPICURUS (341–270), LUCRETIUS presented this atomistic view in his didactic poem *On the Nature of Things*. After adopting the PARMENIDEAN conclusion that *"things cannot be born from nothing, cannot when begotten be brought back from nothing"* [29, p. 4], he postulates basic indestructible elements and empty space in which these elements combine in various formations. Birth and decay are thus but appearances of the fundamental process that unites and dissolves the combination of atoms. The question of how lasting structures are formed—a question that will be taken up much later by NEWTON and LAPLACE—is answered by LUCRETIUS with the notion of atoms that deviate from their paths, causing collisions and subsequently a dense fabric of structure and motion.

> *For verily not by design (...) but because many in number and shifting about in many ways throughout the universe they are driven and*

> *tormented by blows during infinite time past, after trying motions and unions of every kind at length they fall into arrangements such as those out of which this our sum of things has been formed, and by which too it is preserved* [29, p. 13].

In this context, the question of the freedom of the will presents itself to LUCRETIUS. Since his work treats mainly ethical issues, this question will play a fundamental role through most of the poem, especially because it cannot be answered with reference to either arbitrary motion or to motion guided by the force of gravitation. Consequently, LUCRETIUS pursues a course that will "naturalize" free will:

> *besides blows and weights there is another cause of motions, from which this power of free action has been begotten in us, since we see that nothing can come from nothing. (...) the mind itself does not feel an internal necessity in all its actions and is not as it were overmastered and compelled to bear and put up with this, is caused by a minute swerving of first-beginnings [atoms G.N.] at no fixed part of space and no fixed time* [29, p. 18].

In looking back at the accounts given in the third and fourth sections of this epilogue, it will be noticed how each position achieves some balance between the two poles represented by PARMENIDES and HERACLITUS. NEWTON, for instance, utilizes atomism by letting his Creator create indestructible, eternal corpuscles (see I. NEWTON *Opticks*, Quaery 31), which, alongside the mathematical laws, explain the permanence of what lies behind the phenomenal world. At the same time, however, he is compelled to postulate a ceaseless intervention of God in the workings of the universe. The efforts of LEIBNIZ and EINSTEIN can be seen as extensive reformulations of PARMENIDES' unchanging One. HUME'S work may be regarded as the attempt to balance the flux of sense impressions with the constancy of habit. Finally, in KANT'S work the role of the unchanging unity of the subject can be described as ensuring an identity-preserving fixed point in the face of the flux characteristic of all perceptual impressions.

Similarly, a consequent determinism in science represents manifest change in the context of immutable being: Although something is in motion, the motion itself is fixed according to deterministic laws. When viewed from a nontemporal perspective, such motion does not take place at all, and we end up with the PARMENIDEAN denial of motion per se. Every mathematical depiction of motion attempts to reduce motion in this way. It is accepted as real on the one hand, but on the other hand it is represented in a medium (i.e., mathematics) that is itself considered incapable of motion.

Let us return to our original question, the question of the relationship between mathematics and reality as it was described by EINSTEIN. By giving the question a slightly different twist, our focus became less the precarious relationship itself than the historical dynamic underlying it: The

bond linking mathematics and reality has repeatedly required reforging, and the result has in each case taken a different form from all previous linkages. It has become clear, at least in its rudiments, that with respect to this question mathematics and metaphysics have developed in mutual dependence, but neither has dominated over the other. Both attempts refer to each other and both will continue to rely on each other.

It is thus not so much the relation of mathematics to reality that is ultimately of importance; what is astonishing and always worthy of discussion is that such a relation is attempted, and that it, in a sense that again and again needs to be redefined and justified, can be successful at all.

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# List of Symbols and Abbreviations

| | | |
|---|---|---|
| (ACP) | Abstract Cauchy Problem .... 17, 84, 145, 150, 151, 295, 347, 368, 419 | |
| $(\mathrm{ACP}_2)$ | Abstract Second-Order Cauchy Problem | 367 |
| $(\overline{\mathrm{ACP}_2})$ | extended Abstract Second-Order Cauchy Problem | 374 |
| (cACP) | controlled Abstract Cauchy Problem | 452 |
| $(\mathrm{cACP}_2)$ | controlled Abstract Second-Order Cauchy Problem | 455 |
| (iACP) | inhomogeneous Abstract Cauchy Problem | 436 |
| (nACP) | nonautonomous Abstract Cauchy Problem | 227 |
| (nACP) | nonautonomous Abstract Cauchy Problem | 477 |
| (ADDE) | Abstract Delay Differential Equation | 420 |
| (DE) | Differential Equation | 3, 11, 497 |
| (FE) | Functional Equation | 2, 6, 8, 14, 36, 497 |
| (IDE) | Integro-Differential Equation | 449 |
| (IE) | Integral Equation/Variation of Parameters Formula | 161 |
| $(\mathrm{IE}^*)$ | Integral Equation/Variation of Parameters Formula | 161 |
| (cPDE) | Partial Differential Equation with constant coefficients | 404 |
| (vPDE) | Partial Differential Equation with variable coefficients | 404 |
| (SBeGB) | Spectral Bound equal Growth Bound Condition | 281, 282 |
| (SMT) | Spectral Mapping Theorem | 271, 280, 281, 483 |
| (WSMT) | Weak Spectral Mapping Theorem | 32, 271, 281–283, 301 |
| $\mathbb{1}$ | constant one function | 26 |
| $\mathbb{1}_J$ | characteristic function of the set $J$ | 231 |
| $\|\cdot\|_A$ | graph norm for $A$ | 52, 515 |
| $\|\cdot\|_{\mathrm{ess}}$ | essential norm | 249 |
| $\|\cdot\|_{F_\alpha}$ | Favard norm of order $\alpha$ | 129 |
| $\|\cdot\|_{X_\alpha}$ | Hölder norm of order $\alpha$ | 130 |
| $\|\cdot\|_n$ | Sobolev norm of order $n$ | 124 |
| $\langle\cdot,\cdot\rangle$ | canonical bilinear form | 511 |
| $(\cdot\,\vert\,\cdot)$ | inner product | 517 |

| | | |
|---|---|---|
| $\oplus^2_{n\in\mathbb{N}} X_n$ | Hilbert direct sum of the Hilbert spaces $X_n$ | 511 |
| $f \otimes y$ | element of a space of vector-valued functions | 520 |
| $f \otimes T$ | operator on a space of vector-valued functions | 520 |
| $x \otimes x'$ | rank-one operator | 521 |
| $A'$ | (Banach space) adjoint of $A$ | 517 |
| $A^*$ | (Hilbert space) adjoint of $A$ | 517 |
| $\overline{A}$ | closure of $A$ | 516 |
| $A^n$ | $n$th power of $A$ | 519 |
| $A_n$ | part/extension of $A$ in $X_n$ | 124, 126 |
| $A^\alpha$ | $\alpha$th power of $A$ | 140 |
| $A \subset B$ | $A$ is a restriction of $B$ | 516 |
| $A_{\mid Y}$ | part of $A$ in $Y$ | 60 |
| $\mathrm{AC}(J)$ | space of absolutely continuous functions | 64, 510 |
| $A\sigma(A)$ | approximate point spectrum of $A$ | 242 |
| $c$ | space of convergent sequences | 509 |
| $c_0$, $c_0(X)$ | space of null sequences | 509 |
| $\mathrm{C}^\alpha(J)$ | classical Hölder space of order $\alpha$ | 136, 510 |
| $\mathrm{C}^\infty(J)$ | space of infinitely many times differentiable functions | 510 |
| $\mathrm{C}^k(J)$ | space of $k$-times continuously differentiable functions | 510 |
| $\mathrm{C}(\Omega)$ | space of continuous functions | 510 |
| $\mathrm{C}_0(\Omega)$ | space of continuous functions vanishing at infinity | 25, 510 |
| $\mathrm{C}_\mathrm{b}(\Omega)$ | space of bounded continuous functions | 510 |
| $\mathrm{C}_\mathrm{c}(\Omega)$ | space of continuous functions having compact support | 25, 510 |
| $\mathrm{C}_\mathrm{ub}(\Omega)$ | space of bounded, uniformly continuous functions | 510 |
| $\overline{\mathrm{co}}(K)$ | closed convex hull of $K$ | 511 |
| $D(A)$ | domain of $A$ | 49 |
| $D(A^\infty)$ | intersection of the domains of all powers of $A$ | 53, 519 |
| $D(A^n)$ | domain of $A^n$ | 53, 519 |
| ess sup | essential supremum | 32, 524 |
| $\mathcal{F}$ | Fourier transform | 406, 526 |
| $F_\alpha$ | Favard space of order $\alpha$ | 130 |
| $f * g$ | convolution of $f$ with $g$ | 164, 527, 530 |
| $\widehat{f}$ | Fourier transform of $f$ | 406, 526 |
| $\mathrm{fix}(T(t))_{t\ge 0}$ | fixed space of $(T(t))_{t\ge 0}$ | 337 |
| $\mathcal{G}(A)$ | graph of $A$ | 515 |
| $\Gamma$ | unit circle in $\mathbb{C}$ | 13 |
| $\mathrm{h}^\alpha(J)$ | classical little Hölder space of order $\alpha$ | 137, 510 |
| $\mathrm{H}^k(\Omega)$ | classical Sobolev space of order $(k,2)$ | 407, 510 |
| $\mathrm{H}^k_0(J)$ | classical Sobolev space of order $(k,2)$ | 510 |
| $\mathcal{J}(x)$ | duality set for $x \in X$ | 87 |
| $\ker(\Phi)$ | kernel of $\Phi$ | 516 |
| $\mathcal{K}(X)$ | space of all compact linear operators on $X$ | 248 |
| $\mathcal{L}$ | Laplace transform | 530 |
| $\ell^\infty$, $\ell^\infty(X)$ | space of bounded sequences | 509 |
| $\ell^p$ | space of $p$-summable sequences | 510 |
| $\mathrm{Lip_u}(J)$ | space of uniformly Lipschitz continuous functions | 510 |
| $\mathrm{L}^\infty(J,X)$ | space of $X$-valued measurable, essentially bounded functions | 510 |
| $\mathrm{L}^p(J,X)$ | space of $X$-valued $p$-Bochner integrable functions | 510 |
| $\mathrm{L}^\infty(\Omega,\mu)$ | space of measurable, essentially bounded functions | 510 |
| $\mathrm{L}^p(\Omega,\mu)$ | space of $p$-integrable functions | 510 |
| $\mathcal{L}(X)$, $\mathcal{L}(X,Y)$ | space of bounded linear operators | 511, 517 |
| $\mathrm{M_b}(\mathbb{R})$ | space of regular (signed or complex) Borel measures | 510 |

| | | |
|---|---|---|
| $M_q$ | multiplication operator associated to $q$ | 25 |
| $\mathbb{N}_0$ | nonnegative natural numbers | 36 |
| $\omega_0(U)$ | growth bound of the evolution family $(U(t,s))_{t\geq s}$ | 479 |
| $\omega_0(\mathcal{T})$ | growth bound of the semigroup $\mathcal{T}$ | 40, 299 |
| $\omega_{\text{ess}}(\mathcal{T})$ | essential growth bound of the semigroup $\mathcal{T}$ | 258 |
| $P\sigma(A)$ | point spectrum of $A$ | 241 |
| $q_{\text{ess}}(\Omega)$ | essential range of the function $q$ | 31, 524 |
| $\mathrm{r}(A)$ | spectral radius of $A$ | 241 |
| $\mathrm{r}_{\text{ess}}(T)$ | essential spectral radius of $T$ | 249 |
| $R(\lambda, A)$ | resolvent of $A$ in $\lambda$ | 239 |
| $R\sigma(A)$ | residual spectrum of $A$ | 243 |
| $\mathrm{rg}(A)$ | range of $A$ | 517 |
| $\rho(A)$ | resolvent set of $A$ | 239 |
| $\rho_{\mathrm{F}}(T)$ | Fredholm domain of $T$ | 248 |
| $\mathscr{S}(\mathbb{R}^N)$ | Schwartz space of rapidly decreasing functions | 405, 511 |
| $\mathrm{s}(A)$ | spectral bound of $A$ | 57, 250 |
| $\mathcal{S}_{t_0}^{\mathrm{DS}}$ | class of Desch–Schappacher perturbations | 183 |
| $\mathcal{S}_{t_0}^{\mathrm{MV}}$ | class of Miyadera–Voigt perturbations | 196 |
| $\Sigma_\delta$ | sector in $\mathbb{C}$ of angle $\delta$ | 96 |
| $\sigma(A)$ | spectrum of $A$ | 239 |
| $\Sigma(A, B, C)$ | control system | 452 |
| $\sigma_{\text{ess}}(T)$ | essential spectrum of $T$ | 248 |
| $\sigma(X, X')$ | weak topology | 511 |
| $\sigma(X', X)$ | weak* topology | 511 |
| $\sigma_+(A)$ | boundary spectrum of $A$ | 116 |
| $\mathrm{Sp}(\mathcal{T})$ | Arveson spectrum of the group $\mathcal{T}$ | 285 |
| $\mathrm{Sp}(U)$ | Arveson spectrum of the operator $U$ | 287 |
| $\mathrm{supp}\, f$ | support of $f$ | 25 |
| $(T(t))_{t\geq 0}$ | one-parameter semigroup of linear operators | 14 |
| $(T(t))_{t\in\mathbb{R}}$ | one-parameter group of linear operators | 14 |
| $\left(T(t)_{/Y}\right)_{t\geq 0}$ | quotient semigroup of $(T(t))_{t\geq 0}$ in ${}^X/_Y$ | 61 |
| $\left(T(t)_{\vert Y}\right)_{t\geq 0}$ | subspace semigroup of $(T(t))_{t\geq 0}$ in $Y$ | 60 |
| $(T(t)^{\odot})$ | sun dual semigroup of $(T(t))_{t\geq 0}$ | 62 |
| $(T_n(t))_{t\geq 0}$ | restricted/extrapolated semigroup of $(T(t))_{t\geq 0}$ in $X_n$ | 124, 126 |
| $(T_l(t))_{t\geq 0}$ | left translation semigroup | 33 |
| $(T_r(t))_{t\geq 0}$ | right translation semigroup | 33 |
| $(T(z))_{z\in\Sigma_\delta\cup\{0\}}$ | analytic semigroup of angle $\delta$ | 101 |
| $\mathrm{UBV}(\mathbb{R})$ | space of functions with uniformly bounded variation | 511 |
| $(U(t,s))_{t\geq s}$ | evolution family of linear operators | 478 |
| $\mathrm{W}^{1,p}(J, X)$ | Sobolev space of order $(1,p)$ of Bochner $p$-integrable functions | 510 |
| $\mathrm{W}^{k,p}(\Omega, \mu)$ | classical Sobolev space of order $(k,p)$ | 413, 510 |
| $X_\alpha$ | abstract Hölder space of order $\alpha$ | 130 |
| $X_n$ | abstract Sobolev space of order $n$ | 124, 126, 515 |
| $X^{\odot}$ | sun dual of $X$ | 62 |
| $Y \hookrightarrow X$ | $Y$ continuously embedded in $X$ | 60 |

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